Matthew Scroggs Advent Calendar 2024 Answers

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Answers: https://www.mscroggs.co.uk/puzzles/advent2024

Dec 1

Question: Eve writes down five different positive integers. The sum of her integers is 16. What is the product of her integers?

Answer: 144

Explanation: Let A denote any set of five distinct positive integers that sums to 16. If $1 \notin A$ then the smallest possible sum would be

$$2 + 3 + 4 + 5 + 6 = 20. \tag{1}$$

As this exceeds 16, it must be that $1 \in A$. Similarly, if $2 \notin A$, then the smallest possible sum is

$$1 + 3 + 4 + 5 + 6 = 19, \tag{2}$$

which again exceeds 16, so that it must be that $2 \in A$ as well. What if $3 \notin A$? Then, again, the smallest sum becomes

$$1 + 2 + 4 + 5 + 6 = 18, \tag{3}$$

and hence $3 \in A$ as well. Lastly, if $4 \notin A$, then the smallest sum becomes

$$1 + 2 + 3 + 5 + 6 = 17 \tag{4}$$

so that 4 must be in A.

Therefore, $A = \{1, 2, 3, 4, n\}$ where n is some unknown positive integer. Clearly we have

$$\begin{gathered} 1 + 2 + 3 + 4 + n = 16 \\ 10 + n = 16 \\ n = 6 \end{gathered} \tag{5}$$

so that it has to be that

$$A = \{1, 2, 3, 4, 6\}. \tag{6}$$

The product of these is then of course $1 \cdot 2 \cdot 3 \cdot 4 \cdot 6 = 144$, which is our answer. This was also validated using a brute force Python program.

Dec 2

Question: 14 is the smallest even number that cannot be obtained by rolling two 6-sided dice and finding the product of the numbers rolled.

What is the smallest even number that cannot be obtained by rolling one hundred 100-sided dice and finding the product of the numbers rolled?

Answer: 202

Explanation: For a pair of N-sided dice, let $f(N)$ denote the smallest even number that *cannot* be obtained by the product of a roll of the dice. The general claim is that $f(N)$ is $2a$ where a is the smallest prime number greater than N, though we will not prove this in full generality.

So, for the $N = 6$ example, the smallest prime larger than 6 is of course $a = 7$ so that $f(6) = 2a = 2 \cdot 7 = 14$. For $N = 100$, the smallest prime larger than 100 is clearly $a = 101$ so that our answer becomes $f(100) = 2a = 2{\cdot}101 = 202$.

We *will* prove this special case when $a = N + 1$, which applies to both the example and our actual problem. First, let $n = 2a$, noting that n has only the nontrivial factors 2 and a since a is prime. A roll of the N-sided pair of dice cannot produce this even number as a product because the pair would have to be either a 1 and an n or a 2 and an a. However, we have that

$$N < N + 1 = a < 2a = n \tag{7}$$

so that the neither of the required larger factors of n or a appear on any side of either die.

Now, suppose that $m = 2b$ is an even number less than n. Then of course

$$\begin{aligned} 2b = m < n = 2a \\ b < a \\ b < N + 1 \end{aligned} \tag{8}$$

so that $b \leq N$ and so is a number on the side of each die. Hence, the product of the roll of a 2 and a b is of course $2b = m$ so that this is attainable as the product of a roll. Since $m < n$ was arbitrary, this shows that $f(N) = n$ is the *smallest* even number that cannot be obtained from the pair of dice.

These results were verified with a brute force Python program.

Dec 3

Question: There are 5 ways to write 5 as the sum of positive odd numbers:

- $1 + 1 + 1 + 1 + 1$
- $1 + 1 + 3$
- $3 + 1 + 1$
- $1 + 3 + 1$
- 5

How many ways are there to write 14 as the sum of positive odd numbers?

Answer: 377

Explanation: This is similar to the 2018, Dec 1 problem and the general approach will be the same. For any positive integer n, let $f(n)$ denote the number of ways to write n as a sum of positive odd integers. For reasons that will become clear momentarily, we set $f(0) = 1$. Clearly also $1 = 1$ and $2 = 1 + 1$ are the only valid ways to write those numbers so that $f(1) = f(2) = 1$.

Now, if $n = 2m + 1$ is odd, then the first number in the sum can be one of $m + 1$ odd numbers, including n itself. Specifically, these are $2k + 1$ where $0 \leq k \leq m$. Once this number is chosen via k, the remaining value of the sum is

$$n - (2k + 1) = (2m + 1) - (2k + 1) = 2(m - k) \tag{9}$$

such that all the numbers still sum to n, and there are of course $f\left[2(m - k)\right]$ ways to express this value as the sum of odd numbers. Thus, the total number of ways to express n is to sum over all the choices of k, which becomes the recursive formula

$$f(2m + 1) = \sum_{l=0}^{m} f\left[2(m - l)\right]. \tag{10}$$

We can simplify this by setting $k = m - l$ so that $l = 0 \to k = m$ and $l = m \to k = 0$, and hence

$$f(2m+1) = \sum_{k=0}^{m} f(2k). \tag{11}$$

Note that the first number in the sum being n itself corresponds to $l = m$ in (10) and $k = 0$ in (11) so that the remaining value is 0 and hence $f(2k) = f(0) = 1$ since overall there is only one of these sums. Thus, we set $f(0) = 1$ above to make (10) and (11) more concise.

Now suppose that $n = 2m$ is even. Here there are m choices for the first number, namely the odd numbers $2k+1$ where $0 \le k \le m-1$. The remaining sum value is then

$$n - (2k+1) = 2m - 2k - 1 = 2(m-k) - 1. \tag{12}$$

Following the same line of reasoning as the odd case, the recursive formula becomes

$$f(2m) = \sum_{l=0}^{m-1} f\left[2(m-l) - 1\right]. \tag{13}$$

We can again change variables to $k = m - l$ so that $l = 0 \to k = m$ and $l = m - 1 \to k = 1$, and hence

$$f(2m) = \sum_{k=1}^{m} f(2k-1). \tag{14}$$

Now, if again $n = 2m+1$ is odd then we clearly have

$$n - 1 = 2m \tag{15}$$
$$n - 2 = 2m - 1 = 2m - 1 + 2 - 2 = 2(m-1) + 1. \tag{16}$$

Therefore, by (11), we get

$$f(n-2) = f\left[2(m-1)+1\right] = \sum_{k=1}^{m-1} f(2k) \tag{17}$$

so that

$$\begin{aligned} f(n) = f(2m+1) &= \sum_{k=0}^{m} f(2k) \\ &= \sum_{k=0}^{m-1} f(2k) + f(2m) \\ &= f(n-2) + f(n-1) \end{aligned} \tag{18}$$

by (17) and (15), which is the recursive Fibonacci relation! What happens though when $n = 2m$ is even? In this case we have

$$n - 1 = 2m - 1 \tag{19}$$
$$n - 2 = 2m - 2 = 2(m-1) \tag{20}$$

and

$$f(n-2) = f\left[2(m-1)\right] = \sum_{k=1}^{m-1} f(2k-1) \tag{21}$$

by (14). Hence,

$$f(n) = f(2m) = \sum_{k=1}^{m} f(2k-1)$$

$$
\begin{aligned}
&= \sum_{k=1}^{m-1} f(2k-1) + f(2m-1) \\
&= f(n-2) + f(n-1)
\end{aligned} \tag{22}
$$

by (21) and (19), which is the same relation!

Therefore, we have the standard Fibonacci sequence:

$$
\begin{aligned}
f(1) &= 1 \\
f(2) &= 1 \\
f(n) &= f(n-1) + f(n-2).
\end{aligned} \tag{23}
$$

Noting that we can calculate the example at $f(5) = 5$, it is then largely trivial to continue the sequence to calculate our answer of $f(14) = 377$.

These results were verified with a brute force Python program.

Dec 4

Question: The geometric mean of a set of n numbers is computed by mulitplying all the numbers together, then taking the nth root. The factors of 9 are 1, 3, and 9. The geometric mean of these factors is

$$
\sqrt[3]{1 \times 3 \times 9} = \sqrt[3]{27} = 3 \tag{24}
$$

What is the smallest number where the geometric mean of its factors is 13?

Answer: 169

Explanation: We know from the 2021, Dec 1 problem that *only* such number is $13^2 = 169$. This was verified using a brute force Python program.

Dec 5

Question: The sum of 11 consecutive integers is 2024. What is the smallest of the 11 integers?

Answer: 179

Explanation: We can leverage the results of the 2021, Dec 9 problem. There, the sum of the n consecutive integers starting at (and including) a was derived to be

$$
\sum_{k=a}^{a+n-1} k = \frac{1}{2} n(n + 2a - 1) \tag{25}
$$

Suppose that this sum is given as s and that n is also known. Then we can solve (25) for a to get

$$
\begin{aligned}
\frac{1}{2} n(n + 2a - 1) &= s \\
n + 2a - 1 &= \frac{2s}{n} \\
2a &= \frac{2s}{n} - n + 1 \\
a &= \frac{s}{n} + \frac{1-n}{2}
\end{aligned} \tag{26}
$$

In our case we have $n = 11$ and $s = 2024$ so that (26) evaluates to our answer $a = 179$.

Dec 6

Question: Put the digits 1 to 9 (using each digit exactly once) in the boxes so that the sums are correct. The sums should be read left to right and top to bottom ignoring the usual order of operations. For example, 4+3×2 is 14, not 10. Today's number is the product of the numbers in the red boxes. The number n has 55 digits. All of its digits are 9. What is the sum of the digits of n^3?

Answer: 990

Explanation: Suppose that a_k is the k-digit number whose digits are all 9, which is to say that $a_k = 10^k - 1$. Also, let $e(m)$ denote the number of digital nines in m. We show these below for $1 \leq k \leq 10$:

k	a_k	a_k^3	$e(a_k^3)$
1	9	729	1
2	99	970299	3
3	999	997002999	5
4	9999	999700029999	7
5	99999	999970000299999	9
6	999999	999997000002999999	11
7	9999999	999999700000029999999	13
8	99999999	999999970000000299999999	15
9	999999999	999999997000000002999999999	17
10	9999999999	999999999700000000029999999999	19

In the context of the digital sum of a_k^3, we can observe that a_k^3 has one seven, one two, some irrelevant number of zeros, and some number of nines. Though we do not attempt to prove it, it seems that the number of nines $e(a_k^3)$ is simply the odd numbers. That is

$$e(a_k^3) = 2k - 1. \tag{27}$$

Letting $s(m)$ denote the digital sum of m, the digital sum of a_k^3 should then be

$$\begin{aligned} s(a_k^3) &= 9e(a_k^3) + 7 + 2 = 9e(a_k^3) + 9 \\ &= 9(2k-1) + 9 = 9(2k - 1 + 1) \\ &= 18k \end{aligned} \tag{28}$$

This result was verified using a Python program for all $1 \leq k \leq 1000$. This of course included our particular case of $s(n) = s(a_{55}^3) = 18 \cdot 55 = 990$, which is our answer.

Dec 7

Question: What is the obtuse angle in degrees between the minute and hour hands of a clock at 08:22?

Answer: 119

Explanation: In what follows, we take 12 o'clock to be at zero degrees, with increasing angles going clockwise, in opposition to the standard counter-clockwise on the polar plane.

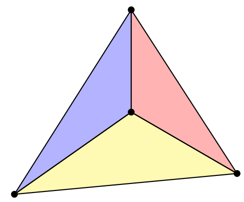
Regarding the minute hand, each revolution of the hand around the clock face is an hour, which is of course 60 minutes. Thus, each minute advances the minute hand by $m = 360/60 = 6$ degrees. This means that position of the minute hand at 08:22 would be $p_m = 22m = 132$ degrees relative to (12 o'clock).

For the hour hand, each revolution around the clock face is 12 hours, during which time $60 \cdot 12 = 720$ minutes pass. Hence, every minute advances the hour hand by $h = 360/720 = 1/2$ a degree. The time 08:22 is of course 8 hours and 22 minutes past midnight (12 o'clock on the clock face), which is equivalent to $m_h = 60 \cdot 8 + 22 = 502$ total minutes. Thus, the position of the hour hand is $p_h = m_h h = 502/2 = 251$ degrees.

The angle between them is then $|p_h - p_m| = |251 - 132| = 119$ degrees, which is indeed obtuse.

Dec 8

Question: It is possible to arrange 4 points on a plane and draw non-intersecting lines between them to form 3 non-overlapping triangles:



It is not possible to make more than 3 triangles with 4 points.

What is the maximum number of non-overlapping triangles that can be made by arranging 290 points on a plane and drawing non-intersecting lines between them?

Answer: 575

Explanation: Suppose that we have n points in a plane forming $T(n)$ triangles with a convex overall boundary. If we then add a point somewhere *inside* one of the triangles (i.e. not on an edge or at a vertex) and connect this new point to each of the three vertices of the triangle, then this divides that triangle into three smaller triangles. Hence, we have created two additional triangles (the third simply replacing the one that was subdivided) so that

$$T(n+1) = T(n) + 2. \tag{29}$$

It is easy to see that this is a way to create the *most* new triangles since putting the new point *outside* the entire shape will only create one new triangle since the shape is convex. The point could also be added on the edge of an existing triangle (but not on an existing vertex) as this subdivides *two* original triangles each into two sub-triangles, again adding two additional triangles overall.

Now, for three points, clearly the maximum number of triangles is simply one, noting that this is also always convex. Therefore, we have

$$T(3) = 1. \tag{30}$$

From (29) and (30) it is easy to deduce that

$$T(n) = 2n - 5. \tag{31}$$

This is also easy to prove using induction. First, clearly for $n = 3$ we have

$$T(n) = T(3) = 2 \cdot 3 - 5 = 6 - 5 = 1, \tag{32}$$

which comports with (30). Now suppose that (31) is true for n. Then we have

$$\begin{aligned} T(n+1) &= T(n) + 2 && \text{(by (29))} \\ &= 2n - 5 + 2 && \text{(by the induction hypothesis)} \\ &= 2(n+1) - 5, && (33) \end{aligned}$$

which shows that (31) is true for $n + 1$ as well, completing the induction.

Finally, the answer we seek is then $T(290) = 2 \cdot 290 - 5 = 575$, noting that also $T(4) = 2 \cdot 4 - 5 = 3$ as expected per the example and diagram above.

Dec 9

Question: Put the digits 1 to 9 (using each digit exactly once) in the boxes so that the sums are correct. The sums should be read left to right and top to bottom ignoring the usual order of operations. For example, $4 + 3 \times 2$ is 14, not 10. Today's number is the product of the numbers in the red boxes.

	−		+		=	8
−		÷		÷		
	+		×		=	9
×		÷		÷		
	÷		×		=	9
=		=		=		
12		1		3		

Answer: 252

Explanation: Logic was used to deduce the solution:

7	−	8	+	9	=	8
−		÷		÷		
5	+	4	×	1	=	9
×		÷		÷		
6	÷	2	×	3	=	9
=		=		=		
12		1		3		

This was verified using Python as well as shown to be unique with a brute force approach.

Dec 10

Question: A number is a palindrome if it's the same when its digits are written in reverse order.

What is the sum of all the numbers between 10 and 100 that are palindromes?

Answer: 495

Explanation: We can ignore 100 since this is clearly not a palindrome. Thus, we consider all two-digit numbers: 10 through 99. It is apparent that these can be a palindrome if and only if both digits are the same. These are exactly the numbers $x_n = 10n + n$ where $1 \leq n \leq 9$. Hence, the sum of these is

$$\sum_{n=1}^{9} x_n = \sum_{n=1}^{9} (10n + n) = \sum_{n=1}^{9} 11n = 11 \sum_{n=1}^{9} n = 11 \frac{9 \cdot 10}{2} = 55 \cdot 9 = 495, \tag{34}$$

which is our answer.

This result was verified with a brute force Python program.

Dec 11

Question: There are 6 sets of integers between 1 and 5 (inclusive) that contain an odd number of numbers whose median value is 3:

- $\{3\}$
- $\{1, 3, 4\}$
- $\{2, 3, 4\}$
- $\{1, 3, 5\}$
- $\{2, 3, 5\}$
- $\{1, 2, 3, 4, 5\}$

How many sets of integers between 1 and 11 (inclusive) are there that contain an odd number of numbers whose median value is 5?

Answer: 210

Explanation: Suppose that we consider the integers between 1 and an odd $N = 2k + 1$ inclusive. Consider the number of sets of odd size $n = 2l + 1$ whose median value is m (where of course $1 \leq m \leq N$). Since m is the median value, there must be l values less than m and l values greater than m. For values less than m there are $m - 1$ possible numbers, from which we must choose l. Thus, there are

$$P_<(m, l) = \binom{m-1}{l} \tag{35}$$

possibilities. For values greater than m there are $N - m$ possible numbers, from which l must be chosen again. Here there are

$$P_>(m, l) = \binom{N-m}{l} = \binom{2k+1-m}{l} \tag{36}$$

possibilities. Since these choices are independent of each other, there are clearly

$$P(m, l) = P_<(m, l) \cdot P_>(m, l) = \binom{m-1}{l}\binom{2k+1-m}{l} \tag{37}$$

total possible sets of $2l + 1$ elements such that m is the median. It then follows that the total number of possible sets of integers in $[1, 2k + 1]$ where m is the median is

$$S(k, m) = \sum_{l=0}^{k} P(m, l) = \sum_{l=0}^{k} \binom{m-1}{l}\binom{2k+1-m}{l}. \tag{38}$$

For the example, this evaluates to the expected $S(2, 3) = 6$, noting that here $N = 5 = 2 \cdot 2 + 1$. For the main problem, we have $N = 11 = 2 \cdot 5 + 1$ so that $S(5, 5) = 210$ is our answer. These were calculated using Python.

Dec 12

Question: Holly picks a three-digit number. She then makes a two-digit number by removing one of the digits. The sum of her two numbers is 309. What was Holly's original three-digit number?

Answer: 281

Explanation: Suppose that the original three-digit number has digits $d_2d_1d_0$, which is to say that the number itself is $100d_2 + 10d_1 + d_0$. First, it must be that the digit that is removed is the least significant d_0 since otherwise the sum of the numbers would end in an even digit. Hence, we have the following addition

$$\begin{array}{r} d_2d_1d_0 \\ +\quad d_2d_1 \\ \hline 3\ 0\ 9 \end{array}$$

Since $9 + 9 = 18 < 19$, the ones place column can only sum to 9 and not 19, thus there cannot be a carry to the tens place. Moreover, noting that d_2 cannot be zero (since then we would not have a three-digit number at all), the tens place column can sum to zero only if the digits d_1 and d_2 sum to 10 since there is no carried value to account for.

It then follows that there is a carried value of 1 added to d_2 in the hundreds place so that $d_2 + 1 = 3$ and, therefore, $d_2 = 2$. From before, we then have $d_1 + d_2 = d_1 + 2 = 10$ so that $d_1 = 8$. Lastly, we know that $d_0 + d_1 = d_0 + 8 = 9$, thus $d_0 = 1$. So our addition has to be

$$\begin{array}{r} 281 \\ +\ 28 \\ \hline 309 \end{array}$$

and Holly's original three-digit number is of course 281.

Dec 13

Question: Today's number is given in this crossnumber. No number in the completed grid starts with 0.

1	2	3
4		
5		

Across

1	Today's number.	(**3**)
4	Two times 5A.	(**3**)
5	A multiple of 1.	(**3**)

Down

1	Sum of digits is 15.	(**3**)
2	Sum of digits is 19.	(**3**)
3	Three times 5A.	(**3**)

Answer: 997

Explanation: Logic was used to deduce the solution:

1 9	2 9	3 7
4 4	7	0
5 2	3	5

This was verified using Python as well as shown to be unique with a brute force approach.

Dec 14

Question: 15^3 is 3375. The last 3 digits of 15^3 are 375.

What are the last 3 digits of $15^{1234567890}$?

Answer: 625

Explanation: Let $D(n)$ denote 15^n modulo 1000 for a natural number n, so that it is effectively only the last three digits of 15^n. We will prove that

$$D(n) = \begin{cases} 375 & n \text{ is odd} \\ 625 & n \text{ is even} \end{cases} \tag{39}$$

for all $n \geq 3$. We show this by induction on n. First, clearly

$$D(3) = 15^3 \pmod{1000} = 3375 \pmod{1000} = 375 \tag{40}$$

so that (39) holds in the base case. Now assume that (39) holds for n. If n is odd then $D(n) = 15^n \pmod{1000} = 375$ so that

$$\begin{aligned} D(n+1) &= 15^{n+1} \pmod{1000} \\ &= 15 \cdot 15^n \pmod{1000} \\ &= 15 \cdot 375 \pmod{1000} \\ &= 5625 \pmod{1000} \\ &= 625, \end{aligned} \tag{41}$$

noting that $n+1$ is even since n is odd. On the other hand, if n is even, then $D(n) = 15^n \pmod{1000} = 625$, and hence

$$\begin{aligned} D(n+1) &= 15^{n+1} \pmod{1000} \\ &= 15 \cdot 15^n \pmod{1000} \\ &= 15 \cdot 625 \pmod{1000} \\ &= 9375 \pmod{1000} \end{aligned}$$

$$= 375, \tag{42}$$

noting that $n+1$ is odd since n is even. Thus $D(n+1)$ holds for either parity of n, completing the induction.

Our answer is then

$$15^{1234567890} \pmod{1000} = D(1234567890) = 625 \tag{43}$$

since 1234567890 is even.

Dec 15

Question: The number 2268 is equal to the product of a square number (whose last digit is not 0) and the same square number with its digits reversed: 36×63.

What is the smallest three-digit number that is equal to the product of a square number (whose last digit is not 0) and the same square number with its digits reversed?

Answer: 976

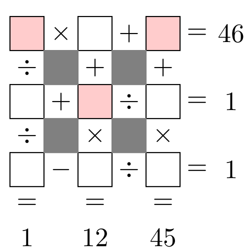
Explanation: Let $R(n)$ denote the number obtained by reversing the digits of n. We can simply enumerate the first ten square numbers and the product of the square and its reversed-digit number:

n	n^2	$R(n^2)$	$n^2 \cdot R(n^2)$	Three digits?
1	1	1	1	No
2	4	4	16	No
3	9	9	81	No
4	16	61	976	Yes
5	25	52	1300	No
6	36	63	2268	No
7	49	94	4606	No
8	64	46	2944	No
9	81	18	1458	No

The next square is $10^2 = 100$ which cannot be used (since it ends in not one, but two zeros), then comes $11^2 = 121$, which is palindromic so that $121 \cdot 121 = 14641$ is itself a square number. Regardless of these interesting properties, clearly we are getting into products that are far too large, and so there is no need to continue. Looking at the table above, evidently there is only *one* square number such that the product of it and its reversed-digit number is exactly three digits! Thus, our answer must be $4^2 \cdot R(4^2) = 16 \cdot 61 = 976$.

Dec 16

Question: Put the digits 1 to 9 (using each digit exactly once) in the boxes so that the sums are correct. The sums should be read left to right and top to bottom ignoring the usual order of operations. For example, $4 + 3 \times 2$ is 14, not 10. Today's number is the product of the numbers in the red boxes.



Answer: 336

Explanation: Logic was used to deduce the solution:

8	×	5	+	6	=	46
÷		+		+		
2	+	7	÷	9	=	1
÷		×		×		
4	−	1	÷	3	=	1
=		=		=		
1		12		45		

This was verified using Python as well as shown to be unique with a brute force approach.

Dec 17

Question: The number 40 has 8 factors: 1, 2, 4, 5, 8, 10, 20, and 40.

How many factors does the number $2^{26} \times 5 \times 7^5 \times 11^2$ have?

Answer: 972

Explanation: Let $N(n)$ denote the number of factors of a positive integer n. If n has the prime factorization of

$$n = \prod_{i=1}^{N} p_i^{e_i}, \tag{44}$$

then the number of factors is

$$N(n) = \prod_{i=1}^{N} (e_i + 1). \tag{45}$$

So, for the example, the prime factorization of 40 is $2^3 \cdot 5$ so that

$$N(40) = (3+1)(1+1) = 4 \cdot 2 = 8. \tag{46}$$

Our actual number is already given in terms of its prime factorization, i.e.

$$n = 2^{26} \cdot 5 \cdot 7^5 \cdot 11^2, \tag{47}$$

so that clearly

$$N(n) = (26+1)(1+1)(5+1)(2+1) = 27 \cdot 2 \cdot 6 \cdot 3 = 972 \tag{48}$$

is our answer.

Dec 18

Question: If $k = 21$, then $28k \div (28+k)$ is an integer.

What is the largest integer k such that $28k \div (28+k)$ is an integer?

Answer: 756

Explanation: Let $\alpha = 28$, so that we have

$$\frac{\alpha k}{\alpha + k} = m \tag{49}$$

for some integer m, noting that m must be positive since k is (since then both $\alpha k > 0$ and $\alpha + k > 0$). We can rearrange (49) to obtain

$$k = \frac{\alpha m}{\alpha - m}. \tag{50}$$

Now, the denominator must be positive since we would like our k to be positive (and finite!). Hence, it has to be that $0 < a - m$ so that $m < \alpha$. Moreover, larger m will result in a larger numerator (i.e. αm) and a smaller denominator (i.e. $\alpha - m$) so that the fraction k will be larger. That is, we want m to be as large as it can be since we are interested in the largest qualifying k.

Since also $m < \alpha$, how about the largest possible m of $m = \alpha - 1 = 27$? Indeed, $m = 27$ results in $k = 756$ from (50), noting that this simply makes the denominator 1 so that of course $k = \alpha m = \alpha(\alpha - 1)$ is an integer. Hence, $k = 756$ has to be the answer we seek.

Dec 19

Question: There are 9 integers below 100 whose digits are all non-zero and add up to 9: 9, 18, 27, 36, 45, 54, 63, 72, and 81.

How many positive integers are there whose digits are all non-zero and add up to 9?

Answer: 256

Explanation: Suppose that $1 \leq s \leq 9$ is a digital sum, and let $N_m(s)$ denote the number of m-digit numbers such that the digits are all non-zero digits and add to s. Let N_s denote the total number of qualifying integers, with any number of digits. Then it was derived in the 2022, Dec 6 problem that

$$N_s = \sum_{m=1}^{s} N_m(s), \tag{51}$$

where $N_m(s)$ is given by the recursive

$$N_m(s) = \sum_{j=m-1}^{s-1} N_{m-1}(j) \tag{52}$$

with the base case

$$N_1(s) = 1. \tag{53}$$

It was discussed in that problem that $N_m(s)$ can be found for successive values of m, needed to evaluate (51), using algebra and summation formulae, but that this quickly gets very tedious. So, instead we implement and evaluate the recursive function and (51) in Python. This results in our answer $N_9 = 256$.

The answer was also verified with a brute force search in the Python program.

Based on this answer ($N_9 = 256$) and that of the 2022, Dec 6 problem ($N_8 = 128$), it would seem that $N_s = 2^{s-1}$. Tests show that indeed this also holds for other values of s below 8. However, no approach was found to be able to prove this, so it remains simply a conjecture.

Dec 20

Question: $p(x)$ is a polynomial with integer coefficients such that:

- $p(0) > 0$;
- if x is a real number, $4x - 9 < p(x) < x^2 - 2x + 2$.

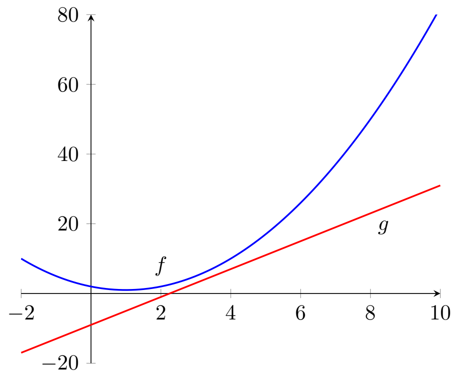
What is $p(23)$?

Answer: 484

Explanation: Define the following functions of real numbers:

$$f(x) = x^2 - 2x + 2 \tag{54}$$
$$g(x) = 4x - 9 \tag{55}$$

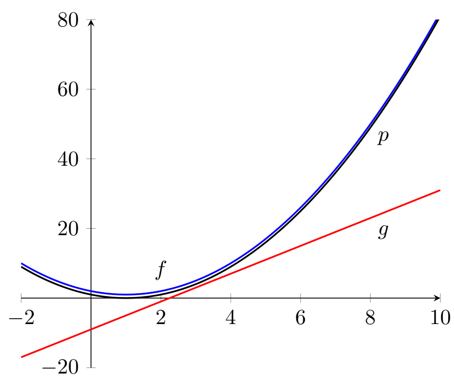
so that our second condition requires that $g(x) < p(x) < f(x)$. Let us have a look at f and g:



Obviously our function p will need to squeeze between these over the entire domain. Even though we do not even know what degree polynomial p needs to be, what if we simply shift f down one, would this work? This of course results in

$$p(x) = x^2 - 2x + 1, \tag{56}$$

noting that this does indeed have integer coefficients. Looking at our functions all together, this seems to fit the bill:



Regarding the first condition on p, clearly we have $p(0) = 1 > 0$ so that this is met. Regarding the second condition, $1 < 2$ obviously implies that

$$\begin{aligned} x^2 - 2x + 1 &< x^2 - 2x + 2 \\ p(x) &< f(x) \end{aligned} \tag{57}$$

so that the upper bound is met. For the lower bound, we take a more roundabout approach by first defining

$$h(x) = x^2 - 6x + 10. \tag{58}$$

We then have the derivatives

$$h'(x) = 2x - 6 \tag{59}$$

$$h''(x) = 2. \tag{60}$$

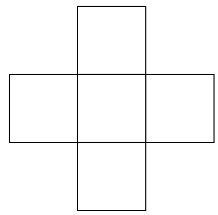
Setting $h'(x) = 0$ and solving gives $x = 3$, and plugging this back in gives $h(3) = 1$. We also note that $h''(3) = 2$ of course. Since this is a parabola, these all mean that its absolute minimum is at $(3, 1)$. So, in particular, we have

$$\begin{aligned} 0 < 1 \le h(x) &= x^2 - 6x + 10 \\ 0 &< x^2 - 2x + 1 - 4x + 9 \\ 4x - 9 &< x^2 - 2x + 1 \\ g(x) &< p(x), \end{aligned} \tag{61}$$

which shows that the lower bound holds as well. So it seems that our parabola p qualifies! We do not prove that this is the only qualifying polynomial, trusting that any other such polynomials (if they exist) would yield the same answer. Our answer is then of course $p(23) = 484$.

Dec 21

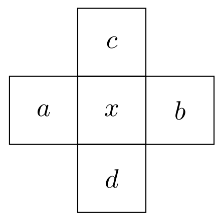
Question: Noel wants to write a different non-zero digit in each of the five boxes below so that the products of the digits of the three-digit numbers reading across and down are the same.



What is the smallest three-digit number that Noel could write in the boxes going across?

Answer: 138

Explanation: First, label the digits in the boxes as



We require that the products of the down and across digits be the same, hence

$$\begin{aligned} axb &= cxd \\ ab &= cd \end{aligned} \tag{62}$$

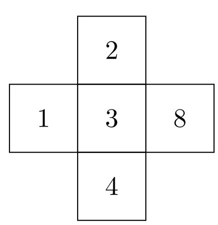
since x is nonzero. As long as the above holds, x can really be any remaining digit. Next, we would like the across number to be as small as possible, so can we get away with $a = 1$? This would mean that

$$b = 1 \cdot b = ab = cd \tag{63}$$

so that the digit b must be a composite number. The only composite digits are 4, 6, 8, and 9. Furthermore $b \in \{4, 9\}$ would require that c and d be the same digit, which is not allowed. Thus $b \in \{6, 8\}$.

We will investigate the implications of these two choices. If $b = 6$ then it has to be that one of c or d has to be 2 and the other 3. Since we do not really care about the vertical three-digit number, it does not matter which is which, so set $c = 2$ and $d = 3$. Since 1, 2, and 3 have already been used, the smallest digit we can use for x is 4 (in the interest of making the across number as small as possible). Thus, we have $(a, x, b) = (1, 4, 6)$ so that our across number is 146.

If instead we set $b = 8$ then it must be that $b = 2$ and $d = 4$, again noting that it does not matter which is which. Now 3 is available to assign to x so that $(a, x, b) = (1, 3, 8)$ so our across number is 138. This is smaller than in the previous case, so this must be our answer! Our completed boxes are then



Dec 22

Question: Put the digits 1 to 9 (using each digit exactly once) in the boxes so that the sums are correct. The sums should be read left to right and top to bottom ignoring the usual order of operations. For example, $4 + 3 \times 2$ is 14, not 10. Today's number is the largest number that can be formed using the three digits in the red boxes.

☐	+	☐	÷	☐	=	3
−		×		×		
☐	+	☐	−	☐	=	6
−		−		−		
☐	×	☐	×	☐	=	16
=		=		=		
1		13		16		

Answer: 851

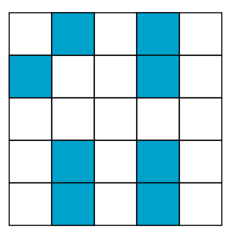
Explanation: Logic was used to deduce the solution:

9	+	3	÷	4	=	3
−		×		×		
7	+	5	−	6	=	6
−		−		−		
1	×	2	×	8	=	16
=		=		=		
1		13		16		

This was verified using Python as well as shown to be unique with a brute force approach.

Dec 23

Question: In a grid of squares, each square is friendly with itself and friendly with every square that is horizontally, vertically, or diagonally adjacent to it (and is not friendly with any other squares). In a 5×5 grid, it is possible to color 8 squares so that every square is friendly with at least two colored squares:

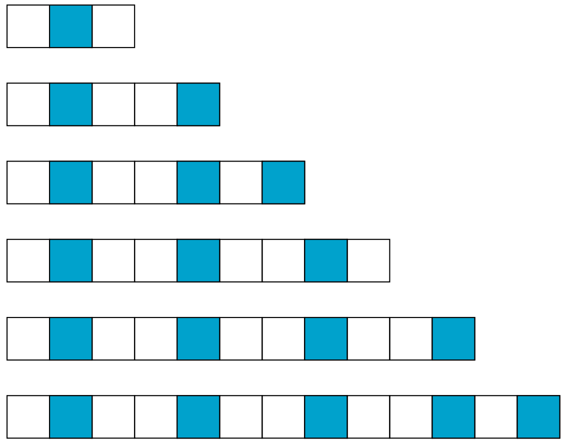


It it not possible to do this by coloring fewer than 8 squares. What is the fewest number of squares that need to be colored in a 23×23 grid so that every square is friendly with at least two colored squares?

Answer: 128

Explanation: Consider an $n \times n$ grid of squares, where we typically take n to be odd. We can decouple the horizontal dimension (for brevity, call this the x dimension) and vertical dimension (call this the y dimension) in the following way: we color our squares in columns in which every column is identical. The colored columns are then spaced such that every square is friendly with at least one colored column. We can symbolize the colored columns with a single row in which colored squares represent colored columns. The question is then how sparsely can the colored squares be spaced such that every square in the row is friendly with at least one colored square in the row?

Well, clearly the second square must be colored in order for the first to be friendly with one. The third square is then friendly with the second so does not need colored. The fourth though is not, so either it or the fifth (if there is one) needs to be colored. In general, the colored squares can be spaced two apart but with the second square and either the second-to-last or the last square needing colored in order for the last square to have a friendly colored square. We show this pattern below for $n \in \{3, 5, 7, 9, 11, 13\}$:

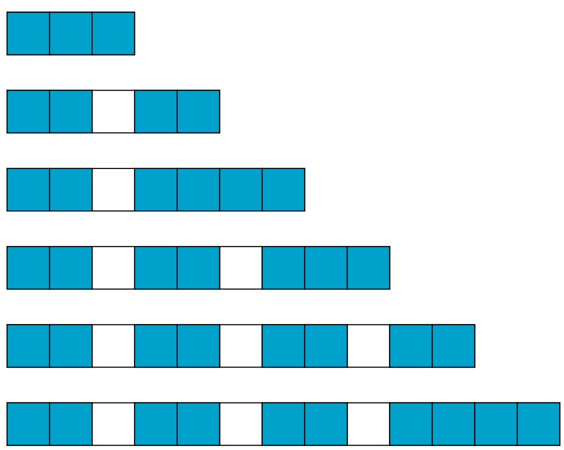


So what is the minimum number of colored squares needed for a given n. We do not prove it, but this was reasoned to be

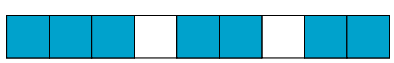
$$h(n) = \left\lceil \frac{n}{3} \right\rceil. \tag{64}$$

If we simply duplicate any of the above rows vertically n times, this will produce a grid that has the desired property (i.e. that every square is friendly with at least two colored squares), but this is not a minimum solution. The important property currently is that every square is friendly with at least one colored column.

Now we change focus to the y dimension, namely what should these columns be to minimize the number of colored squares needed while retaining the desired property? We can repeat the above exercise, representing each colored column with a single row of squares and finding the pattern such that every square is friendly with at least *two* colored squares. Here, the basic pattern involves spaced groups of two colored squares, noting that both the first two and last two squares must always be colored so the that first and last squares are friendly with two colored squares. We again show minimal required patterns for the same set of n:



Note that these minimal patterns are not unique. For example, a valid alternate pattern for $n = 9$ is the following:



However, the number of colored squares is the same for both. While more difficult to determine than the x dimension case, this minimal number of colored squares was reasoned to be

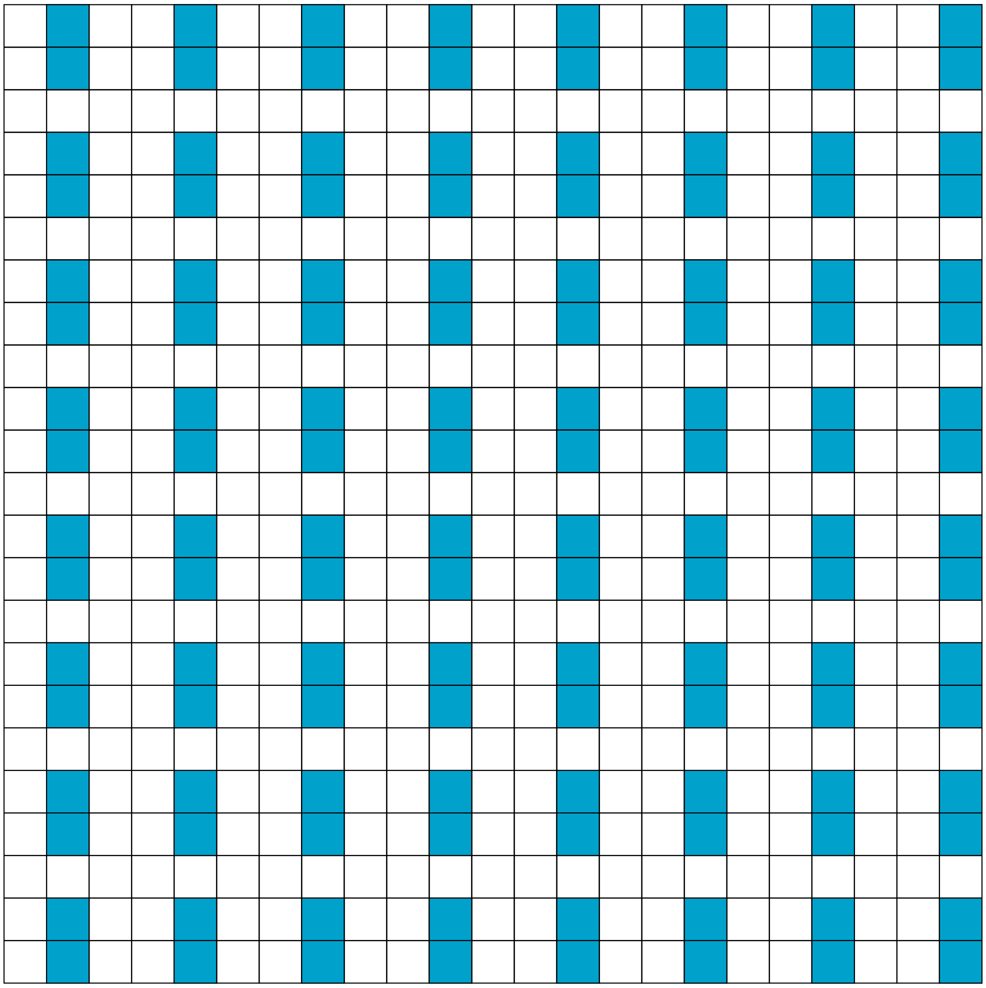
$$v(n) = n - \left\lfloor \frac{n-2}{3} \right\rfloor, \tag{65}$$

noting that the first term is simply the total number of squares and the second, floor term is the number of squares that are *not* colored. Then the total minimum number of colored squares is

$$N(n) = h(n) \cdot v(n) = \left\lceil \frac{n}{3} \right\rceil \left(n - \left\lfloor \frac{n-2}{3} \right\rfloor \right), \tag{66}$$

noting that this evaluates to the expected $N(5) = 8$ in the example case.

Putting the x and y dimension patterns together, a minimal qualifying coloring pattern for our full 23×23 grid is:



Observe the repeated columns, spaced out according to the spacing derived for the x dimension so that every square is friendly with at least one column. The vertical pattern of each column then ensures that every square is friendly with at least *two* other squares over all. Counting the colored squares above then agrees with the evaluation of our answer from (66) above with $N(23) = 128$. It is worth noting that, in all that precedes, the x and y dimensions could have been swapped with no change in the end result.

These results were checked with a brute force Python program, though this still required treating each dimension separately in order for the computation time to be reasonable.

Dec 24

Question: There are 343 three-digit numbers whose digits are all 1, 2, 3, 4, 5, 6, or 7. What is the mean of all these numbers?

Answer: 444

Explanation: Let $D = \{1, 2, 3, 4, 5, 6, 7\} = \{k \mid 1 \le k \le 7\}$ be the set of used digits. Since digits can be repeated, there are $7^3 = 343$ three-digit numbers that can be made from these, which was given. Similarly, there are $7^2 = 49$ two-digit numbers that can be made. Now consider the Cartesian product $N = D \times D \times D$, which is of course the set of ordered triplets, each of the three components being from D. Denote the three components of an $x \in N$ as x_1, x_2, and x_3. Clearly each vector $x \in N$ corresponds to a unique three-digit number of interest. The mean of these three-digit numbers is then

$$m = \frac{1}{7^3} \sum_{x \in N} (100x_1 + 10x_2 + x_3). \tag{67}$$

Now, for a fixed value of, say, x_1, there are 7^2 ordered pairs $(x_2, x_3) \in D \times D$. As a result of this, (67) can be rearranged as

$$\begin{aligned} m &= \frac{1}{7^3} \left[\sum_{d \in D} 7^2 \cdot 100d + \sum_{d \in D} 7^2 \cdot 10d + \sum_{d \in D} 7^2 d \right] \\ &= \frac{7^2}{7^3} \left[100 \sum_{d \in D} d + 10 \sum_{d \in D} d + \sum_{d \in D} d \right] \\ &= \frac{1}{7} [100 + 10 + 1] \sum_{d \in D} d \\ &= \frac{111}{7} \sum_{d=1}^{7} d \\ &= \frac{111}{7} \frac{7 \cdot 8}{2} = 111 \cdot 4 \\ &= 444, \end{aligned} \tag{68}$$

which is our answer.

This solution was verified using a brute force Python program.