

Introduction to Set Theory
Third Edition, Revised and Expanded
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Solutions Manual

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September 17, 2019

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Chapter 5 Cardinal Numbers

§5.1 Cardinal Arithmetic

Exercise 5.1.1

Prove properties (a)-(n) of cardinal arithmetic stated in the text of this section. These are

- (a) $\kappa + \lambda = \lambda + \kappa$
- (b) $\kappa + (\lambda + \mu) = (\kappa + \lambda) + \mu$
- (c) $\kappa \leq \kappa + \lambda$
- (d) If $\kappa_1 \leq \kappa_2$ and $\lambda_1 \leq \lambda_2$, then $\kappa_1 + \lambda_1 \leq \kappa_2 + \lambda_2$
- (e) $\kappa \cdot \lambda = \lambda \cdot \kappa$
- (f) $\kappa \cdot (\lambda \cdot \mu) = (\kappa \cdot \lambda) \cdot \mu$
- (g) $\kappa \cdot (\lambda + \mu) = \kappa \cdot \lambda + \kappa \cdot \mu$
- (h) $\kappa \leq \kappa \cdot \lambda$ if $\lambda > 0$
- (i) If $\kappa_1 \leq \kappa_2$ and $\lambda_1 \leq \lambda_2$, then $\kappa_1 \cdot \lambda_1 \leq \kappa_2 \cdot \lambda_2$
- (j) $\kappa + \kappa = 2 \cdot \kappa$
- (k) $\kappa + \kappa \leq \kappa \cdot \kappa$, whenever $\kappa \geq 2$
- (l) $\kappa \leq \kappa^\lambda$ if $\lambda > 0$
- (m) $\lambda \leq \kappa^\lambda$ if $\kappa > 1$
- (n) If $\kappa_1 \leq \kappa_2$ and $\lambda_1 \leq \lambda_2$, then $\kappa_1^{\lambda_1} \leq \kappa_2^{\lambda_1}$

Solution:

For solutions (a) through (c) suppose that

$$\kappa = |K| \qquad \lambda = |L| \qquad \mu = |M|,$$

where K , L , and M are mutually disjoint sets.

(a)

Proof. It is obvious that

$$\kappa + \lambda = |K \cup L| = |L \cup K| = \lambda + \kappa$$

since $K \cup L = L \cup K$ and K and L are disjoint. □

(b)

Proof. First we note that clearly

$$K \cup (L \cup M) = K \cup L \cup M = (K \cup L) \cup M.$$

Now suppose that there is an $x \in K \cap (L \cup M)$ so that $x \in K$ and $x \in L \cup M$. If $x \in L$ then $x \in K \cap L$ and if $x \in M$ then $x \in K \cap M$, either of which is a contradiction since all three sets are mutually disjoint. Hence K and $L \cup M$ are disjoint. A similar argument shows that $K \cup L$ and M are disjoint. Thus we have the following:

$$\kappa + (\lambda + \mu) = |K| + |L \cup M| = |K \cup (L \cup M)| = |(K \cup L) \cup M| = |K \cup L| + |M| = (\kappa + \lambda) + \mu$$

as desired. □

(c)

Proof. Define the function $f : K \rightarrow K \cup L$ by simply the identity $f(k) = k$ for any $k \in K$. Obviously this is an injective function so that $\kappa = |K| \leq |K \cup L| = \kappa + \lambda$. \square

(d)

Proof. Suppose that

$$\kappa_1 = |K_1| \qquad \kappa_2 = |K_2| \qquad \lambda_1 = |L_1| \qquad \lambda_2 = |L_2|$$

for sets $K_1, K_2, L_1,$ and L_2 where $K_1 \cap L_1 = \emptyset$ and $K_2 \cap L_2 = \emptyset$. Also suppose that $\kappa_1 \leq \kappa_2$ and $\lambda_1 \leq \lambda_2$. Thus $|K_1| = \kappa_1 \leq \kappa_2 = |K_2|$ so that there is an injective function f from K_1 to K_2 . Similarly there is an injective function $g : L_1 \rightarrow L_2$ since $|L_1| = \lambda_1 \leq \lambda_2 = |L_2|$. Now define $h : K_1 \cup L_1 \rightarrow K_2 \cup L_2$ by

$$h(x) = \begin{cases} f(x) & x \in K_1 \\ g(x) & x \in L_1. \end{cases}$$

We show that h is injective so consider x and y in $K_1 \cup L_1$ where $x \neq y$.

Case: $x \in K_1, y \in K_1$. Then

$$h(x) = f(x) \neq f(y) = h(y)$$

since f is injective and $x \neq y$.

Case: $x \in L_1, y \in L_1$. Then

$$h(x) = g(x) \neq g(y) = h(y)$$

since g is injective and $x \neq y$.

Case: $x \in K_1, y \in L_1$. Then we have $h(x) = f(x) \in K_2$ and $h(y) = g(y) \in L_2$ so that $h(x) \neq h(y)$ since K_2 and L_2 are disjoint. Note that this is the same as the case in which $x \in L_1$ and $y \in K_1$ since we simply switch x and y .

Since these cases are exhaustive and $h(x) \neq h(y)$ in each this shows that h is injective. Hence we have demonstrated that

$$\kappa_1 + \lambda_1 = |K_1 \cup L_1| \leq |K_2 \cup L_2| = \kappa_2 + \lambda_2$$

as desired. \square

For solutions (e) through (h) suppose that

$$\kappa = |A| \qquad \lambda = |B| \qquad \mu = |C|$$

for sets $A, B,$ and C .

(e)

Proof. First we show that $|A \times B| = |B \times A|$ by constructing a bijection $f : A \times B \rightarrow B \times A$. For $(a, b) \in A \times B$ define

$$f(a, b) = (b, a) \in B \times A,$$

which is clearly a function. Then for $(a, b) \in A \times B$ and $(c, d) \in A \times B$ where $f(a, b) = f(c, d)$ we have

$$f(a, b) = (b, a) = f(c, d) = (d, c)$$

so that $b = d$ and $a = c$. Hence $(a, b) = (c, d)$ so that f is injective. Now consider any $(b, a) \in B \times A$ so that clearly $f(a, b) = (b, a)$, noting that $(a, b) \in A \times B$. Clearly this shows that f is surjective.

Hence f is bijective so that

$$\kappa \cdot \lambda = |A \times B| = |B \times A| = \lambda \cdot \kappa$$

as required. □

(f)

Proof. Similar part (e) above, it is trivial to find a bijection from $A \times (B \times C)$ to $(A \times B) \times C$ so that

$$\kappa \cdot (\lambda \cdot \mu) = |A \times (B \times C)| = |(A \times B) \times C| = (\kappa \cdot \lambda) \cdot \mu$$

as desired. □

(g)

Proof. Here suppose additionally that $B \cap C = \emptyset$. First we note that since B and C are disjoint that $A \times B$ and $A \times C$ are also disjoint. Suppose that this is not the case so that there is an $(a, b) \in A \times B$ where $(a, b) \in A \times C$ also. Then clearly $b \in B$ and $b \in C$, which is a contradiction since they are disjoint. Now, it is also trivial to show the equality

$$A \times (B \cup C) = (A \times B) \cup (A \times C).$$

Hence we have that

$$\kappa \cdot (\lambda + \mu) = \kappa \cdot |B \cup C| = |A \times (B \cup C)| = |(A \times B) \cup (A \times C)| = |A \times B| + |A \times C| = \kappa \cdot \lambda + \kappa \cdot \mu$$

as desired. □

(h)

Proof. Here suppose that $\lambda > 0$ so that $B \neq \emptyset$. Here we construct a bijection $f : A \rightarrow A \times B$, from which it follows that

$$\kappa = |A| \leq |A \times B| = \kappa \cdot \lambda.$$

Since $B \neq \emptyset$ there exists a $b \in B$. So for any $a \in A$ define

$$f(a) = (a, b),$$

which is clearly a function. So for $a_1, a_2 \in A$ where $a_1 \neq a_2$ we have that

$$f(a_1) = (a_1, b) \neq (a_2, b) = f(a_2)$$

so that f is injective. □

(i)

Proof. Suppose that

$$\kappa_1 = |A_1| \quad \kappa_2 = |A_2| \quad \lambda_1 = |B_1| \quad \lambda_2 = |B_2|$$

for sets A_1, A_2, B_1 , and B_2 where $\kappa_1 = |A_1| \leq |A_2| = \kappa_2$ and $\lambda_1 = |B_1| \leq |B_2| = \lambda_2$. Hence there is an injective function $f : A_1 \rightarrow A_2$ and injective function $g : B_1 \rightarrow B_2$. We shall construct an injective function $h : A_1 \times B_1 \rightarrow A_2 \times B_2$ so that it immediately follows that

$$\kappa_1 \cdot \lambda_1 = |A_1 \times B_1| \leq |A_2 \times B_2| = \kappa_2 \cdot \lambda_2$$

as required. So for $(a, b) \in A_1 \times B_1$ define

$$h(a, b) = (f(a), g(b))$$

Suppose then $(a, b) \in A_1 \times B_1$ and $(c, d) \in A_1 \times B_1$ where $(a, b) \neq (c, d)$. If $a \neq c$ then $f(a) \neq f(c)$ since f is injective so that

$$h(a, b) = (f(a), g(b)) \neq (f(c), g(d)) = h(c, d)$$

Similarly if $b \neq d$ then $g(b) \neq g(d)$ since g is injective. Hence again

$$h(a, b) = (f(a), g(b)) \neq (f(c), g(d)) = h(c, d)$$

Thus in all cases $h(a, b) \neq h(c, d)$ so that h is injective. □

(j) This is adequately proven in the text.

For solutions (k) through (m) suppose that

$$\kappa = |A| \qquad \lambda = |B|$$

for sets A and B .

(k)

Proof. Suppose here that $\kappa \geq 2$. Then $2 \leq \kappa$ and $\kappa \leq \kappa$ so that by property (i) we have

$$2 \cdot \kappa \leq \kappa \cdot \kappa.$$

Then by property (j) we have

$$\kappa + \kappa = 2 \cdot \kappa \leq \kappa \cdot \kappa$$

as desired. □

(l)

Proof. Here suppose that $\lambda = |B| > 0$ so that $B \neq \emptyset$. Hence there exists a $b \in B$. We shall construct an injective $f : A \rightarrow A^B$, from which it follows that

$$\kappa = |A| \leq |A^B| = \kappa^\lambda.$$

So for any $a \in A$ define $f(a) = g$ where $g : B \rightarrow A$ is a function defined by $g(b) = a$ for all $b \in B$, noting that $g \neq \emptyset$ since $B \neq \emptyset$.

Now consider any $a_1, a_2 \in A$ where $a_1 \neq a_2$ so that for any $b \in B$ we have

$$f(a_1)(b) = a_1 \neq a_2 = f(a_2)(b).$$

From this it follows that $f(a_1) \neq f(a_2)$ so that f is injective. □

(m)

Proof. Here suppose that $\kappa = |A| > 1$ so that there are $a_1, a_2 \in A$ where $a_1 \neq a_2$. We shall construct an injective function $f : B \rightarrow A^B$ so that

$$\lambda = |B| \leq |A^B| = \kappa^\lambda.$$

So for any $b \in B$ define $f(b) = g$ where $g : B \rightarrow A$ is a function defined by

$$g(c) = \begin{cases} a_1 & c = b \\ a_2 & c \neq b \end{cases}$$

for $c \in B$. Now suppose that $b_1, b_2 \in B$ where $b_1 \neq b_2$. We then have

$$f(b_1)(b_1) = a_1 \neq a_2 = f(b_2)(b_1)$$

since $b_1 \neq b_2$. From this it follows that $f(b_1) \neq f(b_2)$ so that f is injective. \square

(n)

Proof. Suppose that

$$\kappa_1 = |A_1| \quad \kappa_2 = |A_2| \quad \lambda_1 = |B_1| \quad \lambda_2 = |B_2|$$

for sets A_1, A_2, B_1 , and B_2 where $\kappa_1 = |A_1| \leq |A_2| = \kappa_2$ and $\lambda_1 = |B_1| \leq |B_2| = \lambda_2$.

The theorem as presented in the text is actually not true in full generality. As a counterexample suppose that $A_1 = A_2 = B_1 = \emptyset$ so that $\kappa_1 = \kappa_2 = \lambda_1 = 0$ and $B_2 = 1$ so that $\lambda_2 = 1$. Then certainly the hypotheses above are true but we also have

$$\kappa_1^{\lambda_1} = 0^0 = 1 > 0 = 0^1 = \kappa_2^{\lambda_2}$$

where we have used the results of Exercises 5.1.2 and 5.1.3.

However, if we add the restriction that $\kappa_2 > 0$ then it becomes true. To prove this first note that this implies that $A_2 \neq \emptyset$ so that there is an $a_2 \in A_2$. Also there is an injective function $f : A_1 \rightarrow A_2$ and an injective function $g : B_1 \rightarrow B_2$. We shall construct an injective $F : A_1^{B_1} \rightarrow A_2^{B_2}$, from which it follows that

$$\kappa_1^{\lambda_1} = |A_1^{B_1}| \leq |A_2^{B_2}| = \kappa_2^{\lambda_2}.$$

So for any $h_1 \in A_1^{B_1}$ define $F(h_1) = h_2$ where $h_2 \in A_2^{B_2}$ is defined by

$$h_2(b) = \begin{cases} f(h_1(g^{-1}(b))) & b \in \text{ran}(h_1) \\ a_2 & b \notin \text{ran}(h_1) \end{cases}$$

for any $b \in B_2$, noting that g^{-1} is a function on $\text{ran}(h_1)$ since g is injective. Clearly F is a function but now we show that it is injective.

So consider any $h_1, h_2 \in A_1^{B_1}$ where $h_1 \neq h_2$. Then there is a $b_1 \in B_1$ such that $h_1(b_1) \neq h_2(b_1)$. So let $b_2 = g(b_1)$ so that clearly $b_2 \in \text{ran}(g)$ and $b_1 = g^{-1}(b_2)$. Hence we have

$$F(h_1)(b_2) = f(h_1(g^{-1}(b_2))) = f(h_1(b_1)) \neq f(h_2(b_1)) = f(h_2(g^{-1}(b_2))) = F(h_2)(b_2)$$

since $h_1(b_1) \neq h_2(b_1)$ and f is injective. It thus follows that $F(h_1) \neq F(h_2)$ so that we have shown that F is injective. \square

Exercise 5.1.2

Show that $\kappa^0 = 1$ and $\kappa^1 = \kappa$ for all κ .

Solution:

Proof. Suppose that $\kappa = |A|$ for a set A .

We claim that $A^\emptyset = \{\emptyset\} = 1$ so that clearly

$$\kappa^0 = |A^\emptyset| = |1| = 1.$$

First consider any $f \in A^\emptyset$. Suppose that $f \neq \emptyset$ so that there is a $(b, a) \in f \subseteq \emptyset \times A$. But then $b \in \emptyset$, which is a contradiction. Hence $f = \emptyset$. So if there are any $f \in A^\emptyset$ then $f = \emptyset$ but are there any $f \in A^\emptyset$? Clearly the empty set is a function from \emptyset to A since it is vacuously true that for every $b \in \emptyset$ there is a unique $a \in A$ such that $(b, a) \in \emptyset$. Hence $\emptyset \in A^\emptyset$ so that $A^\emptyset = \{\emptyset\} = 1$.

We also claim that $|A^1| = |A|$ so that

$$\kappa^1 = |A^1| = |A| = \kappa.$$

To this end for any $f \in A^1$ define $F(f) = f(\emptyset)$, noting that $1 = \{\emptyset\}$, so that clearly $F : A^1 \rightarrow A$. Now consider any $f, g \in A^1$ where $f \neq g$. Then it has to be that

$$F(f) = f(\emptyset) \neq g(\emptyset) = F(g)$$

so that F is injective. Now consider any $a \in A$ and define $f \in A^1$ by $f(\emptyset) = a$. Then clearly

$$F(f) = f(\emptyset) = a$$

so that F is surjective. Hence we've shown that F is bijective. □

Exercise 5.1.3

Show that $1^\kappa = 1$ for all κ and $0^\kappa = 0$ for all $\kappa > 0$.

Solution:

Proof. Suppose that $\kappa = |A|$ for a set A .

Note first that if $\kappa = 0$ then by Exercise 5.1.2 it follows that

$$1^\kappa = 1^0 = 1.$$

In the case where $\kappa > 0$ we claim that there is a unique $f \in 1^A$ so that clearly then

$$1^\kappa = |1^A| = 1.$$

For existence define $f : A \rightarrow 1$ by $f(x) = \emptyset$ for all $x \in A$ so that clearly $f \in 1^A$. For uniqueness consider any $f_1, f_2 \in 1^A$. Since $1 = \{\emptyset\}$ it has to be that $f_1(x) = f_2(x) = \emptyset$ for all $x \in A$. Hence $f_1 = f_2$.

Now suppose also that $\kappa > 0$ so that $A \neq \emptyset$. Hence there is an $a \in A$. We claim that in this case that $\emptyset^A = \emptyset$ so that

$$0^\kappa = |\emptyset^A| = |\emptyset| = 0.$$

So suppose that $\emptyset^A \neq \emptyset$ so that there is an $f \in \emptyset^A$. Then it's true that for every $a \in A$ there is a unique $b \in \emptyset$ such that $f(a) = b$. But since there is an $a \in A$ this implies that there is also a $b \in \emptyset$, which is a contradiction. Hence there can be no $f \in \emptyset^A$ so that $\emptyset^A = \emptyset$. □

Exercise 5.1.4

Prove that $\kappa^\kappa \leq 2^{\kappa \cdot \kappa}$.

Solution:

Proof. Suppose that $\kappa = |A|$ for a set A . Then we construct an injective $F : A^A \rightarrow 2^{A \times A}$ so that

$$\kappa^\kappa = |A^A| \leq |2^{A \times A}| = |2|^{|A \times A|} = 2^{\kappa \cdot \kappa}.$$

So for any $f \in A^A$ define $F(f) = g$ where $g \in 2^{A \times A}$ is defined by

$$g(a_1, a_2) = \begin{cases} 0 & f(a_1) \neq a_2 \\ 1 & f(a_1) = a_2 \end{cases}$$

for $(a_1, a_2) \in A \times A$. To show that F is injective consider any $f, g \in A^A$ where $f \neq g$. Then there is an $a \in A$ such that $f(a) \neq g(a)$. Now let $a_1 = f(a)$ and $a_2 = g(a)$ so that

$$f(a) = a_1 \neq a_2 = g(a).$$

Since $f(a) = a_1$ it follows by definition that $F(f)(a, a_1) = 1$. Similarly since $g(a) = a_2 \neq a_1$ it follows that $F(g)(a, a_1) = 0$. Hence we have

$$F(f)(a, a_1) = 1 \neq 0 = F(g)(a, a_1)$$

so that clearly $F(f) \neq F(g)$. Thus F is injective. □

Exercise 5.1.5

If $|A| \leq |B|$ and if $A \neq \emptyset$, then there is a mapping of B onto A . We later show, with the help of the Axiom of Choice, that the converse is also true: If there is a mapping of B onto A , then $|A| \leq |B|$.

Solution:

Proof. Suppose that $|A| \leq |B|$ for sets A and B where $A \neq \emptyset$. Then there is an $a \in A$. There is also an injective $f : A \rightarrow B$ so that f^{-1} is a function from $\text{ran}(f) \rightarrow A$. So let g be a mapping from B to A defined by

$$g(b) = \begin{cases} f^{-1}(b) & b \in \text{ran}(f) \\ a & b \notin \text{ran}(f). \end{cases}$$

To show that g is onto consider any $x \in A$ and let $b = f(x)$. Thus $b \in \text{ran}(f)$ so that

$$g(b) = f^{-1}(b) = f^{-1}(f(x)) = x.$$

Hence g is onto since a was arbitrary. □

Exercise 5.1.6

If there is a mapping of B onto A , then $2^{|A|} \leq 2^{|B|}$. [Hint: Given g mapping B onto A , let $f(X) = g^{-1}[X]$, for all $X \subseteq A$.]

Solution:

Proof. Suppose that f is a mapping from B onto A . We shall construct an injective $F : 2^A \rightarrow 2^B$ so that

$$2^{|A|} = |2^{|A|} = |2^A| \leq |2^B| = |2^{|B|} = 2^{|B|}.$$

So for any $g \in 2^A$ let $F(g) = h$ where $h \in 2^B$ is defined by

$$h(b) = g(f(b))$$

for $b \in B$. To show that F is injective consider any $g_1, g_2 \in 2^A$ where $g_1 \neq g_2$. Then there is an $a \in A$ such that $g_1(a) \neq g_2(a)$. Since $f : B \rightarrow A$ is onto there is a $b \in B$ such that $f(b) = a$. Thus we have

$$F(g_1)(b) = g_1(f(b)) = g_1(a) \neq g_2(a) = g_2(f(b)) = F(g_2)(b)$$

so that $F(g_1) \neq F(g_2)$. Thus F is injective. □

Exercise 5.1.7

Use Cantor's Theorem to show that "the set of all sets" does not exist.

Solution:

Proof. Suppose that X is the set of all sets. Consider any $Z \in \mathcal{P}(X)$. Since clearly Z is a set we have $Z \in X$. Thus since Z was arbitrary it follows that $\mathcal{P}(X) \subseteq X$ so that by Exercise 4.1.3 $|\mathcal{P}(X)| \leq |X|$. However, this contradicts Cantor's Theorem, according to which $|\mathcal{P}(X)| > |X|$. Thus X cannot be the set of all sets. □

Exercise 5.1.8

Let X be a set and let f be a one-to-one mapping of X into itself such that $f[X] \subset X$. Then X is infinite.

Solution:

Proof. For a set X suppose that $f : X \rightarrow X$ is injective. Also suppose that $\text{ran}(f)$ is a proper subset of X .

Now suppose that X is finite so that there is an $n \in \mathbf{N}$ such that there is a bijective $g : n \rightarrow X$. We also note that clearly $g^{-1} : X \rightarrow n$ is also a bijection. Now define a function $h : n \rightarrow n$ by

$$h(k) = (g^{-1} \circ f \circ g)(k) = g^{-1}(f(g(k)))$$

for any $k \in n$. Since g , f , and g^{-1} are all injective it follows from Exercise 2.3.5 that h is also injective.

We now claim that $\text{ran}(h)$ is proper subset of n . First we note that for any $m \in n$ we have

$$g(h(m)) = g(g^{-1}(f(g(m)))) = f(g(m))$$

since g is a function. Now, since $\text{ran}(f) \subset X$ there is an $x \in X$ such that $x \notin \text{ran}(f)$. So let $k = g^{-1}(x)$ so that $g(k) = x$. Now suppose that there is an $m \in n$ such that $h(m) = k$. Then per the above we have

$$f(g(m)) = g(h(m)) = g(k) = x,$$

which is impossible since $x \notin \text{ran}(f)$. So it must be that there is no such m so that $k \notin \text{ran}(h)$. Hence since $k \in n$ it follows that $\text{ran}(g) \subset n$.

Now clearly h is a surjective mapping from n to $\text{ran}(h)$. But since h is also injective it is thus a bijection from n to $\text{ran}(h)$. However, according to Lemma 4.2.2 there is no bijective mapping from n to $\text{ran}(h)$ since $\text{ran}(h) \subset n$. We have thus arrived at a contradiction so that, if the hypotheses hold, then X cannot be finite. Hence by definition X is infinite. \square

Exercise 5.1.9

Every countable set is Dedekind infinite.

Solution:

Lemma 5.1.9.1. *If sets X and Y are equipotent (i.e. $|X| = |Y|$) and Y is Dedekind infinite then X is also Dedekind infinite.*

Proof. Since X and Y are equipotent there is a bijective $f : X \rightarrow Y$ so that f^{-1} is also bijective. Also since Y is Dedekind infinite there is a $Z \subset Y$ such that there is a bijective $g : Y \rightarrow Z$ so that g^{-1} is also bijective. So since $Z \subset Y$ there is a $y \in Y$ such that $y \notin Z$. So let $S = X - \{f^{-1}(y)\}$. Clearly since $f^{-1}(y) \in X$ it follows that $S \subset X$ since $f^{-1}(y) \notin S$. Now define $h : X \rightarrow S$ by

$$h(x) = (f^{-1} \circ g \circ f)(x) = f^{-1}(g(f(x)))$$

for $x \in X$. Since f is a function this implies that

$$f(h(x)) = f(f^{-1}(g(f(x)))) = g(f(x)).$$

Now suppose for a moment that there is an $x \in X$ such that $h(x) = f^{-1}(y)$. Then

$$g(f(x)) = f(h(x)) = f(f^{-1}(y)) = y,$$

which is impossible since $y \notin Z$ but $\text{ran}(g) \subseteq Z$. Hence there is no such x so that h really is a map from X to S (as opposed to X to X).

Now, since f^{-1} , g , and f are all injective it follows from Exercise 2.3.5 that h is injective as well. Then consider any $s \in S$ and let $x = f^{-1}(g^{-1}(f(s)))$, noting that $g^{-1}(f(s))$ exists since $s \neq f^{-1}(y)$. Then we have

$$h(x) = f^{-1}(g(f(f^{-1}(g^{-1}(f(s)))))) = f^{-1}(g(g^{-1}(f(s)))) = f^{-1}(f(s)) = s$$

so that h is surjective since s was arbitrary. Hence h is a bijective map from X to S , and since $S \subset X$ this means that X is Dedekind infinite. \square

Lemma 5.1.9.2. *\mathbf{N} is Dedekind infinite.*

Proof. Let $N = \mathbf{N} - \{0\}$ so that clearly N is proper subset of \mathbf{N} . Then we define the map $f : \mathbf{N} \rightarrow N$ by

$$f(n) = n + 1$$

for $n \in \mathbf{N}$. Consider any $n, m \in \mathbf{N}$ where $n \neq m$. Then clearly

$$\begin{aligned} n &\neq m \\ n + 1 &\neq m + 1 \end{aligned}$$

$$f(n) \neq f(m)$$

so that f is injective. Now consider any $n \in \mathbf{N}$ so that clearly $f(n-1) = (n-1) + 1 = n$, noting that since $n \neq 0$ we have $n \geq 1$ so that $n-1 \geq 0$. Hence $n-1 \in \mathbf{N}$. This shows that f is surjective. Hence f is a bijection from \mathbf{N} onto a proper subset N so that by definition \mathbf{N} is Dedekind infinite. \square

Main Problem.

Proof. Suppose that X is a countable set. Then by definition X is equipotent to \mathbf{N} . Hence since \mathbf{N} is Dedekind infinite (Lemma 5.1.9.2) it follows that X is as well by Lemma 5.1.9.1. \square

Exercise 5.1.10

If X contains a countable subset, then X is Dedekind infinite.

Solution:

Proof. Suppose that X is a set with a countable subset Y . Then by Exercise 5.1.9 Y is Dedekind infinite so that there is a $Z \subset Y \subseteq X$ such that there is a bijective $f : Y \rightarrow Z$. So define the following $g : X \rightarrow X$ by

$$g(x) = \begin{cases} f(x) & x \in Y \\ x & x \notin Y \end{cases}$$

for any $x \in X$. Now since $Z \subset Y$ there is a $y \in Y$ such that $y \notin Z$, noting that since $Y \subseteq X$, $y \in X$.

First we claim that $y \notin \text{ran}(g)$. So suppose that it is so that there is an $x \in X$ such that $g(x) = y$. If $x \in Y$ then by definition $g(x) = f(x) = y$, but this is a contradiction since $f : Y \rightarrow Z$ but $y \notin Z$. On the other hand if $x \notin Y$ then we have $g(x) = x = y$, which is also a contradiction since $y = x \in Y$. Since a contradiction follows in either case it must be that there is no such x so that $y \notin \text{ran}(g)$. Hence $\text{ran}(g) \subset X$.

Clearly g is a surjective map from X to $\text{ran}(g)$ so we now show that it is injective. So consider any $x_1, x_2 \in X$ where $x_1 \neq x_2$.

Case: $x_1 \in Y$ and $x_2 \in Y$. Then

$$g(x_1) = f(x_1) \neq f(x_2) = g(x_2)$$

since f is injective.

Case: $x_1 \notin Y$ and $x_2 \notin Y$. Then

$$g(x_1) = x_1 \neq x_2 = g(x_2).$$

Case: $x_1 \in Y$ and $x_2 \notin Y$. Then $g(x_1) = f(x_1) \in Z$ but $g(x_2) = x_2 \notin Y$ so that $x_2 \notin Z$ either since $Z \subset Y$. Hence $f(x_1) \neq x_2$ so that

$$g(x_1) = f(x_1) \neq x_2 = g(x_2).$$

Thus in all cases $g(x_1) \neq g(x_2)$ so that g is injective since x_1 and x_2 were arbitrary.

Therefore we have shown that g is a bijective map from X to $\text{ran}(g) \subset X$ so that X is Dedekind infinite by definition. \square

Exercise 5.1.11

If X is Dedekind infinite, then it contains a countable subset. [Hint: Let $x \in X - f[X]$; define $x_0 = x$, $x_1 = f(x_0)$, \dots , $x_{n+1} = f(x_n)$, \dots . The set $\{x_n \mid n \in \mathbf{N}\}$ is countable.]

Solution:

Proof. Suppose that X is a Dedekind infinite set. Then there is a $Y \subset X$ such that there is a bijective $f : X \rightarrow Y$. Since $Y \subset X$ there is an $x \in X$ such that $x \notin Y$. So first define $x_0 = x$ and then for $n \in \mathbf{N}$ define $x_{n+1} = f(x_n)$.

We claim that $x_n \neq x_m$ for any $n, m \in \mathbf{N}$ where $n \neq m$, from which it clearly follows that

$$Z = \{x_n \mid n \in \mathbf{N}\}$$

is a countable set. So consider any $n, m \in \mathbf{N}$ where $n \neq m$. Without loss of generality we can assume that $n < m$. Suppose that $x_n = x_m$. We now show by induction that $x_{n-k} = x_{m-k}$ for all $n \geq k \geq 0$. If $k = 0$ then we clearly have

$$x_{n-k} = x_{n-0} = x_n = x_m = x_{m-0} = x_{m-k}.$$

Now suppose that $x_{n-k} = x_{m-k}$. We then have

$$f(x_{n-(k+1)}) = f(x_{n-k-1}) = x_{n-k} = x_{m-k} = f(x_{m-k-1}) = f(x_{m-(k+1)})$$

Since f is injective this implies that $x_{n-(k+1)} = x_{m-(k+1)}$ so that inductive proof is complete. So since this holds for $k = n$ we have that

$$x_0 = x_{n-n} = x_{m-n} = f(x_{m-n-1}),$$

Noting that $m - n - 1 \geq 0$ since $m \geq n + 1$. But $x_0 = x \notin Y$ and $f(x_{m-n-1}) \in Y$ since $f : X \rightarrow Y$ so that we have a contradiction. So it must be that $x_n \neq x_m$. Hence Z is countable. Since also clearly $Z \subseteq X$ the proof is complete. \square

Exercise 5.1.12

If A and B are Dedekind infinite, the $A \cup B$ is Dedekind infinite. [Hint: Use Exercise 1.11.]

Solution:

Proof. Suppose that sets A and B are both Dedekind infinite. Then A contains a countable subset C by Exercise 5.1.11. Clearly $C \subseteq A \cup B$ so that C is a countable subset of $A \cup B$. Hence by Exercise 5.1.10 $A \cup B$ is Dedekind infinite. \square

Exercise 5.1.13

If A and B are Dedekind infinite, then $A \times B$ is Dedekind infinite. [Hint: Use Exercise 1.11.]

Solution:

Proof. Suppose that A and B are both Dedekind infinite. Then A contains a countable subset C by Exercise 5.1.11. Also since B is Dedekind infinite it is not finite by Exercise 5.1.8. Hence $B \neq \emptyset$

so that there is a $b \in B$. Clearly then the set

$$D = \{(a, b) \mid a \in C\}$$

is a countable subset of $A \times B$ so that $A \times B$ is Dedekind infinite by Exercise 5.1.10. \square

Exercise 5.1.14

If A is infinite, then $\mathcal{P}(\mathcal{P}(A))$ is Dedekind infinite. [Hint: For each $n \in \mathbf{N}$, let $S_n = \{X \subset A \mid |X| = n\}$. The set $\{S_n \mid n \in \mathbf{N}\}$ is a countable subset of $\mathcal{P}(\mathcal{P}(A))$.]

Solution:

Lemma 5.1.14.1. *If A is an infinite set then for any $n \in \mathbf{N}$ there is a $B \subseteq A$ such that $|B| = n$.*

Proof. Suppose that A is an infinite set and consider any $n \in \mathbf{N}$. Then $\aleph_0 \leq |A|$ so that there is an injective $f : \mathbf{N} \rightarrow A$. Now $n \in \mathbf{N}$ but also $n \subseteq \mathbf{N}$. So define the set

$$B = \{f(k) \mid k \in n\}.$$

Clearly $B \subseteq A$ and we show that $|B| = n$ by defining a mapping $g : n \rightarrow B$ by

$$g(k) = f(k)$$

for $k \in n$. Since f is injective clearly g is. Now consider any $b \in B$. By definition then there is a $k \in n$ such that $f(k) = b$. Hence $g(k) = f(k) = b$ so that g is surjective. Hence since g is bijective $|B| = n$ as desired. \square

Main Problem.

Proof. Suppose that A is infinite. Then for any $n \in \mathbf{N}$ define

$$S_n = \{X \in \mathcal{P}(A) \mid |X| = n\},$$

noting that $S_n \neq \emptyset$ by Lemma 5.1.14.1. We also note that for $n, m \in \mathbf{N}$ where $n \neq m$ we have $S_n \neq S_m$ since for any $X \in S_n$ and $Y \in S_m$ we have

$$|X| = n \neq m = |Y|$$

so that $X \neq Y$. From this it follows that

$$S = \{S_n \mid n \in \mathbf{N}\}$$

is a countable set. Also, for an $S_n \in S$ we have that each $X \in S_n$ is in $\mathcal{P}(A)$ so that $S_n \subseteq \mathcal{P}(A)$. Hence each $S_n \in \mathcal{P}(\mathcal{P}(A))$. Thus $S \subseteq \mathcal{P}(\mathcal{P}(A))$. Since S is countable it then follows that $\mathcal{P}(\mathcal{P}(A))$ is Dedekind infinite by Exercise 5.1.10. \square

§5.2 The Cardinality of the Continuum

Exercise 5.2.1

Prove that the set of all finite sets of reals has cardinality 2^{\aleph_0} . We remark here that the set of all countable sets of reals also has cardinality 2^{\aleph_0} , but the proof of this requires the Axiom of Choice.

Solution:

Proof. Let F denote the set of all finite sets of reals. First we construct an injective $f : F \rightarrow \mathbf{R}^{\mathbf{N}}$. So consider any $A \in F$. Then $|A| = n$ for an $n \in \mathbf{N}$ so that there is a finite sequence $\langle a_k \mid k \in n \rangle$ where $\text{ran}(a) = A$. Now we define an infinite sequence of reals $\hat{a} \in \mathbf{R}^{\mathbf{N}}$ by

$$\hat{a}_k = \begin{cases} a_k & k \in n \text{ (i.e. } 0 \leq k < n) \\ a_0 & k \notin n \text{ (i.e. } k \geq n) \end{cases}$$

so that clearly we have $\text{ran}(\hat{a}) = A$ as well. Note that this only works if $A \neq \emptyset$ since otherwise there is no a_0 . In the case where $A = \emptyset$ we set $\hat{a}_k = k$ for $k \in \mathbf{N}$ so that $\text{ran}(\hat{a}) = \mathbf{N}$. In any case we set $f(A) = \hat{a}$.

Now we claim that f is injective. So consider any $A, B \in F$ where $A \neq B$. If one of them is the empty set, say A , then since B is finite $m = \max(\lceil \max(B) \rceil + 1, 0)$ exists so that clearly $m \notin B$. Hence $m \notin \text{ran}(f(B)) = B$. However $m \in \text{ran}(f(A)) = \mathbf{N}$ since $m \in \mathbf{N}$. It thus follows that $\text{ran}(f(B)) \neq \text{ran}(f(A))$ so that $f(A) \neq f(B)$. On the other hand if neither A nor B is the empty set (but still $A \neq B$) then there is an $a \in A$ where $a \notin B$ or vice versa. Without loss of generality we need only consider the first case. Clearly then $a \in \text{ran}(f(A)) = A$ but $a \notin \text{ran}(f(B)) = B$ so that again $f(A) \neq f(B)$. Hence in all cases we've shown that f is injective.

Thus we have that

$$|F| \leq |\mathbf{R}^{\mathbf{N}}| = 2^{\aleph_0},$$

where the last equality was shown in Theorem 5.2.3d. Now define

$$E = \{\{x\} \mid x \in \mathbf{R}\}$$

so that clearly $E \subseteq F$ and $|E| = |\mathbf{R}|$. Hence we have

$$2^{\aleph_0} = |\mathbf{R}| = |E| \leq |F|$$

by Exercise 4.1.3. Thus by the Cantor-Bernstein Theorem $|F| = 2^{\aleph_0}$ as required. \square

Exercise 5.2.2

A real number x is *algebraic* if it is a solution of some equation

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0,$$

where a_0, \dots, a_n are integers. If x is not algebraic, it is called *transcendental*. Show that the set of algebraic numbers is countable and hence the set of all transcendental numbers has cardinality 2^{\aleph_0} .

Solution:

I did not prove this here as I have already done so when studying Rudin's Principles of Mathematical Analysis, Exercise 2.2.

Exercise 5.2.3

If a linearly ordered set P has a countable dense subset, then $|P| \leq 2^{\aleph_0}$.

Solution:

Note that the countable dense subset is dense in P and not just in itself as explained in the errata list.

Proof. Suppose that $(P, <)$ is our linearly ordered set and R is the countable dense subset of P . We construct an $f : P \rightarrow \mathcal{P}(R)$ by defining

$$f(x) = \{y \in R \mid y < x\}$$

for any $x \in P$. Clearly for such x we have that $f(x) \subseteq R$ so that $f(x) \in \mathcal{P}(R)$.

Now we claim that f is injective. So consider any $x, y \in P$ such that $x \neq y$. Without loss of generality we can assume that $x < y$. Since R is dense in P there is a $z \in R$ where $x < z < y$. From this it follows that $z \in f(y)$ but that $z \notin f(x)$ since it is not true that $z < x$. Hence clearly $f(x) \neq f(y)$ so that we have shown that f is injective.

Thus we have

$$|P| \leq |\mathcal{P}(R)| = 2^{|R|} = 2^{\aleph_0},$$

where we have used Theorem 5.1.9. □

Exercise 5.2.4

The set of all closed subsets of reals has cardinality 2^{\aleph_0} .

Solution:

Proof. Let C denote all the closed subsets of \mathbf{R} and O the open sets. We form a mapping $f : C \rightarrow O$ defined by

$$f(A) = \mathbf{R} - A$$

for $A \in C$. Clearly by definition $f(A)$ is open for every $A \in C$ since A is closed.

Now consider any $A, B \in C$ where $A \neq B$. Then there is an $a \in A$ such that $a \notin B$ or vice versa. Without loss of generality we can assume the former. Then since $a \in A$ it follows that $a \notin \mathbf{R} - A$. But also since $a \notin B$ (but $a \in \mathbf{R}$) we have that $a \in \mathbf{R} - B$. Thus

$$f(A) = \mathbf{R} - A \neq \mathbf{R} - B = f(B)$$

so that f is injective.

Now consider any $B \in O$ and let $A = \mathbf{R} - B$.

$$f(A) = \mathbf{R} - A = \mathbf{R} - (\mathbf{R} - B) = B$$

so that f is also surjective. Hence we have that

$$|C| = |O| = 2^{\aleph_0}$$

by Theorem 2.6b. □

Exercise 5.2.5

Show that, for $n > 0$, $n \cdot 2^{2^{\aleph_0}} = \aleph_0 \cdot 2^{2^{\aleph_0}} = 2^{\aleph_0} \cdot 2^{2^{\aleph_0}} = 2^{2^{\aleph_0}} \cdot 2^{2^{\aleph_0}} = (2^{2^{\aleph_0}})^n = (2^{2^{\aleph_0}})^{\aleph_0} = (2^{2^{\aleph_0}})^{2^{\aleph_0}} = 2^{2^{2^{\aleph_0}}}$.

Solution:

Lemma 5.2.5.1. For any cardinal number κ

$$1 \cdot \kappa = \kappa.$$

Proof. Suppose that $\kappa = |A|$ for a set A . We define $f : A \rightarrow 1 \times A$ by

$$f(a) = (0, a)$$

for $a \in A$, noting that $1 = \{0\}$. Clearly by simple inspection this is bijective so that

$$1 \cdot \kappa = |1 \times A| = |A| = \kappa$$

as desired. □

Main Problem.

Proof. First we note that clearly since $\aleph_0 \leq 2^{\aleph_0}$ we have

$$2^{\aleph_0} \leq 2^{2^{\aleph_0}}$$

by property (n) in section 5.1. So consider any cardinal $n \in \mathbf{N}$ where $n > 0$ so that $1 \leq n$. We then have

$$\begin{aligned} 2^{2^{\aleph_0}} &= 1 \cdot 2^{2^{\aleph_0}} && \text{(by Lemma 5.2.5.1)} \\ &\leq n \cdot 2^{2^{\aleph_0}} \leq \aleph_0 \cdot 2^{2^{\aleph_0}} \leq 2^{\aleph_0} \cdot 2^{2^{\aleph_0}} \leq 2^{2^{\aleph_0}} \cdot 2^{2^{\aleph_0}} && \text{(repeated property (i) of 5.1)} \\ &= \left(2^{2^{\aleph_0}}\right)^2 && \text{(by property (o) of 5.1)} \\ &= 2^{2 \cdot 2^{\aleph_0}} && \text{(by Theorem 5.1.7b)} \\ &= 2^{2^{\aleph_0}}. && \text{(by Theorem 5.2.2b)} \end{aligned}$$

We also have

$$\begin{aligned} 2^{2^{\aleph_0}} &= \left(2^{2^{\aleph_0}}\right)^1 && \text{(by Exercise 5.1.2)} \\ &\leq \left(2^{2^{\aleph_0}}\right)^n \leq \left(2^{2^{\aleph_0}}\right)^{\aleph_0} \leq \left(2^{2^{\aleph_0}}\right)^{2^{\aleph_0}} && \text{(repeated property (n) of 5.1)} \\ &= 2^{\aleph_0 \cdot 2^{\aleph_0}} && \text{(by Theorem 5.1.7b)} \\ &= 2^{2^{\aleph_0}} && \text{(by Theorem 5.2.2b)} \end{aligned}$$

Clearly these together with the Cantor-Bernstein Theorem shows the desired result. □

Exercise 5.2.6

The cardinality of the set of all discontinuous functions is $2^{2^{\aleph_0}}$. [Hint: Using Exercise 2.5, show that $|\mathbf{R}^{\mathbf{R}} - C| = 2^{2^{\aleph_0}}$ whenever $|C| \leq 2^{\aleph_0}$.]

Solution:

Lemma 5.2.6.1. *If B is a set with $|B| = 2^{2^{\aleph_0}}$ and A is a subset of B with $|A| \leq 2^{\aleph_0}$ then $|B - A| = 2^{2^{\aleph_0}}$.*

Proof. The proof is analogous to that of Theorem 5.2.4. So suppose that C is a set with $|C| = 2^{2^{\aleph_0}}$. Let $B = C \times C$ so that by Exercise 5.2.5 we have

$$|B| = |C \times C| = 2^{2^{\aleph_0}} \cdot 2^{2^{\aleph_0}} = 2^{2^{\aleph_0}}.$$

Also suppose that $A \subseteq B$ where $|A| = 2^{\aleph_0}$. Now define a set

$$P = \{x \in C \mid \exists y \in C((x, y) \in A)\}.$$

Clearly then $|P| \leq |A| = 2^{\aleph_0}$. Since also $|C| = 2^{2^{\aleph_0}}$ but $P \subseteq C$ it follows that there is an $x_0 \in C$ where $x_0 \notin P$. If we let $X = \{x_0\} \times C$ then any $(x, y) \in X$ is not in A so that $(x, y) \in C \times C - A = B - A$. Hence $X \subseteq B - A$ but also since there is an obvious bijection between X and C we have

$$2^{2^{\aleph_0}} = |C| = |X| \leq |B - A|.$$

Since also clearly $B - A \subseteq B$ we also have that

$$|B - A| \leq |B| = 2^{2^{\aleph_0}}.$$

Hence by the Cantor-Bernstein Theorem $|B - A| = 2^{2^{\aleph_0}}$ as desired. □

Main Problem.

Proof. By Lemma 5.2.7 $|\mathbf{R}^{\mathbf{R}}| = 2^{2^{\aleph_0}}$. Also by Theorem 5.2.6a the set C of all continuous $f : \mathbf{R} \rightarrow \mathbf{R}$ has cardinality of 2^{\aleph_0} . Thus clearly the set of all discontinuous functions from $\mathbf{R} \rightarrow \mathbf{R}$ is simply

$$D = \mathbf{R}^{\mathbf{R}} - C.$$

But then by Lemma 5.2.6.1 above we have that

$$|D| = |\mathbf{R}^{\mathbf{R}}| = 2^{2^{\aleph_0}}$$

as desired. □

Exercise 5.2.7

Construct a one-to-one mapping of $\mathbf{R} \times \mathbf{R}$ onto \mathbf{R} . [Hint: If $a, b \in [0, 1]$ have decimal expansions $0.a_1a_2a_3 \dots$ and $0.b_1b_2b_3 \dots$, map the ordered pair (a, b) onto $0.a_1b_1a_2b_2a_3b_3 \dots \in [0, 1]$. Make adjustments to avoid sequences where the digit 9 appears from some place onward.]

Solution:

Skipping this problem due to the obviousness of it in principle but the fact that the details are tedious, and I have proved similar problems when studying Rudin's Principles of Mathematical Analysis.

Chapter 6 Ordinal Numbers

§6.1 Well-Ordered Sets

Exercise 6.1.1

Give an example of a linearly ordered set $(L, <)$ and an initial segment S of L which is not of the form $\{x \mid x < a\}$, for any $a \in L$.

Solution:

We claim that $L = \mathbf{R}$ and $S = \{x \in L \mid x \leq 0\}$ with the usual order meet the criteria.

Proof. First, clearly $L = \mathbf{R}$ is a linearly ordered set. So consider any $a \in S$ and any $x < a$ so that we have

$$x < a \leq 0.$$

Hence $x \in S$ also so that by definition S is an initial segment of \mathbf{R} . Now suppose that S does have the form

$$S = \{x \in L \mid x < a\}$$

for some $a \in L$. Since $0 \leq 0$ clearly $0 \in S$ by the original definition so that by the above $0 < a$. But now consider $a/2$, which is clearly in $L = \mathbf{R}$. By the above $a/2 < a$ since $a > 0$ so $a/2 \in S$ but we also have $0 < a/2$ (hence it is not true that $a/2 \leq 0$) since $0 < a$ so that by the original definition $a/2 \notin S$. Since we have a contradiction it must be that S cannot be expressed in such a form. \square

Exercise 6.1.2

$\omega + 1$ is not isomorphic to ω (in the well-ordering by \in).

Solution:

Proof. Since $\omega = \mathbf{N}$ and $\omega + 1 = \omega \cup \{\omega\}$ clearly ω is a proper subset of $\omega + 1$ (since $\omega \notin \omega$ but $\omega \in \omega + 1$). Now consider any $a \in \omega = \mathbf{N}$ and any $x < a$. Then clearly also $x \in \mathbf{N}$ so $x \in \omega$. Thus ω is an initial segment of $\omega + 1$. Then, since it has already been shown that both ω and $\omega + 1$ are well-ordered sets, it follows from Corollary 6.1.5a that they cannot be isomorphic. \square

Exercise 6.1.3

There exist 2^{\aleph_0} well-orderings of the set of natural numbers.

Solution:

Lemma 6.1.3.1. *Suppose that A is a subset of \mathbf{N} (including $A = \mathbf{N}$). Then every initial segment of A with the standard ordering is finite.*

Proof. Consider any initial segment S of $(A, <)$. Then by Lemma 6.1.2 there is an $n \in A \subseteq \mathbf{N}$ such that $S = \{k \in A \mid k < n\}$. So consider any $k \in S$ so that $k \in A \subseteq \mathbf{N}$ and $k < n$. Then by the definition of $<$ we have that $k \in n$. Since k was arbitrary this shows that $S \subseteq n$ so that $|S| \leq n$. From this it clearly follows that S is finite since n is. \square

Main Problem.

Proof. Throughout the following let $<$ denote the standard well-ordering on \mathbf{N} and let R be the set of all well-orderings defined on \mathbf{N} .

First we construct an injective $F : R \rightarrow \mathbf{N}^{\mathbf{N}}$. So for any $\prec \in R$ we have that $(\mathbf{N}, <)$ and (\mathbf{N}, \prec) are two well-orderings of \mathbf{N} . Consider then Theorem 6.1.3. We show that (c) cannot be the case, i.e. that an initial segment of $(\mathbf{N}, <)$ cannot be isomorphic to (\mathbf{N}, \prec) . So suppose that this is the case so that f is an isomorphism from an initial segment S of $(\mathbf{N}, <)$ to (\mathbf{N}, \prec) . Then by Lemma 6.1.3.1, S is finite whereas \mathbf{N} is infinite, but since f is a bijection this is impossible since it would imply that $|S| = |\mathbf{N}| = \aleph_0$. Hence it must be that (a) $(\mathbf{N}, <)$ is isomorphic to (\mathbf{N}, \prec) or (b) the former is isomorphic to an initial segment of the latter. In either case such an isomorphism f is unique by Corollary 6.1.5c. So define $F(\prec) = f$, noting that clearly $f \in \mathbf{N}^{\mathbf{N}}$.

Now we show that F is injective by considering two $\prec_1, \prec_2 \in R$ where $\prec_1 \neq \prec_2$. Without loss of generality we can assume that there is an $(n, m) \in \prec_1$ where $(n, m) \notin \prec_2$. Thus $n \prec_1 m$ but since \prec_2 is a linear, strict ordering and $\neg(n \prec_2 m)$ it has to be that $m \prec_2 n$ since $n \neq m$. Now let $f_1 = F(\prec_1)$ and $f_2 = F(\prec_2)$. Since both f_1 and f_2 are bijective there are $k_1, l_1, k_2, l_2 \in \mathbf{N}$ such that

$$\begin{array}{ll} f_1(k_1) = n & f_2(k_2) = n \\ f_1(l_1) = m & f_2(l_2) = m \end{array}$$

Since f_1 is an isomorphism and $f_1(k_1) = n \prec_1 m = f_1(l_1)$ it follows that

$$k_1 < l_1$$

and similarly since f_2 is an isomorphism and $f_2(l_2) = m \prec_2 n = f_2(k_2)$ it follows that

$$l_2 < k_2.$$

Now we claim that either $f_1(k_1) \neq f_2(k_1)$ or $f_1(l_1) \neq f_2(l_1)$. Either case shows that $F(\prec_1) = f_1 \neq f_2 = F(\prec_2)$ so that F is injective. To this end suppose that $f_1(k_1) = f_2(k_1) = n = f_2(k_2)$. Then since f_2 is injective it follows that $k_1 = k_2$. Hence with the above we have

$$l_2 < k_2 = k_1 < l_1$$

so that $m = f_2(l_2) \prec_2 f_2(l_1)$ and hence $m \neq f_2(l_1)$. Thus we have $f_1(l_1) = m \neq f_2(l_1)$ so that the disjunction is shown (since $\neg P \rightarrow Q \equiv P \vee Q$) and F is injective.

Hence since $F : R \rightarrow \mathbf{N}^{\mathbf{N}}$ is injective we have that

$$|R| \leq |\mathbf{N}^{\mathbf{N}}| = \aleph_0^{\aleph_0} = 2^{\aleph_0}$$

by Theorem 5.2.2c.

Now suppose that B is the set of all bijections from \mathbf{N} to \mathbf{N} . We then construct an injective $G : 2^{\mathbf{N}} \rightarrow B$. So for any infinite sequence $a \in 2^{\mathbf{N}}$ we define an $f \in \mathbf{N}^{\mathbf{N}}$ by

$$f(2n) = \begin{cases} 2n & a_n = 0 \\ 2n + 1 & a_n = 1 \end{cases} \quad f(2n + 1) = \begin{cases} 2n + 1 & a_n = 0 \\ 2n & a_n = 1. \end{cases}$$

for $n \in \mathbf{N}$, i.e we swap $2n$ and $2n + 1$ if $a_n = 1$ and leave them alone if $a_n = 0$. We then assign $G(a) = f$. It is trivial but tedious to show that f is bijective so that indeed $f \in B$.

Now consider any $a, b \in 2^{\mathbf{N}}$ where $a \neq b$ and let $f = G(a)$ and $g = G(b)$. Since $a \neq b$ there is an $n \in \mathbf{N}$ where $a_n \neq b_n$. Without loss of generality we can assume that $a_n = 0 \neq 1 = b_n$. Then we have

$$f(2n) = 2n \neq 2n + 1 = g(2n)$$

since $a_1 = 0$ but $b_n = 1$. Hence $f \neq g$ so that G is injective, from which it follows that $|2^{\mathbf{N}}| \leq |B|$. Lastly we construct an injective $H : B \rightarrow R$. So for an $f \in B$ define

$$\prec = \{(f(n), f(m)) \mid (n, m) \in \mathbf{N} \times \mathbf{N} \wedge n < m\}$$

and set $H(f) = \prec$. Clearly by definition since f is bijective it is an isomorphism from $(\mathbf{N}, <)$ to (\mathbf{N}, \prec) . This means that (\mathbf{N}, \prec) is isomorphic to $(\mathbf{N}, <)$ so that clearly \prec is a well-ordering since $<$ is. Hence indeed $H(f) = \prec \in R$.

Now we show that H is injective. So consider $f_1, f_2 \in B$ where $\prec_1 = H(f_1) = H(f_2) = \prec_2$. Then $f_1^{-1} \circ f_2$ is an isomorphism from (\mathbf{N}, \prec_1) to (\mathbf{N}, \prec_2) . But since $\prec_1 = \prec_2$ these are the same well-ordered set so that it follows from Corollary 6.1.5b that the only isomorphism between them is the identity $i_{\mathbf{N}}$. Hence $f_1^{-1} \circ f_2 = i_{\mathbf{N}}$, from which it follows that $f_1 = f_2$. Therefore H is injective so that $|B| \leq |R|$.

Putting this together results in

$$2^{\aleph_0} = |2^{\mathbf{N}}| \leq |B| \leq |R|.$$

It then follows from the Cantor-Bernstein Theorem that $|R| = 2^{\aleph_0}$ as desired. \square

Exercise 6.1.4

For every infinite subset A of \mathbf{N} , $(A, <)$ is isomorphic to $(\mathbf{N}, <)$.

Solution:

Proof. Let A be an infinite subset of \mathbf{N} . Then $(A, <)$ (where $<$ is the standard well-ordering of \mathbf{N}) is a well-ordering since any $B \subseteq A$ is also a subset of \mathbf{N} and therefore has a least element. Hence by Theorem 6.1.3 either:

1. $(A, <)$ and $(\mathbf{N}, <)$ are isomorphic,
2. An initial segment of $(A, <)$ is isomorphic to $(\mathbf{N}, <)$, or
3. $(A, <)$ is isomorphic to an initial segment of $(\mathbf{N}, <)$

We show that they must be isomorphic (1) by showing that (2) and (3) lead to contradictions.

Suppose (2), i.e. that an initial segment S of $(A, <)$ is isomorphic to $(\mathbf{N}, <)$. Then since $A \subseteq \mathbf{N}$ it follows from Lemma 6.1.3.1 that S is finite. But since this is isomorphic to \mathbf{N} it means that $|S| = |\mathbf{N}| = \aleph_0$, which is a contradiction!

Now suppose (3) so that $(A, <)$ is isomorphic to an initial segment S of $(\mathbf{N}, <)$. Again Lemma 6.1.3.1 tells us that S is finite whereas A is infinite. But since they are isomorphic this implies that $|S| = |A| = \aleph_0$, which is again a contradiction!

Hence it has to be that $(A, <)$ and $(\mathbf{N}, <)$ are isomorphic. \square

Exercise 6.1.5

Let $(W_1, <_1)$ and $(W_2, <_2)$ be disjoint well-ordered sets, each isomorphic to $(\mathbf{N}, <)$. Show that the sum of the two linearly ordered sets (as defined in Lemma 4.5 in Chapter 4) is a well-ordering, and is isomorphic to the ordinal number $\omega + \omega = \{0, 1, 2, \dots, \omega, \omega + 1, \omega + 2, \dots\}$.

Solution:

Proof. Suppose that (W, \prec) is the sum and associated order as defined in Lemma 4.4.5. By that lemma \prec is a linear ordering but we must show that it is also a well-ordering.

First we note that clearly W_1 and W_2 are both well-orderings since they are both isomorphic to $(\mathbf{N}, <)$. So consider any non-empty subset of A of $W = W_1 \cup W_2$. Let $A_1 = A \cap W_1$ and $A_2 = A \cap W_2$ so that clearly they are disjoint since W_1 and W_2 are and $A_1 \subseteq W_1$ and $A_2 \subseteq W_2$. Also since A is not empty either A_1 or A_2 (or both) are also not empty. If A_1 is not empty then since $A_1 \subseteq W_1$ and $(W_1, <_1)$ is a well-ordering there is a least element $a \in A_1$. Otherwise if A_1 is empty then A_2 is not and it has a least element a since it is a non-empty subset of the well-ordered $(W_2, <_2)$. Now consider any $b \in A$ so that also $b \in W$. If $b \in W_1$ then $b \in A_1$ so that A_1 is not empty. In this case since a is the least element of A_1 we have $a \leq_1 b$ so that by definition $a \preceq b$. On the other hand if $b \in W_2$ then $b \in A_2$. If A_1 was empty then a is the least element of A_2 and $b \in A_2$ so that again $a \leq_2 b$, hence by definition $a \preceq b$. If A_1 is not empty then $a \in A_1 \subseteq W_1$ so that by the definition of the sum (W, \prec) we have that $a \prec b$ since $b \in W_2$. Hence also $a \preceq b$. Thus in all cases $a \preceq b$ so that a is the least element of A since b was arbitrary.

Now we show that (W, \prec) is isomorphic to $(\omega + \omega, <)$. First, since $(W_1, <_1)$ and $(W_2, <_2)$ are both isomorphic to $(\mathbf{N}, <)$ let $f_1 : W_1 \rightarrow \mathbf{N}$ and $f_2 : W_2 \rightarrow \mathbf{N}$ be isomorphisms. Now we define $g : W \rightarrow \omega + \omega$ by

$$g(w) = \begin{cases} f_1(w) & w \in W_1 \\ \omega + f_2(w) & w \in W_2 \end{cases}$$

for $w \in W = W_1 \cup W_2$, noting that g is well defined since W_1 and W_2 are disjoint. Clearly since $\text{ran}(f_1) = \text{ran}(f_2) = \mathbf{N}$ we have that $g(w) \in \omega + \omega$ for all $w \in W$.

Consider any $k \in \omega + \omega$ so that $k \in \mathbf{N}$ or $k = \omega + n$ for some $n \in \mathbf{N}$. In the former case let $w = f_1^{-1}(k)$, which exists since f_1 is bijective. Thus $w \in W_1$ so that by definition $g(w) = f_1(w) = k$. In the latter case let $w = f_2^{-1}(n)$, which exists since f_2 is bijective. Thus $w \in W_2$ so that by definition $g(w) = \omega + f_2(w) = \omega + n = k$. This shows that g is surjective.

Now we show that g is an increasing function. So consider any $w_1, w_2 \in W$ where $w_1 \prec w_2$.

Case: $w_1, w_2 \in W_1$. Then since $w_1 \prec w_2$ we have that $w_1 <_1 w_2$. It then follows that $g(w_1) = f_1(w_1) < f_1(w_2) = g(w_2)$ since f_1 is an isomorphism.

Case: $w_1, w_2 \in W_2$. Then since $w_1 \prec w_2$ we have that $w_1 <_2 w_2$. It then follows that $f_2(w_1) < f_2(w_2)$ since f_2 is an isomorphism. Hence we clearly then have $g(w_1) = \omega + f_2(w_1) < \omega + f_2(w_2) = g(w_2)$.

Case: $w_1 \in W_1$ and $w_2 \in W_2$. Then we have that $g(w_1) = f_1(w_1) \in \mathbf{N}$ and $g(w_2) = \omega + f_2(w_2)$ so that clearly $g(w_1) < \omega \leq g(w_2)$ since $f_2(w_2) \in \mathbf{N}$.

Case: $w_2 \in W_1$ and $w_1 \in W_2$. If this were the case then by the definition of \prec we would have that $w_2 \prec w_1$, which contradicts the established hypothesis that $w_1 \prec w_2$. Hence this case is impossible.

Hence in all cases $g(w_1) < g(w_2)$ so that g is increasing. Therefore it is also injective and an isomorphism (since we've shown that it is surjective as well). Thus we've shown that W is isomorphic to $\omega + \omega$ as desired. \square

Exercise 6.1.6

Show that the lexicographic product $(\mathbf{N} \times \mathbf{N}, <)$ (see Lemma 4.6 in Chapter 4) is isomorphic to $\omega \cdot \omega$.

Solution:

Proof. Suppose that \prec is the lexicographic ordering of $\mathbf{N} \times \mathbf{N}$. Now we define $f : \mathbf{N} \times \mathbf{N} \rightarrow \omega \cdot \omega$ by

$$f(n, m) = \omega \cdot n + m$$

for any $(n, m) \in \mathbf{N} \times \mathbf{N}$. Clearly $f(n, m) \in \omega \times \omega$.

First we show that f is surjective. So consider any $k \in \omega \cdot \omega$ so that there are $n, m \in \mathbf{N}$ where $k = \omega \cdot n + m$. Then we clearly have that $f(n, m) = \omega \cdot n + m = k$. Since clearly $(n, m) \in \mathbf{N} \times \mathbf{N}$ it follows that f is surjective.

Now we show that f is an increasing function. To this end consider any $(n_1, m_1), (n_2, m_2) \in \mathbf{N} \times \mathbf{N}$ where $(n_1, m_1) \prec (n_2, m_2)$.

Case: $n_1 = n_2$. Then since $(n_1, m_1) \prec (n_2, m_2)$ it must be that $m_1 < m_2$. Hence we have that $f(n_1, m_1) = \omega \cdot n_1 + m_1 = \omega \cdot n_2 + m_1 < \omega \cdot n_2 + m_2 = f(n_2, m_2)$.

Case: $n_1 \neq n_2$. Then since $(n_1, m_1) \prec (n_2, m_2)$ it must be that $n_1 < n_2$. Hence we have that $f(n_1, m_1) = \omega \cdot n_1 + m_1 < \omega \cdot n_2 \leq \omega \cdot n_2 + m_2 = f(n_2, m_2)$.

Thus in all cases $f(n_1, m_1) < f(n_2, m_2)$ so that f is increasing. It then follows that f is injective and isomorphic. Hence $(\mathbf{N} \times \mathbf{N}, \prec)$ is isomorphic to $\omega \cdot \omega$. \square

Exercise 6.1.7

Let $(W, <)$ be a well-ordered set, and let $a \notin W$. Extend $<$ to $W' = W \cup \{a\}$ by making a greater than all $x \in W$. Then W has a smaller order type than W' .

Solution:

This problem is looking ahead to future sections where order types and how to compare them are defined.

Proof. Suppose that α is the order type of W and that $f : W \rightarrow \alpha$ is the isomorphism. Now let $\beta = S(\alpha) = \alpha \cup \{\alpha\}$. We then claim that W' is isomorphic to β . So define a $g : W' \rightarrow \beta$ by

$$g(w) = \begin{cases} f(w) & w \in W \\ \alpha & w \notin W \end{cases}$$

for $w \in W'$. Clearly we have that $g(w) \in \beta$ for any $w \in W'$.

Now consider any $x \in \beta$. If $x = \alpha$ then set $w = a \notin W$ so that $g(w) = \alpha = x$. If $x \neq \alpha$ then $x \in \alpha$ so set $w = f^{-1}(x)$ so that then $w \in W$. We then have that $g(w) = f(w) = f(f^{-1}(x)) = x$. Therefore g is surjective.

Now consider any $w_1, w_2 \in W'$ where $w_1 < w_2$

Case: $w_1, w_2 \in W$. Then since f is an isomorphism and $w_1 < w_2$ we have that $g(w_1) = f(w_1) < f(w_2) = g(w_2)$.

Case: $w_1 \in W$ and $w_2 = a$. Then $g(w_1) = f(w_1) \in \alpha$ and $w_2 \notin W$ so that $g(w_2) = \alpha$. Hence $g(w_1) \in g(w_2)$ so that by the definition of $<$ we have that $g(w_1) < g(w_2)$.

Note that these cases are exhaustive since it can't be that $w_1 = w_2 = \alpha$ since $w_1 < w_2$ (and therefore $w_1 \neq w_2$). It also cannot be that $w_2 \in W$ but $w_1 = a$ since then it would be that $w_2 \leq w_1$ since a is the greatest element of W' , which contradicts $w_1 < w_2$. Thus in all cases $g(w_1) < g(w_2)$ so that g is increasing, and therefore injective and an isomorphism.

Hence β is the order type of W' , α is the order type of W , and $\alpha < \beta$ since $\alpha \in \beta$. \square

Exercise 6.1.8

The sets $W = \mathbf{N} \times \{0, 1\}$ and $W' = \{0, 1\} \times \mathbf{N}$, ordered lexicographically, are nonisomorphic well-ordered sets. (See the remark following Theorem 4.7 in Chapter 4.)

Solution:

Proof. Let \prec be the lexicographic ordering of $W = \mathbf{N} \times \{0, 1\}$ and \prec' be the lexicographic ordering of $W' = \{0, 1\} \times \mathbf{N}$.

First we define $f : W \rightarrow \omega$ by

$$f(n, m) = 2n + m$$

for $(n, m) \in W$. Clearly each $f(n, m) \in \mathbf{N} = \omega$.

Now consider any $k \in \omega = \mathbf{N}$. If k is even then $k = 2n$ for some $n \in \mathbf{N}$ so set $w = (n, 0) \in W$. Then clearly $f(w) = f(n, 0) = 2n = k$. On the other hand if k is odd then $k = 2n + 1$ for some $n \in \mathbf{N}$ so set $w = (n, 1) \in W$. Then clearly $f(w) = f(n, 1) = 2n + 1 = k$. This shows that f is surjective.

Now consider any $w_1 = (n_1, m_1)$ and $w_2 = (n_2, m_2)$ in W where $w_1 \prec w_2$.

Case: $n_1 = n_2$. Then since $w_1 \prec w_2$ it has to be that $m_1 < m_2$, and since $m_1, m_2 \in \{0, 1\}$ it has to be that $m_1 = 0$ and $m_2 = 1$. From this it follows that

$$f(w_1) = f(n_1, m_1) = f(n_1, 0) = 2n_1 < 2n_1 + 1 = 2n_2 + 1 = f(n_2, 1) = f(n_2, m_2) = f(w_2).$$

Case: $n_1 \neq n_2$. Then since $w_1 \prec w_2$ it has to be that $n_1 < n_2$. Then $n_1 + 1 \leq n_2$ and since also $m_1 < 2$ we have

$$f(w_1) = f(n_1, m_1) = 2n_1 + m_1 < 2n_1 + 2 = 2(n_1 + 1) \leq 2n_2 \leq 2n_2 + m_2 = f(n_2, m_2) = f(w_2).$$

Hence in all cases $f(w_1) < f(w_2)$ so that f is increasing and therefore injective and isomorphic. Therefore W is isomorphic to ω .

Now we define $g : W' \rightarrow \omega + \omega$ by

$$g(n, m) = \begin{cases} m & n = 0 \\ \omega + m & n = 1 \end{cases}$$

for $(n, m) \in W'$. Clearly since $m \in \mathbf{N}$ we have that $g(n, m) \in \omega + \omega$ for all $(n, m) \in W'$.

Now consider any $\alpha \in \omega + \omega$. If $\alpha \in \omega = \mathbf{N}$ then $(0, \alpha) \in W'$ and $g(0, \alpha) = \alpha$. On the other hand if $\alpha = \omega + m$ for some $m \in \mathbf{N}$ then $(1, m) \in W'$ and $g(1, m) = \omega + m = \alpha$. Therefore g is surjective.

Now consider any $w_1 = (n_1, m_1)$ and $w_2 = (n_2, m_2)$ in W' where $w_1 \prec' w_2$.

Case: $n_1 = n_2$. Then since $w_1 \prec' w_2$ it has to be that $m_1 < m_2$. If $n_1 = n_2 = 0$ then

$$g(w_1) = g(n_1, m_1) = g(0, m_1) = m_1 < m_2 = g(0, m_2) = g(n_2, m_2) = g(w_2).$$

On the other hand if $n_1 = n_2 = 1$ then

$$g(w_1) = g(n_1, m_1) = g(1, m_1) = \omega + m_1 < \omega + m_2 = g(1, m_2) = g(n_2, m_2) = g(w_2).$$

Case: $n_1 \neq n_2$. Then since $w_1 \prec' w_2$ it has to be that $n_1 < n_2$. Moreover since $n_1, n_2 \in \{0, 1\}$ it has to be that $n_1 = 0$ and $n_2 = 1$ so that

$$g(w_1) = g(n_1, m_1) = g(0, m_1) = m_1 < \omega + m_2 = g(1, m_2) = g(n_2, m_2) = g(w_2).$$

Hence in all cases $g(w_1) < g(w_2)$ so that g is increasing and therefore injective and an isomorphism. Therefore W' is isomorphic to $\omega + \omega$.

Now, since $w \in \omega + \omega$ we have that $\omega < \omega + \omega$ and so are distinct ordinals. Therefore by the remarks following Theorem 6.2.10 ω and $\omega + \omega$ are not isomorphic. If W and W' were isomorphic with h as the isomorphism then $g \circ h \circ f^{-1}$ would be an isomorphism from ω to $\omega + \omega$, which is impossible. So it must be that W and W' are not isomorphic. \square

§6.2 Ordinal Numbers

Exercise 6.2.1

A set X is transitive if and only if $X \subseteq \mathcal{P}(X)$.

Solution:

Proof. (\rightarrow) Suppose that X is a transitive set and consider any $x \in X$. Then $x \subseteq X$ since X is transitive. Thus $x \in \mathcal{P}(X)$ so that, since x was arbitrary, $X \subseteq \mathcal{P}(X)$.

(\leftarrow) Now suppose that $X \subseteq \mathcal{P}(X)$ and consider any $x \in X$. Then also $x \in \mathcal{P}(X)$ so that $x \subseteq X$. Hence, since x was arbitrary, X is transitive by definition. \square

Exercise 6.2.2

A set X is transitive if and only if $\bigcup X \subseteq X$.

Solution:

Proof. (\rightarrow) Suppose that X is transitive and consider any $y \in \bigcup X$. Then there is an $x \in X$ such that $y \in x$. Since X is transitive and $x \in X$ we have that $x \subseteq X$ so that $y \in X$ as well. Since y was arbitrary this shows that $\bigcup X \subseteq X$.

(\leftarrow) Now suppose that $\bigcup X \subseteq X$ and consider any $x \in X$. If $x = \emptyset$ then clearly $x \subseteq X$. So suppose that $x \neq \emptyset$ and consider any $y \in x$. Then since $x \in X$ it follows that $y \in \bigcup X$ so that also $y \in X$. So since y was arbitrary it follows that $x \subseteq X$. Since x was arbitrary by definition X is transitive. \square

Exercise 6.2.3

Are the following sets transitive?

- (a) $\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}$,
- (b) $\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$,
- (c) $\{\emptyset, \{\{\emptyset\}\}\}$.

Solution:

(a) We claim that $X = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}$ is transitive.

Proof. Suppose $x \in X$. If $x = \emptyset$ then obviously $x \subseteq X$. If $x = \{\emptyset\}$ then $x \subseteq X$ since $\emptyset \in X$. If $x = \{\{\emptyset\}\}$ then $x \subseteq X$ since $\{\emptyset\} \in X$. Thus since the cases are exhaustive we've shown that $x \subseteq X$ so that X is transitive by definition. \square

(b) We claim that $X = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}$ is transitive.

Proof. For $x \in X$ the three cases in part (a) above have the same results and, if $x = \{\emptyset, \{\emptyset\}\}$, then $x \subseteq X$ since $\emptyset \in X$ and $\{\emptyset\} \in X$. Hence again X is transitive by definition. \square

(c) We claim that $X = \{\emptyset, \{\{\emptyset\}\}$ is *not* transitive.

Proof. If $x = \{\{\emptyset\}\}$ we have that x is not a subset of X since $\{\emptyset\} \in x$ but $\{\emptyset\} \notin X$. Hence X is not transitive. \square

Exercise 6.2.4

Which of the following statements are true?

- (a) If X and Y are transitive, then $X \cup Y$ is transitive.
- (b) If X and Y are transitive, then $X \cap Y$ is transitive.
- (c) If $X \in Y$ and Y is transitive, then X is transitive.
- (d) If $X \subseteq Y$ and Y is transitive, then X is transitive.
- (e) If Y is transitive and $S \subseteq \mathcal{P}(Y)$, then $Y \cup S$ is transitive.

Solution:

(a) We claim that this is true.

Proof. Consider any $x \in X \cup Y$. If $x \in X$ then $x \subseteq X$ since X is transitive. Since also $X \subseteq X \cup Y$ we clearly have that $x \subseteq X \subseteq X \cup Y$. We can make the same argument if it is the case that $x \in Y$. Hence since x was arbitrary this shows that $X \cup Y$ is indeed transitive. \square

(b) We claim that this is true.

Proof. Consider any $x \in X \cap Y$. Then $x \in X$ and $x \in Y$. Since X and Y are transitive this means that $x \subseteq X$ and $x \subseteq Y$. So consider any $y \in x$ then $y \in X$ and $y \in Y$ so that $y \in X \cap Y$. Hence since y was arbitrary it follows that $x \subseteq X \cap Y$ so that $X \cap Y$ is transitive since x was arbitrary. \square

(c) We claim that this is *not* true.

Proof. It was shown in Exercise 6.2.3 part (a) that $Y = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}$ is transitive. So let $X = \{\{\emptyset\}\}$ so that clearly $X \in Y$. If then $x = \{\emptyset\}$ then $x \in X$ but x is not a subset of X since $\emptyset \notin X$. Hence the original hypothesis is not true. \square

(d) We claim that this is *not* true.

Proof. Again $Y = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}$ is transitive so let $X = \{\{\emptyset\}, \{\{\emptyset\}\}$ so that clearly $X \subseteq Y$. Then if $x = \{\emptyset\}$ then $x \in X$ but x is not a subset of X because $\emptyset \notin X$. Thus the original hypothesis is false. \square

(e) We claim that this is true.

Proof. Consider any $x \in Y \cup S$. If $x \in Y$ then since Y is transitive $x \subseteq Y$. Hence $x \subseteq Y \subseteq Y \cup S$. On the other hand if $x \in S$ then $x \in \mathcal{P}(Y)$ since $S \subseteq \mathcal{P}(Y)$ Hence $x \subseteq Y \subseteq Y \cup S$. Since in all cases $x \subseteq Y \cup S$ and x was arbitrary this shows that $Y \cup S$ is transitive by definition. \square

Exercise 6.2.5

If every $X \in S$ is transitive, then $\bigcup S$ is transitive.

Solution:

Proof. Consider any $x \in \bigcup S$. Then there is an $X \in S$ where $x \in X$. Since X is transitive it follows that $x \subseteq X$. So consider any $y \in x$ so that also $y \in X$. Thus also $y \in \bigcup S$ since $X \in S$. Since y was arbitrary this shows that $x \subseteq \bigcup S$. Since x was arbitrary this shows by definition that $\bigcup S$ is transitive. \square

Exercise 6.2.6

An ordinal α is a natural number if and only if every nonempty subset of α has a greatest element.

Solution:

Proof. (\rightarrow) Suppose that n is a natural number and consider any nonempty subset A of n . Since $A \subseteq n$ it follows that $|A| \leq |n| = n$ so that A is finite. Thus A is a finite set of natural numbers and so has a greatest element. This can be proven by a trivial inductive argument.

(\leftarrow) We show this by contrapositive. Suppose that α is an ordinal such that $\alpha \notin \mathbf{N}$. Then $\alpha \notin \omega = \mathbf{N}$ so that $\alpha \not\leq \omega$, from which it follows that $\alpha \geq \omega$. Hence $\alpha = \omega$ or $\alpha > \omega$, in which case $\omega \in \alpha$ so that $\omega \subseteq \alpha$ since α is transitive. Thus in either case $\mathbf{N} = \omega \subseteq \alpha$. Clearly \mathbf{N} has no greatest element (since if n were such a greatest element then $n + 1 \in \mathbf{N}$ but $n < n + 1$). Thus there is a nonempty subset A of α such that A has no greatest element. \square

Exercise 6.2.7

If a set of ordinals X does not have a greatest element, then $\sup X$ is a limit ordinal.

Solution:

Lemma 6.2.7.1. *If α and β are ordinals and $\alpha < \beta$ then $\alpha + 1 \leq \beta$.*

Proof. To the contrary, suppose that $\alpha + 1 > \beta$. Then by the definition of $<$ we have that $\beta \in \alpha + 1 = \alpha \cup \{\alpha\}$ and since $\beta \neq \alpha$ it has to be that $\beta \in \alpha$. But then $\beta < \alpha$, which is a contradiction. \square

Lemma 6.2.7.2. *If α and β are ordinals and $\alpha < \beta + 1$ then $\alpha \leq \beta$.*

Proof. Since $\alpha < \beta + 1$ we have that $\alpha \in \beta + 1 = \beta \cup \{\beta\}$. Hence $\alpha \in \beta$ or $\alpha = \beta$. Thus $\alpha < \beta$ or $\alpha = \beta$, i.e. $\alpha \leq \beta$. \square

Main Problem.

Proof. Suppose that X is a set of ordinals with no greatest element. Let $\beta = \sup X = \bigcup X$. Then by the remarks following the proof of Theorem 6.2.6 $\beta \notin X$ since X has no greatest element. Now also suppose that β is a successor so that there is an ordinal α such that $\beta = \alpha + 1$.

If $\alpha \in X$ then since X has no greatest element there is a $\gamma \in X$ such that $\alpha < \gamma$. Then by Lemma 6.2.7.1 $\beta = \alpha + 1 \leq \gamma$. It cannot be that $\gamma = \beta$ since $\gamma \in X$ but $\beta \notin X$ so it must be that $\beta < \gamma$. But then since β is an upper bound of X it follows that γ is also. However, since $\gamma \in X$ this would make γ the greatest element of X , which is a contradiction.

On the other hand if $\alpha \notin X$ then consider any $\gamma \in X$. Then $\gamma < \beta = \alpha + 1$ so that by Lemma 6.2.7.2 $\gamma < \alpha$. Since γ was arbitrary this shows that α is an upper bound of X . However, since $\alpha < \beta$ this contradicts the definition of β as being the least upper bound of X , according to which $\alpha \geq \beta$.

Since all cases lead to a contradiction it cannot be that $\beta = \sup X$ is a successor and therefore by definition is a limit ordinal. \square

Exercise 6.2.8

If X is a nonempty set of ordinals, then $\bigcap X$ is an ordinal. Moreover $\bigcap X$ is the least element of X .

Solution:

Proof. Suppose that X is a set of ordinals. Then by Theorem 6.2.6d X has a least element α . We shall show that $\alpha = \bigcap X$, which simultaneously shows that $\bigcap X$ is an ordinal and the least element of X .

Consider any $\beta \in X$. Since α is the least element $\alpha \leq \beta$ so that $\alpha = \beta$ or $\alpha < \beta$. Clearly $\alpha \subseteq \alpha = \beta$ in the former case. In the latter case we have $\alpha \in \beta$ so that $\alpha \subseteq \beta$ as well since β is transitive (since it is an ordinal). Since β was arbitrary any $x \in \alpha$ is also in every $\beta \in X$ so that $x \in \bigcap X$ so that $\alpha \subseteq \bigcap X$ since x was arbitrary.

Now consider any $x \in \bigcap X$. Then clearly $x \in \alpha$ since $\alpha \in X$ so that $\bigcap X \subseteq \alpha$ since x was arbitrary.

Thus we have shown that $\alpha = \bigcap X$ as desired. \square

§6.3 The Axiom of Replacement

Exercise 6.3.1

Let $\mathbf{P}(x, y)$ be a property such that for every x there is at most one y for which $\mathbf{P}(x, y)$ holds. Then for every set A there is a set B such that, for all $x \in A$, if $\mathbf{P}(x, y)$ holds for some y , then $\mathbf{P}(x, y)$ holds for some $y \in B$.

Solution:

Proof. Define a property $\mathbf{R}(x, y)$ such that $\mathbf{R}(x, y)$ holds if and only if

1. $\mathbf{P}(x, y)$ holds, or
2. $y = \emptyset$ and there is not a z such that $\mathbf{P}(x, z)$ holds.

Clearly this property is such that for every x there is a unique y for which $\mathbf{R}(x, y)$ holds.

Now consider any set A . Then by the Axiom Schema of Replacement there is a set B such that, for every $x \in A$, there is a $y \in B$ for which $\mathbf{R}(x, y)$ holds. Consider any $x \in A$. Then by the above there is a $y \in B$ such that $\mathbf{R}(x, y)$ holds. Now suppose that $\mathbf{P}(x, z)$ holds for some z . Then option 2 above cannot be the case so that, $\mathbf{P}(x, y)$ holds (option 1) since $\mathbf{R}(x, y)$ does. Thus $\mathbf{P}(x, y)$ holds for some $y \in B$ as we were required to show. \square

Exercise 6.3.2

Use Theorem 6.3.6 to prove the existence of

- (a) The set $\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\{\{\emptyset\}\}\}, \dots\}$.
- (b) The set $\{\mathbf{N}, \mathcal{P}(\mathbf{N}), \mathcal{P}(\mathcal{P}(\mathbf{N})), \dots\}$.
- (c) The set $\omega + \omega = \omega \cup \{\omega, \omega + 1, (\omega + 1) + 1, \dots\}$.

Solution:

(a)

Proof. Define the operation $\mathbf{G}(x, n)$ for set x and $n \in \mathbf{N}$ by

$$\mathbf{G}(x, n) = \{x\} .$$

Then by Theorem 6.3.6 there is a unique sequence $\langle a_n \mid n \in \mathbf{N} \rangle$ where

$$\begin{aligned} a_0 &= \emptyset \\ a_{n+1} &= \mathbf{G}(a_n, n) = \{a_n\} \end{aligned}$$

for all $n \in \mathbf{N}$. Clearly the range of $\langle a_n \rangle$ is the set we seek. \square

(b)

Proof. Similarly define the operation $\mathbf{G}(x, n)$ for a set x and $n \in \mathbf{N}$ by

$$\mathbf{G}(x, n) = \mathcal{P}(x) ,$$

noting that this set exists by the Axiom of Power Set. Then by Theorem 6.3.6 there is a sequence $\langle a_n \mid n \in \mathbf{N} \rangle$ defined by

$$\begin{aligned} a_0 &= \mathbf{N} \\ a_{n+1} &= \mathbf{G}(a_n, n) = \mathcal{P}(a_n) , \end{aligned}$$

noting that $a_0 = \mathbf{N}$ exists by the Axiom of Infinity. Clearly then the range of $\langle a_n \rangle$ is the set we seek. \square

(c)

Proof. Define the operation $\mathbf{G}(x, n)$ for a set x and $n \in \mathbf{N}$ by

$$\mathbf{G}(x, n) = S(x) = x \cup \{x\} ,$$

Then by Theorem 6.3.6 there is a sequence $\langle a_n \mid n \in \mathbf{N} \rangle$ defined by

$$\begin{aligned} a_0 &= \omega \\ a_{n+1} &= \mathbf{G}(a_n, n) = S(a_n) = a_n + 1 , \end{aligned}$$

noting that $a_0 = \omega = \mathbf{N}$ exists by the Axiom of Infinity. Clearly then the range of $\langle a_n \rangle$ is $A = \{\omega, \omega + 1, \omega + 2, \dots\}$. It then follows that $\omega + \omega = \omega \cup A$ is the set we seek. \square

Exercise 6.3.3

Use Theorem 6.3.6 to define

$$\begin{aligned}V_0 &= \emptyset; \\V_{n+1} &= \mathcal{P}(V_n) \quad (n \in \omega); \\V_\omega &= \bigcup_{n \in \omega} V_n.\end{aligned}$$

Solution:

Proof. Define the operation $\mathbf{G}(x, n)$ for a set x and $n \in \mathbf{N}$ by

$$\mathbf{G}(x, n) = \mathcal{P}(x),$$

noting that this set exists by the Axiom of Power Set. Then by Theorem 6.3.6 there is a sequence $\langle V_n \mid n \in \mathbf{N} \rangle$ defined by

$$\begin{aligned}V_0 &= \emptyset \\V_{n+1} &= \mathbf{G}(V_n, n) = \mathcal{P}(V_n),\end{aligned}$$

noting that $V_0 = \emptyset$ exists by the Axiom of Existence. Then we let

$$V_\omega = \bigcup_{n \in \omega} V_n,$$

noting that $\omega = \mathbf{N}$. This set exists by the Axiom of Union. □

Exercise 6.3.4

- (a) Every $x \in V_\omega$ is finite.
- (b) V_ω is transitive.
- (c) V_ω is an inductive set.

Solution:

(a)

Proof. First we show by induction that every V_n is finite (for $n \in \mathbf{N}$). For $n = 0$ we have $V_n = V_0 = \emptyset$, which is clearly finite. Now suppose that V_n is finite then we have $V_{n+1} = \mathcal{P}(V_n)$, which is finite by Theorem 4.2.8.

Now consider any $x \in V_\omega = \bigcup_{n \in \omega} V_n$ so that there is an $n \in \omega$ such that $x \in V_n$. We note that $n \neq 0$ since $V_0 = \emptyset$ so it cannot be that $x \in V_0 = \emptyset$. Hence V_{n-1} is a set and moreover $V_n = \mathcal{P}(V_{n-1})$. So since $x \in V_n$ it follows that $x \in \mathcal{P}(V_{n-1})$ so that $x \subseteq V_{n-1}$. Thus it follows that $|x| \leq |V_{n-1}|$ so that clearly x is finite since V_{n-1} is (shown above). □

(b)

Proof. Consider any $x \in V_\omega$. Then by the same argument as in part (a) above it follows that $x \in V_n$ where $n \neq 0$. Hence again V_{n-1} is a set and $V_n = \mathcal{P}(V_{n-1})$ so that $x \subseteq V_{n-1}$. Then for any $y \in x$ we have that $y \in V_{n-1}$, from which it follows that clearly $y \in \bigcup_{k \in \omega} V_k = V_\omega$. Hence since y was arbitrary $x \subseteq V_\omega$, and since x was arbitrary this shows that V_ω is transitive by definition. □

(c)

Proof. First we show by induction that each V_n (where $n \in \omega$) is transitive. For $n = 0$ we have $V_n = V_0 = \emptyset$, which is clearly vacuously transitive. Now suppose that V_n is transitive and consider any $x \in V_{n+1} = \mathcal{P}(V_n)$ so that $x \subseteq V_n$. Now consider any $y \in x$ so that also $y \in V_n$. But since V_n is transitive $y \subseteq V_n$ so that $y \in \mathcal{P}(V_n) = V_{n+1}$. Hence since y was arbitrary this shows that $x \subseteq V_{n+1}$ and since x was arbitrary this shows by definition that V_{n+1} is transitive, thereby completing the inductive proof.

Now we show that V_ω is inductive. So first note that $V_1 = \mathcal{P}(V_0) = \mathcal{P}(\emptyset) = \{\emptyset\}$ so that $0 = \emptyset \in V_1$. From this it clearly follows that $0 \in \bigcup_{n \in \omega} V_n = V_\omega$.

Now suppose that $n \in V_\omega = \bigcup_{k \in \omega} V_k$ so that there is an $m \in \omega$ such that $n \in V_m$. Since it was shown above that V_m is transitive we have that $n \subseteq V_m$ as well. So consider any $x \in n+1 = n \cup \{n\}$. If $x \in n$ then also $x \in V_m$ since $n \subseteq V_m$. On the other hand if $x \in \{n\}$ then $x = n \in V_m$. Since x was arbitrary this shows that $n+1 \subseteq V_m$ so that $n+1 \in \mathcal{P}(V_m) = V_{m+1}$. From this it clearly follows that $n+1 \in \bigcup_{k \in \omega} V_k = V_\omega$. This shows that V_ω is inductive by definition. \square

Exercise 6.3.5

- (a) If $x \in V_\omega$ and $y \in V_\omega$, then $\{x, y\} \in V_\omega$.
- (b) If $X \in V_\omega$, then $\bigcup X \in V_\omega$ and $\mathcal{P}(X) \in V_\omega$.
- (c) If $A \in V_\omega$ and f is a function on A such that $f(x) \in V_\omega$ for each $x \in A$, then $f[X] \in V_\omega$.
- (d) If X is a finite subset of V_ω , then $X \in V_\omega$.

Note that part (c) differs slightly from the book; see the Errata List.

Solution:

(a)

Proof. First we show that if $x \in V_n$ for some $n \in \omega$ then $x \in V_m$ for all $m \geq n$. We show this by induction on m . So for $m = n$ clearly $x \in V_n = V_m$. Now suppose that $x \in V_m$. Then it was shown in Exercise 6.3.4 part (c) that V_m is transitive so that $x \subseteq V_m$. Hence $x \in \mathcal{P}(V_m) = V_{m+1}$, thereby completing the inductive proof.

Now suppose that $x, y \in V_\omega$. Then there are $n, m \in \omega$ such that $x \in V_n$ and $y \in V_m$. Without loss of generality we can assume that $n \leq m$ (since if this is not the case then we simply reverse the roles of x and y). So since $m \geq n$ it follows from what was shown above that $x \in V_m$ as well. Hence we have that clearly $\{x, y\} \subseteq V_m$ since both $x \in V_m$ and $y \in V_m$. Then $\{x, y\} \in \mathcal{P}(V_m) = V_{m+1}$ from which it clearly follows that $\{x, y\} \in \bigcup_{k \in \omega} V_k = V_\omega$. \square

(b)

Proof. Suppose that $X \in V_\omega = \bigcup_{k \in \omega} V_k$. Then there is an $n \in \omega$ such that $X \in V_n$. It was shown in Exercise 6.3.4 part (c) that V_n is transitive so that $X \subseteq V_n$.

First we show that $\bigcup X \in V_\omega$. So consider any $x \in \bigcup X$ so that there is a $Y \in X$ such that $x \in Y$. Then since $X \subseteq V_n$ we have that $Y \in V_n$. Since again V_n is transitive we have that $Y \subseteq V_n$ so that $x \in V_n$ since $x \in Y$. Since x was arbitrary it follows that $\bigcup X \subseteq V_n$ so that $\bigcup X \in \mathcal{P}(V_n) = V_{n+1}$. From this it clearly follows that $\bigcup X \in \bigcup_{k \in \omega} V_k = V_\omega$.

Next we show that $\mathcal{P}(X) \in V_\omega$. So consider any $Y \in \mathcal{P}(X)$ so that $Y \subseteq X$. Now consider any $y \in Y$ so that also $y \in X$. Since $X \subseteq V_n$ we have that $y \in V_n$. But since $y \in Y$ was arbitrary

it follows that $Y \subseteq V_n$ so that $Y \in \mathcal{P}(V_n) = V_{n+1}$. Then since $Y \in \mathcal{P}(X)$ was arbitrary it follows that $\mathcal{P}(X) \subseteq V_{n+1}$ so that $\mathcal{P}(X) \in \mathcal{P}(V_{n+1}) = V_{n+2}$. From this it clearly follows that $\mathcal{P}(X) \in \bigcup_{k \in \omega} V_k = V_\omega$. \square

(c)

Proof. Note the issue with this part in the errata list. Since $A \in V_\omega$ we have by Exercise 6.3.4 part (a) that A is finite. Then by Theorem 2.2.5 it follows that $f[A]$ is finite. Also clearly $f[A]$ is a subset of V_ω and hence is a finite subset. Therefore by part (d) below $f[A] \in V_\omega$. \square

(d)

Proof. Consider any finite $X \subseteq V_\omega$. Suppose then that $|X| = n$ for some $n \in \mathbf{N}$. Then for each $x_k \in X$, where $k \in n$, we have that $x_k \in V_\omega = \bigcup_{m \in \omega} V_m$ so that there is an $m_k \in \omega$ where $x_k \in V_{m_k}$. Now let $m = \max_{k \in n} m_k$, which exists since n is finite. Then, for any $k \in n$, by what was shown in Exercise 6.3.5 part (a) we have $x_k \in V_m$ since $x_k \in V_{m_k}$ and $m \geq m_k$. Hence it follows that $X \subseteq V_m$ so that $X \in \mathcal{P}(V_m) = V_{m+1}$. Clearly then $X \in \bigcup_{k \in \omega} V_k = V_\omega$. \square

§6.4 Transfinite Induction and Recursion

Exercise 6.4.1

Prove a more general Transfinite Recursion Theorem (Double Recursion Theorem): Let \mathbf{G} be an operation in two variables. Then there is an operation \mathbf{F} such that $\mathbf{F}(\alpha, \beta) = \mathbf{G}(\mathbf{F} \upharpoonright (\beta \times \alpha))$ for all ordinals β and α . [Hint: Computations are functions on $(\beta + 1) \times (\alpha + 1)$.]

Solution:

Proof. Since in these Recursion Theorems the arguments of interest of the given operation \mathbf{G} are typically functions, we assume that \mathbf{G} still takes a single variable despite what the text says. Hence for each x there is a unique y such that $y = \mathbf{G}(x)$. It just happens that in this case the variables of interest are functions of two variables.

For ordinals α and β we let $f_{\alpha, \beta}$ be the isomorphism from the ordinal $(\beta + 1) \cdot (\alpha + 1)$ to the lexicographic ordering of $(\alpha + 1) \times (\beta + 1)$, which exists by Theorem 6.5.8. We also note that according to Exercise 6.5.2 we have

$$(\beta + 1) \cdot (\alpha + 1) = (\beta + 1) \cdot \alpha + (\beta + 1) \cdot 1 = (\beta + 1) \cdot \alpha + (\beta + 1) = [(\beta + 1) \cdot \alpha + \beta] + 1$$

so that $(\beta + 1) \cdot (\alpha + 1)$ is a successor ordinal. Then for $\gamma < (\beta + 1) \cdot (\alpha + 1)$ define

$$X_{\alpha, \beta, \gamma} : \{(\delta, \varepsilon) \in (\alpha + 1) \times (\beta + 1) \mid (\delta, \varepsilon) \prec f_{\alpha, \beta}(\gamma)\}$$

Then for $\gamma < (\beta + 1) \cdot (\alpha + 1)$ we say that t is a computation of length γ if t is a transfinite sequence whose domain is $\gamma + 1$ and such that $t(\delta) = \mathbf{G}(t \circ (f_{\alpha, \beta}^{-1} \upharpoonright X_{\alpha, \beta, \delta}))$ for all $\delta \leq \gamma$. We note that since $(\beta + 1) \cdot (\alpha + 1)$ is a successor that $\gamma + 1 \leq (\beta + 1) \cdot (\alpha + 1)$.

Now we define the property $\mathbf{P}(x, y, z)$ such that $\mathbf{P}(x, y, z)$ holds if and only if

1. x and y are ordinal numbers and $z = t(f_{x, y}^{-1}(x, y))$ for some computation t of length $(y + 1) \cdot x + y$ (with respect to \mathbf{G}), or

2. x or y is *not* an ordinal and $z = \emptyset$.

We prove that \mathbf{P} defines an operation. Hence we have to show that for any x and y there is a unique z such that $\mathbf{P}(x, y, z)$ holds. So consider any sets α and β . If α or β is not an ordinal than clearly $\mathbf{P}(\alpha, \beta, \emptyset)$ holds and \emptyset is unique. So suppose that both α and β are ordinals. Then it suffices to show that there is a unique computation of length $(y + 1) \cdot x + y$ (with respect to \mathbf{G}) since this will make $z = t(f_{\alpha, \beta}^{-1}(\alpha, \beta))$ unique. We show this via transfinite induction.

So consider any ordinal $\gamma < (\beta + 1) \cdot (\alpha + 1)$ so that $\gamma \leq (y + 1) \cdot x + y$ and assume that for all $\delta < \gamma$ that there is a unique computation of length δ and we show that there exists a unique computation of length γ , which completes the proof that \mathbf{P} defines an operation.

Existence. First define a property $\mathbf{R}(x, y)$ such that $\mathbf{R}(x, y)$ holds if and only if

1. x is an ordinal where $x < \gamma$ and y is a computation of length x (with respect to \mathbf{G}), or
2. x is is an ordinal and $x \geq \gamma$ and $y = \emptyset$, or
3. x is not an ordinal and $y = \emptyset$.

Clearly by the induction hypothesis this property has a unique y for every x . Hence we can apply the Axiom Schema of Replacement, according to which there is a set T such that for every $\delta \in \gamma$ (so that $\delta < \gamma$) there is a t in T such that $\mathbf{R}(\delta, t)$ holds. That is

$$T = \{t \mid t \text{ is the unique computation of length } \delta \text{ for all } \delta < \gamma\}$$

Now, T is a system of transfinite sequences (which are functions) so define $\bar{t} = \bigcup T$ and let $\tau = \bar{t} \cup \left\{ (\gamma, \mathbf{G}(\bar{t} \circ (f_{\alpha, \beta}^{-1} \upharpoonright X_{\alpha, \beta, \gamma}))) \right\}$.

Claim 1: $\text{dom}(\tau) = \gamma + 1$. So consider any $\varepsilon \in \text{dom}(\tau)$. Clearly if $\varepsilon = \gamma$ then $\varepsilon \in \gamma + 1$. On the other hand if $\varepsilon \in \text{dom}(\bar{t})$ then there is a $t \in T$ such that $\varepsilon \in \text{dom}(t)$. But since t is a computation of length δ and $\delta < \gamma$ it follows that $\varepsilon \leq \delta < \gamma < \gamma + 1$ so that $\varepsilon \in \gamma + 1$. Hence since ε was arbitrary $\text{dom}(\tau) \subseteq \gamma + 1$.

Now consider any $\varepsilon \in \gamma + 1$ so that $\varepsilon \leq \gamma$. If $\varepsilon = \gamma$ then clearly by definition $\varepsilon \in \text{dom}(\tau)$. On the other hand if $\varepsilon \neq \gamma$ then $\varepsilon < \gamma$. So consider the $t \in T$ where t is the unique computation of length ε (which exists since $\varepsilon < \gamma$). Then clearly $\varepsilon \in \text{dom}(t)$ so that $\varepsilon \in \text{dom}(\bar{t})$. From this it follows that clearly $\varepsilon \in \text{dom}(\tau)$ so that $\gamma + 1 \subseteq \text{dom}(\tau)$ since ε was arbitrary. This proves the claim.

Claim 2: τ is a function. Consider any $\varepsilon \in \text{dom}(\tau) = \gamma + 1$ so that again $\varepsilon \leq \gamma$. If $\varepsilon = \gamma$ then clearly $\tau(\varepsilon) = \tau(\gamma) = \mathbf{G}(\bar{t} \circ (f_{\alpha, \beta}^{-1} \upharpoonright X_{\alpha, \beta, \gamma}))$ is unique since \mathbf{G} is an operation. On the other hand if $\varepsilon < \gamma$ then τ is a function so long as \bar{t} is, and this is the case so long as T is a compatible system of functions since $\bar{t} = \bigcup T$. We show this presently.

So consider any arbitrary $t_1, t_2 \in T$ where t_1 is the computation of length ε_1 and t_2 is the computation of length ε_2 . Without loss of generality we can assume that $\varepsilon_1 \leq \varepsilon_2$. We must show that $t_1(\delta) = t_2(\delta)$ for all $\delta \leq \varepsilon_1$. This we show by transfinite induction. So suppose that $t_1(\kappa) = t_2(\kappa)$ for all $\kappa < \delta \leq \varepsilon_1$. Then clearly $t_1 \upharpoonright \delta = t_2 \upharpoonright \delta$, from which it follows that $t_1 \circ (f_{\alpha, \beta}^{-1} \upharpoonright X_{\alpha, \beta, \delta}) = t_2 \circ (f_{\alpha, \beta}^{-1} \upharpoonright X_{\alpha, \beta, \delta})$ and since \mathbf{G} is an operation we have $t_1(\delta) = \mathbf{G}(t_1 \circ (f_{\alpha, \beta}^{-1} \upharpoonright X_{\alpha, \beta, \delta})) = \mathbf{G}(t_2 \circ (f_{\alpha, \beta}^{-1} \upharpoonright X_{\alpha, \beta, \delta})) = t_2(\delta)$. This completes the proof of the claim.

Claim 3: $\tau(\delta) = \mathbf{G}(\tau \circ (f_{\alpha, \beta}^{-1} \upharpoonright X_{\alpha, \beta, \delta}))$ for all $\delta \leq \gamma$. So consider any such δ . If $\delta = \gamma$ then since $\tau \upharpoonright \gamma = \bar{t}$ we clearly have $\tau(\delta) = \tau(\gamma) = \mathbf{G}(\bar{t} \circ (f_{\alpha, \beta}^{-1} \upharpoonright X_{\alpha, \beta, \gamma})) = \mathbf{G}(\tau \circ (f_{\alpha, \beta}^{-1} \upharpoonright X_{\alpha, \beta, \gamma})) = \mathbf{G}(\tau \circ (f_{\alpha, \beta}^{-1} \upharpoonright X_{\alpha, \beta, \delta}))$. On the other hand if $\delta < \gamma$ then let $t \in T$ be the computation of length δ (which exists since $\delta < \gamma$). Then $\tau(\delta) = t(\delta) = \mathbf{G}(t \circ (f_{\alpha, \beta}^{-1} \upharpoonright X_{\alpha, \beta, \delta})) = \mathbf{G}(\tau \circ (f_{\alpha, \beta}^{-1} \upharpoonright X_{\alpha, \beta, \delta}))$ since t is a computation (with respect to \mathbf{G}) and clearly $t \subseteq \tau$.

Claims 1 through 3 show that τ is a computation of length γ and hence that such a computation exists.

Uniqueness. Now let σ be another computation of length γ . We show that $\sigma = \tau$, which proves uniqueness. Since both σ and τ are functions with $\text{dom}(\sigma) = \gamma + 1 = \text{dom}(\tau)$ it suffices to show that $\sigma(\delta) = \tau(\delta)$ for all $\delta \leq \gamma$. We show this once again by using transfinite induction. So suppose that $\sigma(\varepsilon) = \tau(\varepsilon)$ for all $\varepsilon < \delta \leq \gamma$. It then follows that $\sigma \upharpoonright \delta = \tau \upharpoonright \delta$ so that $\sigma \circ (f_{\alpha,\beta}^{-1} \upharpoonright X_{\alpha,\beta,\delta}) = \tau \circ (f_{\alpha,\beta}^{-1} \upharpoonright X_{\alpha,\beta,\delta})$. Then since σ and τ are computations we have that $\sigma(\delta) = \mathbf{G}(\sigma \circ (f_{\alpha,\beta}^{-1} \upharpoonright X_{\alpha,\beta,\delta})) = \mathbf{G}(\tau \circ (f_{\alpha,\beta}^{-1} \upharpoonright X_{\alpha,\beta,\delta})) = \tau(\delta)$, thereby completing the uniqueness proof.

This completes the proof that \mathbf{P} defines an operation.

So let \mathbf{F} be the operation defined by \mathbf{P} . The last thing we need to show to complete the proof of the entire theorem is that $\mathbf{F}(\alpha, \beta) = \mathbf{G}(\mathbf{F} \upharpoonright (\alpha \times \beta))$ for all ordinals α and β , noting that we are treating \mathbf{F} as a function even though it is an operation. Thus, for any set X , $\mathbf{F} \upharpoonright X$ denotes the set $\{(x, \mathbf{F}(x)) \mid x \in X\}$, which forms a function with domain X . The range of this function is a set whose existence is guaranteed by the Axiom Schema of Replacement since \mathbf{F} is an operation.

So consider any ordinals α and β and the unique computation t of length $(\beta + 1) \cdot \alpha + \beta$. Then clearly for any $\gamma \leq (\beta + 1) \cdot \alpha + \beta$ we have that $t_\gamma = t \upharpoonright (\gamma + 1)$ is a computation of length γ . Since this computation is the unique computation of length γ , by the definition of \mathbf{F} as it relates to \mathbf{P} we have $\mathbf{F}(f(\gamma)) = t_\gamma(\gamma) = t(\gamma)$. Since γ was arbitrary this shows that $\mathbf{F} \upharpoonright (\alpha \times \beta) = t \upharpoonright (\beta + 1) \cdot \alpha + \beta$. Then clearly we have $\mathbf{F}(\alpha, \beta) = t((\beta + 1) \cdot \alpha + \beta) = \mathbf{G}(t \circ (f_{\alpha,\beta}^{-1} \upharpoonright X_{\alpha,\beta,(\beta+1)\cdot\alpha+\beta})) = \mathbf{G}(t \upharpoonright (\beta + 1) \cdot \alpha + \beta) = \mathbf{G}(\mathbf{F} \upharpoonright (\alpha \times \beta))$ by what was just shown above. \square

NOTE: This is a very nasty exercise and there may be a simpler way to do this.

Exercise 6.4.2

Using the Recursion Theorem 6.4.9 show that there is a binary operation \mathbf{F} such that

- (a) $\mathbf{F}(x, 1) = 0$ for all x .
- (b) $\mathbf{F}(x, n + 1) = 0$ if and only if there exist y and z such that $x = (y, z)$ and $\mathbf{F}(y, n) = 0$.

We say that x is an n -tuple (where $n \in \omega$, $n > 0$) if $\mathbf{F}(x, n) = 0$. Prove that this definition of n -tuples coincides with the one given in Exercise 5.17 in Chapter 3.

Solution:

Proof. Note that there is no such operation that can exactly satisfy both conditions as they actually contradict each other. To see this suppose there is such an operation \mathbf{F} . Then define the set $x = \emptyset$ and $n = 0$. Then by (a) we have that $\mathbf{F}(x, n + 1) = \mathbf{F}(x, 1) = 0$. It then follows from (b) that there are a y and z such that $x = (y, z)$, but clearly this is not the case for $x = \emptyset$. Hence a contradiction.

To remedy this we simply add a condition to (b), which when restated becomes

- (b) $n > 0$ and $\mathbf{F}(x, n + 1) = 0$ if and only if there exists y and z such that $x = (y, z)$ and $\mathbf{F}(y, n) = 0$.

Now, define an operation \mathbf{G} by $z = \mathbf{G}(x, y_u)$ if and only if either

1. y_u is a function with parameter u , $\text{dom}(y_u) = 1$, and $z = 0$, or
2. y_u is a function with parameter u , $\text{dom}(y_u) = \alpha + 1$ for some ordinal α , there are p and q where $x = (p, q)$, $y_p(\alpha) = 0$, and $z = 0$ or
3. None of the above hold and $z = 1$.

Then by Theorem 6.4.9 there is an operation \mathbf{F} such that $\mathbf{F}(x, \alpha) = \mathbf{G}(x, \mathbf{F}_x \upharpoonright \alpha)$ for all ordinals α and sets x .

Then for any set x we have that clearly $\mathbf{F} \upharpoonright 1$ is a function with domain 1 (and parameter x) so that by definition

$$\mathbf{F}(x, 1) = \mathbf{G}(x, \mathbf{F}_x \upharpoonright 1) = 0.$$

This shows (a).

To show (b) consider any ordinal n and set x .

(\rightarrow) Suppose that $n > 0$ and $\mathbf{F}(x, n+1) = 0$ so that clearly (3) above cannot be the case. Also (1) cannot be the case since $\mathbf{F}(x, n+1) = \mathbf{G}(x, \mathbf{F}_x \upharpoonright n+1)$ and $\text{dom}(\mathbf{F}_x \upharpoonright n+1) = n+1 > 1$. Hence (2) is the case so that there are y and z such that $x = (y, z)$ and $(\mathbf{F}_y \upharpoonright n+1)(n) = 0$. Hence it follows that $F(y, n) = 0$.

(\leftarrow) Now suppose that there are y and z such that $x = (y, z)$ and $F(y, n) = 0$. Then $F(y, n) = G(y, \mathbf{F}_y \upharpoonright n) = 0$.

So if $n = 0$ then $\text{dom}(\mathbf{F}_y \upharpoonright n) = \text{dom}(\mathbf{F}_y \upharpoonright 0) = 0 \neq 1$ so (1) cannot be the case. Also since 0 is not a successor ordinal (2) cannot be the case either (since $\text{dom}(\mathbf{F}_y \upharpoonright 0) \neq \alpha + 1$ for any ordinal α). Hence (3) must be the case, but this implies that $F(y, n) = 1$, which is a contradiction. So we must have that $n \neq 0$ so $n > 0$.

Since $\mathbf{F}(y, n) = 0$ clearly we have that $(\mathbf{F}_y \upharpoonright n+1)(n) = 0$. Since also $\text{dom}(\mathbf{F}_x \upharpoonright n+1) = n+1$ we find that (2) holds for $\mathbf{G}(x, \mathbf{F}_x \upharpoonright n+1)$. From this it follows that $\mathbf{F}(x, n+1) = \mathbf{G}(x, \mathbf{F}_x \upharpoonright n+1) = 0$.

This completes the proof. \square

§6.5 Ordinal Arithmetic

Exercise 6.5.1

Prove the associate law $(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$.

Solution:

Lemma 6.5.1.1. $0 \cdot \alpha = 0$ for all ordinals α .

Proof. We show this by transfinite induction on α . For $\alpha = 0$ we have $0 \cdot \alpha = 0 \cdot 0 = 0$ by Definition 6.5.6a. Now suppose that $0 \cdot \alpha = 0$ so that we have

$$0 \cdot (\alpha + 1) = 0 \cdot \alpha + 0 = 0 + 0 = 0$$

by the induction hypothesis, Definition 6.5.6b, and Definition 6.5.1a. Lastly, suppose that $\alpha \neq 0$ is a limit ordinal and that $0 \cdot \beta = 0$ for all $\beta < \alpha$. Then by Definition 6.5.6c we have that

$$0 \cdot \alpha = \sup \{0 \cdot \beta \mid \beta < \alpha\} = \sup \{0 \mid \beta < \alpha\} = \sup \{0\} = 0$$

by the induction hypothesis. This completes the proof. \square

Lemma 6.5.1.2. Ordinal α is a limit ordinal if and only if $\beta + 1 < \alpha$ for every ordinal $\beta < \alpha$.

Proof. (\rightarrow) We show this by contrapositive. So suppose that there is a $\beta < \alpha$ such that $\beta + 1 \geq \alpha$. Then by Lemma 6.2.7.1 we have also that $\beta + 1 \leq \alpha$, from which it follows that $\alpha = \beta + 1$ so that α is a successor ordinal and not a limit ordinal.

(\leftarrow) Suppose that $\beta + 1 < \alpha$ for every $\beta < \alpha$. Suppose also that $\alpha = \gamma + 1$ is a successor ordinal. Then clearly $\gamma < \gamma + 1 = \alpha$ so that also $\gamma + 1 < \alpha$, which is an immediate contradiction. Hence it must be that α is a limit ordinal.

Note that the bi-conditional is vacuously true for $\alpha = 0$. □

Lemma 6.5.1.3. *Suppose that $\alpha \neq 0$ is an ordinal and $\beta \neq 0$ is a limit ordinal. Then $\alpha \cdot \beta$ is a limit ordinal and $\alpha \cdot \beta \neq 0$.*

Proof. Since $\alpha \neq 0$ we have that $\alpha \geq 1$. Now consider any $\gamma < \alpha \cdot \beta$. We claim that $\gamma \leq \alpha \cdot \delta$ for some $\delta < \beta$. Suppose to the contrary that $\gamma > \alpha \cdot \delta$ for all $\delta < \beta$. Then γ is an upper bound of the set $\{\alpha \cdot \delta \mid \delta < \beta\}$. But from this it follows that

$$\gamma \geq \sup \{\alpha \cdot \delta \mid \delta < \beta\} = \alpha \cdot \beta$$

by Definition 6.5.6c, which contradicts the definition of γ . Hence the claim is true so that $\gamma \leq \alpha \cdot \delta$ for some $\delta < \beta$. Then we have by Lemma 6.5.4a and the fact that $1 \leq \alpha$ that

$$\gamma + 1 \leq \alpha \cdot \delta + 1 \leq \alpha \cdot \delta + \alpha = \alpha \cdot (\delta + 1) < \alpha \cdot \beta$$

by Exercise 6.5.7a since $\delta + 1 < \beta$ since β is a limit ordinal. Note that we also used Definition 6.5.6b. Thus since $\gamma + 1 < \alpha \cdot \beta$ and γ was arbitrary it follows from Lemma 6.5.1.2 that $\alpha \cdot \beta$ is a limit ordinal.

We also have that $\alpha \neq 0$ and $0 < \beta$ (since $0 \neq \beta$) so that by Exercise 6.5.7a above and Definition 6.5.6a

$$0 = \alpha \cdot 0 < \alpha \cdot \beta.$$

Thus $\alpha \cdot \beta \neq 0$. □

Main Problem.

Proof. We show this by transfinite induction on γ .

First for $\gamma = 0$ we have

$$(\alpha \cdot \beta) \cdot \gamma = (\alpha \cdot \beta) \cdot 0 = 0 = \alpha \cdot 0 = \alpha \cdot (\beta \cdot 0) = \alpha \cdot (\beta \cdot \gamma),$$

where we have used Definition 6.5.6a repeatedly. Now suppose that $(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$ for ordinal γ . Then

$$\begin{aligned} (\alpha \cdot \beta) \cdot (\gamma + 1) &= (\alpha \cdot \beta) \cdot \gamma + \alpha \cdot \beta && \text{(by Definition 6.5.6b)} \\ &= \alpha \cdot (\beta \cdot \gamma) + \alpha \cdot \beta && \text{(by the induction hypothesis)} \\ &= \alpha \cdot [\beta \cdot \gamma + \beta] && \text{(by the distributive law, see Exercise 6.5.2)} \\ &= \alpha \cdot [\beta \cdot (\gamma + 1)] && \text{(by Definition 6.5.6b)} \end{aligned}$$

Now suppose that $\gamma \neq 0$ is a limit ordinal and $(\alpha \cdot \beta) \cdot \delta = \alpha \cdot (\beta \cdot \delta)$ for all $\delta < \gamma$. First if $\beta = 0$ then

$$(\alpha \cdot \beta) \cdot \gamma = (\alpha \cdot 0) \cdot \gamma = 0 \cdot \gamma = 0 = \alpha \cdot 0 = \alpha \cdot (0 \cdot \gamma) = \alpha \cdot (\beta \cdot \gamma)$$

where we have used Definition 6.5.6a and Lemma 6.5.1.1. So assume that $\beta \neq 0$. Then we have $(\alpha \cdot \beta) \cdot \gamma = \sup \{(\alpha \cdot \beta) \cdot \delta \mid \delta < \gamma\} = \sup \{\alpha \cdot (\beta \cdot \delta) \mid \delta < \gamma\}$ by Definition 6.5.6c and the induction hypothesis. Then, since $\beta \neq 0$, by Lemma 6.5.1.3 we have that $\beta \cdot \gamma$ is a limit ordinal and $\beta \cdot \gamma \neq 0$. From this and Definition 6.5.6c we have that

$$(\alpha \cdot \beta) \cdot \gamma = \sup \{\alpha \cdot (\beta \cdot \delta) \mid \delta < \gamma\} = \sup \{\alpha \cdot \delta \mid \delta < \beta \cdot \gamma\} = \alpha \cdot (\beta \cdot \gamma)$$

as desired. This completes the inductive proof. □

Exercise 6.5.2

Prove the distributive law $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$.

Solution:

Proof. We show this by transfinite induction on γ . So for $\gamma = 0$ we have

$$\begin{aligned}\alpha \cdot (\beta + \gamma) &= \alpha \cdot (\beta + 0) = \alpha \cdot \beta && \text{(by Definition 6.5.1a)} \\ &= \alpha \cdot \beta + 0 && \text{(by Definition 6.5.1a)} \\ &= \alpha \cdot \beta + \alpha \cdot 0 && \text{(by Definition 6.5.6a)} \\ &= \alpha \cdot \beta + \alpha \cdot \gamma.\end{aligned}$$

Now suppose that $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$ for ordinal γ . We then have

$$\begin{aligned}\alpha \cdot [\beta + (\gamma + 1)] &= \alpha \cdot [(\beta + \gamma) + 1] && \text{(by Definition 6.5.1b)} \\ &= \alpha \cdot (\beta + \gamma) + \alpha && \text{(by Definition 6.5.6b)} \\ &= (\alpha \cdot \beta + \alpha \cdot \gamma) + \alpha && \text{(by the induction hypothesis)} \\ &= \alpha \cdot \beta + (\alpha \cdot \gamma + \alpha) && \text{(by the associativity of addition, Lemma 6.5.4c)} \\ &= \alpha \cdot \beta + \alpha \cdot (\gamma + 1). && \text{(by Definition 6.5.6b)}\end{aligned}$$

Lastly, suppose that $\gamma \neq 0$ is a limit ordinal and that $\alpha \cdot (\beta + \delta) = \alpha \cdot \beta + \alpha \cdot \delta$ for all $\delta < \gamma$. If $\alpha = 0$ then we have

$$\alpha \cdot (\beta + \gamma) = 0 \cdot (\beta + \gamma) = 0 = 0 + 0 = 0 \cdot \beta + 0 \cdot \gamma = \alpha \cdot \beta + \alpha \cdot \gamma,$$

where we have used Lemma 6.5.1.1 above. So suppose that $\alpha \neq 0$ so that $\alpha \geq 1$. Now, if $\xi < \beta + \gamma$ then $\xi \leq \beta + \delta$ for some $\delta < \gamma$. Then by Lemma 6.5.4 we have that $\xi + 1 \leq (\beta + \delta) + 1 = \beta + (\delta + 1) < \beta + \gamma$ since γ is a limit ordinal. Hence $\beta + \gamma$ is a limit ordinal so that by Definition 6.5.6c we have $\alpha \cdot (\beta + \gamma) = \sup \{\alpha \cdot \xi \mid \xi < \beta + \gamma\}$. But then by Definition 6.5.1c we have that $\beta + \gamma = \sup \{\beta + \delta \mid \delta < \gamma\}$. It then follows that

$$\alpha \cdot (\beta + \gamma) = \sup \{\alpha \cdot \xi \mid \xi < \beta + \gamma\} = \sup \{\alpha \cdot (\beta + \delta) \mid \delta < \gamma\} = \sup \{\alpha \cdot \beta + \alpha \cdot \delta \mid \delta < \gamma\}$$

by the induction hypothesis. Then by Definition 6.5.6c we have that $\sup \{\alpha \cdot \delta \mid \delta < \gamma\} = \alpha \cdot \gamma$ since γ is a limit ordinal. It then follows from Lemma 6.5.1.3 that $\alpha \cdot \gamma$ is also a limit ordinal and $\alpha \cdot \gamma \neq 0$ since $\alpha \neq 0$. From this and Definition 6.5.1c we have that

$$\alpha \cdot (\beta + \gamma) = \sup \{\alpha \cdot \beta + \alpha \cdot \delta \mid \delta < \gamma\} = \sup \{\alpha \cdot \beta + \delta \mid \delta < \alpha \cdot \gamma\} = \alpha \cdot \beta + \alpha \cdot \gamma$$

as desired. This completes the inductive proof. \square

Exercise 6.5.3

Simplify

- (a) $(\omega + 1) + \omega$.
- (b) $\omega + \omega^2$.
- (c) $(\omega + 1) \cdot \omega^2$.

Solution:

(a) By Lemma 6.5.4c we have

$$(\omega + 1) + \omega = \omega + (1 + \omega) = \omega + \omega = \omega \cdot 2.$$

(b) We simply have

$$\omega + \omega^2 = \omega \cdot 1 + \omega \cdot \omega = \omega \cdot (1 + \omega) = \omega \cdot \omega = \omega^2$$

(c)

Lemma 6.5.3.1. $(\omega + 1) \cdot n = \omega \cdot n + 1$ for all $n \in \omega$ where $n > 0$.

Proof. We show this by standard (as opposed to transfinite) induction on n . For $n = 1$ we have

$$(\omega + 1) \cdot n = (\omega + 1) \cdot 1 = \omega + 1 = \omega \cdot 1 + 1 = \omega \cdot n + 1.$$

Now suppose that $(\omega + 1) \cdot n = \omega \cdot n + 1$ so that we have

$$\begin{aligned} (\omega + 1) \cdot (n + 1) &= (\omega + 1) \cdot n + (\omega + 1) && \text{(by Definition 6.5.6b)} \\ &= (\omega \cdot n + 1) + (\omega + 1) && \text{(by the induction hypothesis)} \\ &= \omega \cdot n + (1 + \omega) + 1 && \text{(by the associativity of addition)} \\ &= \omega \cdot n + \omega + 1 \\ &= \omega \cdot (n + 1) + 1. && \text{(by Definition 6.5.6b)} \end{aligned}$$

This completes the inductive proof. □

Lemma 6.5.3.2. $(\omega + 1) \cdot \omega = \omega^2$.

Proof. Since ω is a limit ordinal we have by Definition 6.5.6c

$$(\omega + 1) \cdot \omega = \sup \{(\omega + 1) \cdot n \mid n \in \omega\} = \sup \{\omega \cdot n + 1 \mid n \in \omega\}$$

by Lemma 6.5.3.1. Now, since we have

$$\omega \cdot n + 1 < (\omega \cdot n + 1) + \omega = \omega \cdot n + (1 + \omega) = \omega \cdot n + \omega = \omega \cdot (n + 1)$$

it is clear that we have

$$(\omega + 1) \cdot \omega = \sup \{\omega \cdot n + 1 \mid n \in \omega\} = \sup \{\omega \cdot n \mid n \in \omega\} = \omega \cdot \omega = \omega^2$$

by Definition 6.5.6c. □

Main Problem.

We have

$$(\omega + 1) \cdot \omega^2 = (\omega + 1) \cdot (\omega \cdot \omega) = [(\omega + 1) \cdot \omega] \cdot \omega = \omega^2 \cdot \omega = \omega^{2+1} = \omega^3,$$

where we have used Lemma 6.5.3.2 and Definition 6.5.9b.

Exercise 6.5.4

For every ordinal α , there is a unique limit ordinal β and a unique natural number n such that $\alpha = \beta + n$. [Hint: $\beta = \sup \{\gamma \leq \alpha \mid \gamma \text{ is limit}\}$.]

Solution:

Lemma 6.5.4.1. *If α and β are ordinals and $\beta \neq 0$ is a limit ordinal then $\alpha + \beta$ is a limit ordinal.*

Proof. First we note that since $\beta \neq 0$ we have $1 \leq \beta$. Consider any $\gamma < \alpha + \beta$. If $\gamma < \alpha$ then $\gamma + 1 \leq \alpha < \alpha + 1 \leq \alpha + \beta$. On the other hand if $\gamma \geq \alpha$ then by Lemma 6.5.5 there is an ordinal δ such that $\alpha + \delta = \gamma$. Then since $\alpha + \delta = \gamma < \alpha + \beta$ it follows from Lemma 6.5.4a that $\delta < \beta$, and so by Lemma 6.5.1.2 we have that $\delta + 1 < \beta$ since β is a limit ordinal. Therefore we have $\gamma + 1 = (\alpha + \delta) + 1 = \alpha + (\delta + 1) < \alpha + \beta$ by Lemma 6.5.4 parts c and a. Hence in all cases $\gamma + 1 < \alpha + \beta$ so that $\alpha + \beta$ is a limit ordinal by Lemma 6.5.1.2 since γ was arbitrary. \square

Lemma 6.5.4.2. *If α is a limit ordinal and β is another ordinal such that $\beta < \alpha$ then $\beta + n < \alpha$ for any natural number n .*

Proof. We show this by normal (not transfinite) induction on n . For $n = 0$ we clearly have $\beta + n = \beta + 0 = \beta < \alpha$. So suppose that $\beta + n < \alpha$ so that we have

$$\beta + (n + 1) = (\beta + n) + 1 < \alpha$$

by Definition 6.5.1b. The inequality follows from Lemma 6.5.1.2 and the induction hypothesis since α is a limit ordinal. This completes the inductive proof. \square

Main Problem.

Proof. Existence. First for any ordinal α let

$$B = \{\gamma \in \alpha + 1 \mid \gamma \text{ is a limit ordinal}\}$$

and define $\beta = \sup B$.

First we show that β is a limit ordinal. To this end consider any $\delta < \beta$. Then since β is the least upper bound of B it follows that δ is not an upper bound of B so that there is a $\gamma \in B$ such that $\delta < \gamma$. Since $\gamma \in B$ it is a limit ordinal so that also $\delta + 1 < \gamma$ by Lemma 6.5.1.2. Then since β is an upper bound of B we have that $\delta + 1 < \gamma \leq \beta$ so that β is a limit ordinal by Lemma 6.5.1.2 since δ was arbitrary.

Now, since each $\gamma \in B$ is in $\alpha + 1$ we have that $\gamma < \alpha + 1$ so that $\gamma \leq \alpha$. Hence α is an upper bound of B , and since β is the *least* upper bound of B it follows that $\beta \leq \alpha$. Because of this there is a unique ordinal ξ such that $\beta + \xi = \alpha$ by Lemma 6.5.5.

We claim that $\xi < \omega$ so that ξ is a natural number. To the contrary, suppose that $\xi \geq \omega$. It then follows from Lemma 6.5.4 that $\beta + \omega \leq \beta + \xi = \alpha$. Also, $\beta + \omega$ is a limit ordinal by Lemma 6.5.4.1 above since ω is so that $\beta + \omega \in B$ since also $\beta + \omega \in \alpha + 1$ (since $\beta + \omega \leq \alpha$). Then $\beta + \omega \leq \beta$ since β is an upper bound of B , but this is a contradiction since clearly $\beta = \beta + 0 < \beta + \omega$ by Lemma 6.5.4a since $0 < \omega$. Hence it must be that $\xi < \omega$.

Thus $\alpha = \beta + \xi$ for a limit ordinal β and natural number ξ , thereby proving existence.

Uniqueness. Suppose that $\alpha = \beta_1 + n_1 = \beta_2 + n_2$ where β_1 and β_2 are limit ordinals and n_1 and n_2 natural numbers. First suppose that $\beta_1 \neq \beta_2$ so that without loss of generality we can assume that $\beta_1 < \beta_2$. It then follows from Lemma 6.5.4.2 that $\beta_1 + n_1 < \beta_2$ as well since β_2 is a limit ordinal and n_1 is a natural number. But then we have $\beta_1 + n_1 < \beta_2 = \beta_2 + 0 \leq \beta_2 + n_2$, which contradicts the fact that $\beta_1 + n_1 = \alpha = \beta_2 + n_2$. So it must be that in fact $\beta_1 = \beta_2$. But then it follows from Lemma 6.5.4b that $n_1 = n_2$ also, which shows uniqueness. \square

Exercise 6.5.5

Let $\alpha \leq \beta$. The equation $\xi + \alpha = \beta$ may have 0, 1, or infinitely many solutions.

Solution:

Lemma 6.5.5.1. *If α and β are ordinals and n a natural number then $\alpha < \beta$ if and only if $\alpha + n < \beta + n$.*

Proof. (\rightarrow) We show this by induction on n . So for $n = 0$ and any ordinals α and β where $\alpha < \beta$ we clearly have

$$\alpha + n = \alpha + 0 = \alpha < \beta = \beta + 0 = \beta + n.$$

Now suppose that $\alpha + n < \beta + n$ for any ordinals α and β where $\alpha < \beta$. Suppose that α and β are such ordinals so that by Lemma 6.2.7.1 we have $\alpha + 1 \leq \beta < \beta + 1$. Hence $\alpha + 1$ and $\beta + 1$ are ordinals such that $\alpha + 1 < \beta + 1$. It then follows from the induction hypothesis that

$$\alpha + (n + 1) = \alpha + (1 + n) = (\alpha + 1) + n < (\beta + 1) + n = \beta + (1 + n) = \beta + (n + 1),$$

noting that clearly natural numbers commute with respect to addition. This completes the proof by induction.

(\leftarrow) Suppose that $\alpha + n < \beta + n$ for ordinals α and β and natural number n . It cannot be that $\beta < \alpha$ for then it would follow that $\beta + n < \alpha + n$ by what was just shown. Nor can it be that $\alpha = \beta$ since then clearly $\alpha + n = \beta + n$. Hence by the linearity of the ordinal ordering it follows that $\alpha < \beta$. \square

Corollary 6.5.5.2. *If α and β are ordinals and n a natural number then $\alpha = \beta$ if and only if $\alpha + n = \beta + n$.*

Proof. This follows directly from Lemma 6.5.5.1 in the same way as the proof of Lemma 6.5.4b. \square

Lemma 6.5.5.3. *If $\alpha \geq \omega$ is an ordinal and n a natural number then $n + \alpha = \alpha$.*

Proof. For any natural number n we show this by transfinite induction on α . So if $\alpha = \omega$ then as explained in the text we have $n + \alpha = n + \omega = \omega = \alpha$. Now suppose that $n + \alpha = \alpha$ so that we have $n + (\alpha + 1) = (n + \alpha) + 1 = \alpha + 1$. Lastly, suppose that $\alpha > \omega$ is a limit ordinal and that $n + \gamma = \gamma$ for all $\gamma < \alpha$. We then have by Definition 6.5.1c that

$$n + \alpha = \sup \{n + \gamma \mid \gamma < \alpha\} = \sup \{\gamma \mid \gamma < \alpha\} = \alpha$$

by the induction hypothesis and comments in the text after Theorem 6.2.10. This completes the inductive proof. \square

Main Problem.

Proof. In what follows suppose generally that $\alpha = \gamma + n$ and $\beta = \delta + m$ where γ and δ are limit ordinals and n and m are natural numbers. Note that α and β can be expressed in this way uniquely by Exercise 6.5.4.

Case: $\beta = 0$. If $\alpha = 0$ then clearly $\xi = 0$ is the only solution since for any other $\xi \neq 0$ we have $\xi + \alpha = \xi + 0 = \xi \neq 0 = \beta$. On the other hand if $\alpha \neq 0$ then there is no solution since for any ξ we have by Lemma 6.5.4 that $\xi + \alpha > \xi + 0 = \xi \geq 0 = \beta$ so that $\xi + \alpha \neq \beta$.

Case: β is a successor. From this it follows that clearly $m \neq 0$ since otherwise $\beta = \delta + m = \delta + 0 = \delta$ would be a limit ordinal.

Now, if $\alpha = 0$ then clearly $\xi = \beta$ is the only solution so that $\xi + \alpha = \xi + 0 = \xi = \beta$.

If α is a successor then similarly $n \neq 0$. Suppose that $\gamma = 0$ so that since $\alpha \leq \beta$ we have $0 + n = n \leq \delta + m = \beta$. Then if $n \leq m$ then $\xi = \delta + (m - n)$ is a solution since then

$$\xi + \alpha = [\delta + (m - n)] + n = \delta + [(m - n) + n] = \delta + m = \beta.$$

Moreover clearly this is the only solution since if $\xi \neq \delta + (m - n)$ then

$$\xi + \alpha = \xi + n \neq [\delta + (m - n)] + n = \delta + m = \beta$$

by Corollary 6.5.5.2. On the other hand if $n > m$ then for any ordinal $\xi = \varepsilon + k$ where ε is a limit ordinal and k is a natural number, we have that $k + n > 0 + n = n > m$ so that clearly $k + n \neq m$. From this it follows that

$$\xi + \alpha = (\varepsilon + k) + n = \varepsilon + (k + n) \neq \delta + m = \beta$$

since both the limit ordinal and the natural number part of ordinals must be equal for the overall ordinals to be equal. Hence this case has no solutions. Now suppose that $\gamma \neq 0$. Then if $n \neq m$ then there are no solutions since for any ordinal ξ we have

$$\xi + \alpha = \xi + (\gamma + n) = (\xi + \gamma) + n \neq \delta + m$$

since $n \neq m$ and $\xi + \gamma$ is a limit ordinal by Lemma 6.5.4.1. On the other hand if $n = m$ then since $\gamma + n = \alpha \leq \beta = \delta + m = \delta + n$ it follows from Lemma 6.5.5.1 and Corollary 6.5.5.2 that $\gamma \leq \delta$, and since $\gamma \neq 0$ we have that $0 < \gamma \leq \delta$.

Then we have that ξ is a solution if and only if $\xi + \gamma = \delta$. For, supposing that $\xi + \gamma = \delta$, we have

$$\xi + \alpha = \xi + (\gamma + n) = (\xi + \gamma) + n = \delta + n = \delta + m = \beta$$

and if it is not the case then

$$\xi + \alpha = \xi + (\gamma + n) = (\xi + \gamma) + n \neq \delta + n = \delta + m = \beta$$

by Corollary 6.5.5.2. Since γ and δ are both nonzero limit ordinals it then follows from the final case below (where both α and β are nonzero limit ordinals) there are either zero or infinite such ξ .

If α is a nonzero limit ordinal then clearly there are no solutions since, for any ordinal ξ , Lemma 6.5.4.1 tells us that $\xi + \alpha$ is a limit ordinal whereas β is a successor so that it has to be that $\xi + \alpha \neq \beta$.

Case: β is a nonzero limit ordinal.

If $\alpha = 0$ then clearly $\xi = \beta$ is the only solution since $\xi + \alpha = \xi + 0 = \xi$.

If α is a successor ordinal then $n \neq 0$ and for any ordinal ξ we have that $\xi + \alpha = \xi + (\gamma + n) = (\xi + \gamma) + n$, which is a successor ordinal as well since $\xi + \gamma$ is a limit ordinal again by Lemma 6.5.4.1. Hence there are no solutions since β is a limit ordinal and ξ was arbitrary.

Lastly, suppose that α is also a nonzero limit ordinal. Suppose that there at least one solution ξ such that $\xi + \alpha = \beta$. Now consider any natural number k so that we have

$$(\xi + k) + \alpha = \xi + (k + \alpha) = \xi + \alpha = \beta$$

by Lemma 6.5.5.3 since α is a nonzero limit ordinal and therefore clearly $\alpha \geq \omega$. Hence $\xi + k$ is also a solution and so there are an infinite number of solutions since k was arbitrary. Thus there are either zero solutions or an infinite number of solutions (since a nonzero number of solutions implies an infinite number of solutions). This completes the case structure.

These results are summarized in the following table:

β	α	Other	Other	Solutions
0	0			1 ($\xi = 0$)
	> 0			0
Successor	0			1 ($\xi = \beta$)
	Successor	$\gamma = 0$	$n \leq m$	1 ($\xi = \delta + m - n$)
			$n > m$	0
	Successor	$\gamma > 0$	$n \neq m$	0
			$n = m$	0 or ∞
Limit > 0			0	
Limit > 0	0			1 ($\xi = \beta$)
	Successor			0
	Limit > 0			0 or ∞

Since this case structure is exhaustive the result follows. \square

Exercise 6.5.6

Find the least $\alpha > \omega$ such that $\xi + \alpha = \alpha$ for all $\xi < \alpha$.

Solution:

Definition 6.5.6.1. We call an ordinal α an additive ordinal if it has the property that $\beta + \alpha = \alpha$ for all $\beta < \alpha$.

Lemma 6.5.6.2. If α is a limit ordinal and β is any ordinal then $\alpha \cdot \beta$ is a limit ordinal.

Proof. We show this by transfinite induction on β .

So for $\beta = 0$ we clearly have $\alpha \cdot \beta = \alpha \cdot 0 = 0$, which is a limit ordinal.

Now suppose that $\alpha \cdot \beta$ is a limit ordinal. Then we have $\alpha \cdot (\beta + 1) = \alpha \cdot \beta + \alpha$ by Definition 6.5.6b. If $\alpha = 0$ then $\alpha \cdot (\beta + 1) = \alpha \cdot \beta + \alpha = \alpha \cdot \beta + 0 = \alpha \cdot \beta$, which is a limit ordinal by the induction hypothesis. On the other hand if $\alpha \neq 0$ then by Lemma 6.5.4.1 $\alpha \cdot (\beta + 1) = \alpha \cdot \beta + \alpha$ is a limit ordinal since α is.

Lastly suppose that β is a limit ordinal and that $\alpha \cdot \gamma$ is a limit ordinal for all $\gamma < \beta$. Let $A = \{\alpha \cdot \gamma \mid \gamma < \beta\}$ so that by Definition 6.5.6b we have $\alpha \cdot \beta = \sup A$. Consider any $\delta < \alpha \cdot \beta$ so that δ is not an upper bound of A . Hence there is a $\gamma < \beta$ such that $\delta < \alpha \cdot \gamma$. Then by the induction hypothesis $\alpha \cdot \gamma$ is a limit ordinal so that $\delta + 1 < \alpha \cdot \gamma$ by Lemma 6.5.1.2. But then we have $\delta + 1 < \alpha \cdot \gamma \leq \sup A = \alpha \cdot \beta$. Thus by Lemma 6.5.1.2 this shows that $\alpha \cdot \beta$ is also a limit ordinal, which completes the inductive proof. \square

Lemma 6.5.6.3. An ordinal $\alpha = \omega \cdot n$ for a natural number n if and only if α is a limit ordinal and $\alpha < \omega^2$.

Proof. (\rightarrow) Suppose that $\alpha = \omega \cdot n$ for natural number n . Then by Lemma 6.5.6.2 α is a limit ordinal since ω is. Also clearly by Exercise 6.5.7a $\alpha = \omega \cdot n < \omega \cdot (n + 1) \leq \sup \{\omega \cdot k \mid k < \omega\} = \omega \cdot \omega = \omega^2$.

(\leftarrow) Now suppose that α is a limit ordinal and $\alpha < \omega^2$. We have $\alpha < \omega^2 = \omega \cdot \omega = \sup \{\omega \cdot n \mid n < \omega\}$ so that α is not an upper bound of $\{\omega \cdot n \mid n < \omega\}$. Hence there is a $k < \omega$ such that $\alpha < \omega \cdot k$. It therefore follows that the set $A = \{n \in \omega \mid \alpha \leq \omega \cdot n\}$ is nonempty. Since A is nonempty set of natural numbers (which are well-ordered) it follows that A has a least element n . If $n = 0$ it follows that $\alpha = 0 = \omega \cdot 0$ since $\alpha \leq \omega \cdot n = \omega \cdot 0 = 0$ and 0 is the only ordinal for which this is true. Then if $n > 0$ we have that $n - 1$ is a natural number and moreover it follows that $\omega \cdot (n - 1) < \alpha$ since otherwise $n - 1$ would have been the least element of A .

Thus we have that $\omega \cdot (n-1) < \alpha \leq \omega \cdot n$. But since $\omega \cdot n = \omega \cdot [(n-1) + 1] = \omega \cdot (n-1) + \omega$ clearly any ordinal γ where $\omega \cdot (n-1) + 0 = \omega \cdot (n-1) < \gamma < \omega \cdot n = \omega \cdot (n-1) + \omega$ must have the form $\omega \cdot (n-1) + m$ for a natural number $m > 0$. From this it clearly follows that γ is a successor ordinal. Since α is a limit ordinal it can thus not be such a γ so that it cannot be that $\alpha < \omega \cdot n$. But since we have established that $\alpha \leq \omega \cdot n$ (since $n \in A$) it has to be that $\alpha = \omega \cdot n$. \square

Lemma 6.5.6.4. *An ordinal $\alpha = \omega \cdot n + k$ for natural numbers n and k if and only if $\alpha < \omega^2$.*

Proof. (\rightarrow) Suppose that $\alpha = \omega \cdot n + k$ for natural numbers n and k . We then have

$$\begin{aligned}
 \alpha &= \omega \cdot n + k = (\omega \cdot n + k) + 0 \\
 &< (\omega \cdot n + k) + \omega && \text{(by Lemma 6.5.4a since } 0 < \omega) \\
 &= \omega \cdot n + (k + \omega) && \text{(by the associative property)} \\
 &= \omega \cdot n + \omega && \text{(by Lemma 6.5.5.3)} \\
 &= \omega \cdot (n + 1) && \text{(by Definition 6.5.6b)} \\
 &\leq \sup \{ \omega \cdot m \mid m < \omega \} \\
 &= \omega \cdot \omega && \text{(Definition 6.5.6c)} \\
 &= \omega^2 && \text{(by Example 6.5.10a)}
 \end{aligned}$$

as desired.

(\leftarrow) Now suppose that $\alpha < \omega^2$. By Exercise 6.5.4 we have that $\alpha = \beta + k$ for a unique limit ordinal β and natural number k . We also have by Lemma 6.5.4 that $\beta + 0 \leq \beta + k = \alpha < \omega^2$ since obviously $0 \leq k$. Hence β is a limit ordinal such that $\beta < \omega^2$ so that by Lemma 6.5.6.3 there is a natural number n such that $\beta = \omega \cdot n$, thereby proving the result since this means that $\alpha = \beta + k = \omega \cdot n + k$. \square

Main Problem.

We claim that ω^2 is the first additive ordinal after ω .

Proof. To see this we first show that ω^2 is a limit ordinal. So consider any $\alpha < \omega^2$ so that by Lemma 6.5.6.4 there are natural numbers n and k such that $\alpha = \omega \cdot n + k$. We then have $\alpha + 1 = (\omega \cdot n + k) + 1 = \omega \cdot n + (k + 1) < \omega^2$ again by Lemma 6.5.6.4 since $k + 1$ is a natural number. Hence ω^2 is a limit ordinal by Lemma 6.5.1.2.

Next we show that ω^2 is an additive ordinal. So again consider $\alpha < \omega^2$ so that by Lemma 6.5.6.4 there are natural numbers n and k such that $\alpha = \omega \cdot n + k$. We then have

$$\alpha + \omega^2 = (\omega \cdot n + k) + \omega^2 = \omega \cdot n + (k + \omega^2) = \omega \cdot n + \omega^2 = \omega \cdot n + \omega \cdot \omega = \omega \cdot (n + \omega) = \omega \cdot \omega = \omega^2$$

since $k + \omega^2 = \omega^2$ and $n + \omega = \omega$ by Lemma 6.5.5.3.

Lastly we show that if $\omega < \alpha < \omega^2$ then α is not an additive ordinal. Clearly by Lemma 6.5.6.4 there are natural numbers n and k such that $\alpha = \omega \cdot n + k$. Now let $\beta = \omega$ so that clearly $\beta < \alpha$. We then have that

$$\beta + \alpha = \omega + \omega \cdot n + k = \omega \cdot 1 + \omega \cdot n + k = \omega \cdot (1 + n) + k = \omega \cdot (n + 1) + k.$$

We also clearly have

$$\begin{aligned}
 n &< n + 1 \\
 \omega \cdot n &< \omega \cdot (n + 1) && \text{(by Exercise 6.5.7a)} \\
 \omega \cdot n + k &< \omega \cdot (n + 1) + k && \text{(by Lemma 6.5.5.1)} \\
 \alpha &< \beta + \alpha
 \end{aligned}$$

so that $\beta + \alpha \neq \alpha$. Since $\beta < \alpha$ this shows that α is not an additive ordinal. \square

Exercise 6.5.7

- (a) If α_1, α_2 , and β are ordinals and $\beta \neq 0$, then $\alpha_1 < \alpha_2$ if and only if $\beta \cdot \alpha_1 < \beta \cdot \alpha_2$.
 (b) For all ordinals α_1, α_2 , and $\beta \neq 0$, $\beta \cdot \alpha_1 = \beta \cdot \alpha_2$ if and only if $\alpha_1 = \alpha_2$.

Solution:

(a)

Proof. (\rightarrow) We show this by transfinite induction on α_2 . So suppose and that $\alpha_1 < \delta$ implies that $\beta \cdot \alpha_1 < \beta \cdot \delta$ for all $\delta < \alpha_2$ and that $\alpha_1 < \alpha_2$. If α_2 is a successor ordinal then $\alpha_2 = \delta + 1$ for some ordinal δ where $\alpha_1 \leq \delta$ since $\alpha_1 < \alpha_2$. Then we have

$$\begin{aligned} \beta \cdot \alpha_1 &\leq \beta \cdot \delta && \text{(by the induction hypothesis if } \alpha_1 < \delta \text{ and trivially if } \alpha_1 = \delta) \\ &< \beta \cdot \delta + \beta && \text{(by Lemma 6.5.4a since } 0 < \beta) \\ &= \beta \cdot (\delta + 1) && \text{(by Definition 6.5.6b)} \\ &= \beta \cdot \alpha_2. \end{aligned}$$

On the other hand if α_2 is a limit ordinal then $\alpha_1 + 1 < \alpha_2$ since $\alpha_1 < \alpha_2$. Then we have that

$$\begin{aligned} \beta \cdot \alpha_1 &< \beta \cdot (\alpha_1 + 1) && \text{(by the induction hypothesis since } \alpha_1 < \alpha_1 + 1 < \alpha_2) \\ &\leq \sup_{\delta < \alpha_2} \beta \cdot \delta && \text{(since the supremum is an upper bound)} \\ &= \beta \cdot \alpha_2. && \text{(by Definition 6.5.6c)} \end{aligned}$$

This completes the inductive proof.

(\leftarrow) For this we assume that $\beta \cdot \alpha_1 < \beta \cdot \alpha_2$. If it were the case that $\alpha_1 > \alpha_2$ than it would follow by the implication already shown that $\beta \cdot \alpha_1 > \beta \cdot \alpha_2$, which is a contradiction. Similarly if $\alpha_1 = \alpha_2$ then $\beta \cdot \alpha_1 = \beta \cdot \alpha_2$, another contradiction. Hence by the linearity of the order $<$ it must be that $\alpha_1 < \alpha_2$ as desired. \square

(b)

Proof. This follows from part (a) in the same way as the proof of Lemma 6.5.4b but is repeated for completeness.

(\rightarrow) We prove this part by contrapositive, so suppose that $\alpha_1 \neq \alpha_2$. Then either $\alpha_1 < \alpha_2$ or $\alpha_1 > \alpha_2$. In the former case then part (a) implies that $\beta \cdot \alpha_1 < \beta \cdot \alpha_2$ so that $\beta \cdot \alpha_1 \neq \beta \cdot \alpha_2$. In the latter case part (a) similarly implies that $\beta \cdot \alpha_1 > \beta \cdot \alpha_2$ so that again $\beta \cdot \alpha_1 \neq \beta \cdot \alpha_2$.

(\leftarrow) If $\alpha_1 = \alpha_2$ then we trivially have $\beta \cdot \alpha_1 = \beta \cdot \alpha_2$. \square

Exercise 6.5.8

Let α, β , and γ be ordinals, and let $\alpha < \beta$. Then

- (a) $\alpha + \gamma \leq \beta + \gamma$
 (b) $\alpha \cdot \gamma \leq \beta \cdot \gamma$.

Solution:

(a)

Proof. We show this by transfinite induction on γ . For $\gamma = 0$ we clearly have

$$\alpha + \gamma = \alpha + 0 = \alpha < \beta = \beta + 0 = \beta + \gamma$$

so that $\alpha + \gamma \leq \beta + \gamma$ is true. Now suppose that $\alpha + \gamma \leq \beta + \gamma$ so that we have

$$\alpha + (\gamma + 1) = (\alpha + \gamma) + 1 \leq (\beta + \gamma) + 1 = \beta + (\gamma + 1)$$

by Lemma 6.5.5.1 and Corollary 6.5.5.2 since $\alpha + \gamma \leq \beta + \gamma$ (induction hypothesis) and 1 is a natural number.

Lastly, suppose that γ is a nonzero limit ordinal and that $\alpha + \delta \leq \beta + \delta$ for all $\delta < \gamma$. Let $A = \{\alpha + \delta \mid \delta < \gamma\}$ and tentatively suppose that $\alpha + \gamma > \beta + \gamma$. Then $\beta + \gamma < \alpha + \gamma = \sup A$ so that $\beta + \gamma$ is not an upper bound of A . Hence there is a $\delta < \gamma$ such that $\beta + \gamma < \alpha + \delta$. We then have that

$$\beta + \delta \leq \sup\{\beta + \varepsilon \mid \varepsilon < \gamma\} = \beta + \gamma < \alpha + \delta,$$

but this contradicts the induction hypothesis since $\delta < \gamma$. So it has to be that $\alpha + \gamma \leq \beta + \gamma$ as desired. This completes the inductive proof.

Furthermore we give an example that shows that the \leq cannot be replaced with $<$ in the conclusion. Let $\alpha = 1$, $\beta = 2$, and $\gamma = \omega$. Then clearly $\alpha = 1 < 2 = \beta$ but we also have

$$\alpha + \gamma = 1 + \omega = \omega = 2 + \omega = \beta + \gamma$$

so that clearly $\alpha + \gamma < \beta + \gamma$ is not true since they are equal.

Note also that clearly if $\alpha = \beta$ then $\alpha + \gamma = \beta + \gamma$ so that $\alpha + \gamma \leq \beta + \gamma$ is still true. Hence the conclusion is also true in the slightly more general case of $\alpha \leq \beta$. \square

(b)

Proof. We also show this by transfinite induction on γ . For $\gamma = 0$ we clearly have

$$\alpha \cdot \gamma = \alpha \cdot 0 = 0 = \beta \cdot 0 = \beta \cdot \gamma$$

so that $\alpha \cdot \gamma \leq \beta \cdot \gamma$ is true. Now suppose that $\alpha \cdot \gamma \leq \beta \cdot \gamma$ so that we have

$$\begin{aligned} \alpha \cdot (\gamma + 1) &= \alpha \cdot \gamma + \alpha && \text{(by Definition 6.5.6b)} \\ &< \alpha \cdot \gamma + \beta && \text{(by Lemma 6.5.4 since } \alpha < \beta \text{)} \\ &\leq \beta \cdot \gamma + \beta && \text{(by part (a) and the induction hypothesis)} \\ &= \beta \cdot (\gamma + 1) && \text{(by Definition 6.5.6b again).} \end{aligned}$$

Lastly, suppose that γ is a nonzero limit ordinal and that $\alpha \cdot \delta < \beta \cdot \delta$ for all $\delta < \gamma$. The argument is analogous to that in part (a). Let $A = \{\alpha \cdot \delta \mid \delta < \gamma\}$ and tentatively suppose that $\alpha \cdot \gamma > \beta \cdot \gamma$. Then $\beta \cdot \gamma < \alpha \cdot \gamma = \sup A$ so that $\beta \cdot \gamma$ is not an upper bound of A . Hence there is a $\delta < \gamma$ such that $\beta \cdot \gamma < \alpha \cdot \delta$. We then have that

$$\beta \cdot \delta \leq \sup\{\beta \cdot \varepsilon \mid \varepsilon < \gamma\} = \beta \cdot \gamma < \alpha \cdot \delta,$$

but this contradicts the induction hypothesis since $\delta < \gamma$. So it has to be that $\alpha \cdot \gamma \leq \beta \cdot \gamma$ as desired. This completes the inductive proof. \square

A case in which $\alpha < \beta$ but $\alpha \cdot \gamma = \beta \cdot \gamma$ is clearly provided when $\gamma = 0$ so that \leq in the conclusion cannot be replaced with $<$. Similarly here if $\alpha = \beta$ then $\alpha \cdot \gamma = \beta \cdot \gamma$ so that $\alpha \cdot \gamma \leq \beta \cdot \gamma$ is still true. Hence the conclusion is also true in the slightly more general case of $\alpha \leq \beta$.

Exercise 6.5.9

Show that the following rules do not hold for all ordinals α , β , and γ :

- (a) If $\alpha + \gamma = \beta + \gamma$ then $\alpha = \beta$.
- (b) If $\gamma > 0$ and $\alpha \cdot \gamma = \beta \cdot \gamma$, then $\alpha = \beta$.
- (c) $(\beta + \gamma) \cdot \alpha = \beta \cdot \alpha + \gamma \cdot \alpha$.

Solution:

Lemma 6.5.9.1. *If $n > 0$ is a natural number then $n \cdot \omega = \omega$.*

Proof. By Definition 6.5.6 we have that $n \cdot \omega = \sup \{n \cdot k \mid k < \omega\}$ but clearly $n \cdot k$ is a natural number for any $k < \omega$ so that $n \cdot \omega = \sup \{n \cdot k \mid k < \omega\} = \sup \{k \mid k < \omega\} = \omega$. \square

Main Problem.

- (a) Let $\alpha = 1$, $\beta = 2$, and $\gamma = \omega$ so that

$$\alpha + \gamma = 1 + \omega = \omega = 2 + \omega = \beta + \gamma$$

by Lemma 6.5.5.3 but $\alpha = 1 < 2 = \beta$ so that $\alpha \neq \beta$.

- (b) Again let $\alpha = 1$, $\beta = 2$, and $\gamma = \omega$ so that $\gamma = \omega > 0$ and

$$\alpha \cdot \gamma = 1 \cdot \omega = \omega = 2 \cdot \omega = \beta \cdot \gamma$$

by Lemma 6.5.9.1. Clearly though $\alpha = 1 < 2 = \beta$ so that $\alpha \neq \beta$.

- (c) Here let $\alpha = \omega$, $\beta = 1$, and $\gamma = 2$. Then we have

$$(\beta + \gamma) \cdot \alpha = (1 + 2) \cdot \omega = 3 \cdot \omega = \omega$$

by Lemma 6.5.9.1, whereas

$$\beta \cdot \alpha + \gamma \cdot \alpha = 1 \cdot \omega + 2 \cdot \omega = \omega + \omega = \omega \cdot 2,$$

where we have used Lemma 6.5.9.1 here as well as Example 6.5.7b. That $(\beta + \gamma) \cdot \alpha = \omega = \omega \cdot 1 \neq \omega \cdot 2 = \beta \cdot \alpha + \gamma \cdot \alpha$ follows from Exercise 6.5.7b.

Exercise 6.5.10

An ordinal α is a limit ordinal if and only if $\alpha = \omega \cdot \beta$ for some β .

Solution:

Lemma 6.5.10.1. *If α is an ordinal then $\alpha < \alpha + \omega$ and $\alpha + \omega$ is the next limit ordinal after α , i.e. every ordinal β such that $\alpha < \beta < \alpha + \omega$ is a successor ordinal.*

Proof. Consider any ordinal α . First we note that clearly $\alpha = \alpha + 0 < \alpha + \omega$ by Lemma 6.5.6a since $0 < \omega$. It also clearly follows from Lemma 6.5.4.1 that $\alpha + \omega$ is a limit ordinal.

Now suppose that β is any ordinal such that $\alpha < \beta < \alpha + \omega$. Since $\alpha < \beta$ there is a unique ordinal γ such that $\alpha + \gamma = \beta$ by Lemma 6.5.5. Now, since $\alpha + 0 = \alpha < \beta = \alpha + \gamma$ it follows again from Lemma 6.5.6a that $0 < \gamma$. Similarly we have $\alpha + \gamma = \beta < \alpha + \omega$ so that by the same lemma $\gamma < \omega$. Hence $0 < \gamma < \omega$ so that γ is a nonzero natural number. In particular $\gamma \geq 1$ so that $n = \gamma - 1$ is

also a natural number and $\gamma = n + 1$. We then have that $\beta = \alpha + \gamma = \alpha + (n + 1) = (\alpha + n) + 1$ so that clearly β is a successor ordinal. \square

Main Problem.

Proof. (\rightarrow) Suppose to the contrary that not every limit ordinal is equal to $\omega \cdot \beta$ for some β . Then let α be a limit ordinal such that $\alpha \neq \omega \cdot \gamma$ for every ordinal γ . Now let β be the set of ordinals δ such that $\omega \cdot \delta < \alpha$, noting that it could potentially be the empty set. We claim that β is an ordinal number and that moreover it is a limit ordinal.

Clearly if $\beta = \emptyset = 0$ then it is a limit ordinal, so assume that $\beta \neq \emptyset$. Then consider any $\delta \in \beta$ so that $\omega \cdot \delta < \alpha$. Also consider any $x \in \delta$ so that x is also an ordinal number by Lemma 6.2.8 and moreover that $x < \delta$. It then follows from Exercise 6.5.7a that $\omega \cdot x < \omega \cdot \delta < \alpha$. Hence we have that $x \in \beta$ so that $\delta \subseteq \beta$ since x was arbitrary. Since δ was also arbitrary this shows that β is transitive. Also since β is nonempty set of ordinals it follows from Theorem 6.2.6d that β is well-ordered. Thus by definition β is an ordinal number.

Now consider any $\delta < \beta$ so that $\delta \in \beta$ so that $\omega \cdot \delta < \alpha$. By Lemma 6.5.10.1 we have that $\omega \cdot \delta + \omega = \omega \cdot (\delta + 1)$ is the next limit ordinal after $\omega \cdot \delta$ (i.e. there are no limit ordinals between them) so it has to be that $\omega \cdot (\delta + 1) \leq \alpha$ since otherwise α would be a limit ordinal between $\omega \cdot \delta$ and $\omega \cdot (\delta + 1)$. But since $\alpha \neq \omega \cdot \gamma$ for all ordinals γ it cannot be that $\alpha = \omega \cdot (\delta + 1)$. Thus it must be that $\omega \cdot (\delta + 1) < \alpha$ so that $\delta + 1 \in \beta$ so that $\delta + 1 < \beta$. This shows that β is a limit ordinal by Lemma 6.5.1.2.

Now we claim that $\omega \cdot \beta = \alpha$, which is of course is a contradiction that proves the desired result. To see this let $A = \{\omega \cdot \delta \mid \delta < \beta\}$ so that by Definition 6.5.6c we have that $\omega \cdot \beta = \sup A$ since β is a limit ordinal. Now since $\delta < \beta$ means that $\delta \in \beta$ so that $\omega \cdot \delta < \alpha$ by the definition of β , clearly α is an upper bound of A so that $\omega \cdot \beta = \sup A \leq \alpha$. However, if it were the case that $\omega \cdot \beta < \alpha$ then we would have by definition that $\beta \in \beta$ which contradicts Lemma 6.2.7 since we have shown that β is an ordinal. Thus the only possibility is that $\omega \cdot \beta = \alpha$, which gives rise to the contradiction.

Moreover, we can show that β is unique. To see this, consider β_1 and β_2 where $\alpha = \omega \cdot \beta_1$ and $\alpha = \omega \cdot \beta_2$. Then clearly $\omega \cdot \beta_1 = \omega \cdot \beta_2$ so that $\beta_1 = \beta_2$ by Exercise 6.5.7b since clearly $\omega \neq 0$.

(\leftarrow) Suppose that $\alpha = \omega \cdot \beta$ for some ordinal β . Then the result that α is a limit ordinal follows immediately from Lemma 6.5.6.2 since ω is a limit ordinal. \square

Exercise 6.5.11

Find a set A of rational numbers such that $(A, \leq_{\mathbf{Q}})$ is isomorphic to (α, \leq) where

- (a) $\alpha = \omega + 1$,
- (b) $\alpha = \omega \cdot 2$,
- (c) $\alpha = \omega \cdot 3$,
- (d) $\alpha = \omega^\omega$,
- (e) $\alpha = \varepsilon$.

[Hint: $\{n - 1/m \mid m, n \in \mathbf{N} - \{0\}\}$ is isomorphic to ω^2 , etc.]

Solution:

Lemma 6.5.11.1. *If β is an initial segment of an ordinal α then β is also an ordinal.*

Proof. By Lemma 6.1.2 there is an $a \in \alpha$ such that $\beta = \{x \in \alpha \mid x < a\}$. Also, since $a \in \alpha$ and α is an ordinal, a is an ordinal as well by Lemma 6.2.8. Finally, by the comments after Theorem 6.2.10, β

is simply the ordinal a since $\beta = \{x \in \alpha \mid x < a\}$ and each x in that set is an ordinal (by Lemma 6.2.8 since each $x \in \alpha$ and α is an ordinal). \square

Lemma 6.5.11.2. *If A is a set of ordinals then ordinal $\alpha = \sup A$ if and only if α is the least upper bound of A , i.e. α is an upper bound of A and β is not an upper bound of A for any $\beta < \alpha$.*

Proof. (\rightarrow) First suppose that $\alpha = \sup A$. Then by the remarks following the proof of Theorem 6.2.6 in the text α is an upper bound of A and if β is an upper bound of A then $\alpha \leq \beta$. This last statement is simply the contrapositive of the statement that $\beta < \alpha$ implies that β is *not* an upper bound of A and hence is logically equivalent.

(\leftarrow) We show that an ordinal α with the least upper bound property for A is unique, which suffices to show the result since if β has this property then $\beta = \sup A$ since $\sup A$ does as well (by what was just shown above) and the ordinal having this property is unique.

So suppose that ordinals α and β both have the least upper bound property for A but that $\alpha \neq \beta$. Without loss of generality we can assume then that $\alpha < \beta$. But then, since β has the least upper bound property, α cannot be an upper bound of A , which contradicts the fact that α also has the least upper bound property! Hence it has to be that $\alpha = \beta$, which shows the uniqueness. \square

Next we need to build up a little theory.

Definition 6.5.11.3. *For an ordinal α we call a set E_α an embedding of α if it has the following properties:*

1. E_α is a subset of \mathbf{Q} .
2. E_α is order isomorphic to α under the usual ordering of the rationals.
3. For some a and b in \mathbf{Q} , $a \leq x < b$ for every $x \in E_\alpha$. We denote this by saying $a \leq E_\alpha < b$ or that E_α is an embedding in $[a, b)$.

We call U_α a unit embedding of α if it is an embedding of α in $[0, 1)$.

Theorem 6.5.11.4. *If E_α is an embedding of α in $[a, b)$ then for each $x \in E_\alpha$ there is a $\Delta x \in \mathbf{Q}^+$ such that $x + \Delta x \leq b$ and if $y \in \mathbf{Q}$ and $x < y < x + \Delta x$ then y is not in E_α . That is, x is not a limit point from the right.*

Proof. Consider any $x \in E_\alpha$, noting that $x < b$.

If x is the greatest element of E_α then let $\Delta x = b - x$, noting that $\Delta x > 0$ since $b > x$. We also have

$$x + \Delta x = x + b - x = b \leq b.$$

Then consider any $y \in \mathbf{Q}$ where $x < y < x + \Delta x$ so that clearly $y \notin E_\alpha$ since otherwise x would not be the greatest element of E_α .

On the other hand if x is not the greatest element of E_α then let f be the isomorphism between α and E_α and let $\beta = f^{-1}(x)$. It follows that β is not the greatest element of α so that $\beta + 1 \in \alpha$ as well. Then let $\Delta x = f(\beta + 1) - x$, noting that $\Delta x > 0$ since $f(\beta + 1) > f(\beta) = x$ since f is an isomorphism. We also have

$$x + \Delta x = x + f(\beta + 1) - x = f(\beta + 1) < b$$

since $f(\beta + 1) \in E_\alpha$. Now consider any $y \in \mathbf{Q}$ where $f(\beta) = x < y < x + \Delta x = f(\beta + 1)$. If it were the case that $y \in E_\alpha$ then we would have that $\beta < f^{-1}(y) < \beta + 1$ since f is an isomorphism, which is impossible since β is an ordinal. So it must be that $y \notin E_\alpha$ as desired. \square

Corollary 6.5.11.5. *If E_α is an embedding and x and y are in E_α where $x < y$ then $x + \Delta x \leq y$, where $\Delta x \in \mathbf{Q}^+$ is that guaranteed by Theorem 6.5.11.4.*

Proof. If it were the case that $x + \Delta x > y$ then we have that $x < y < x + \Delta x$, which is in direct contradiction to Theorem 6.5.11.4 since $y \in E_\alpha$. \square

For an embedding E_α consider $p \in \mathbf{Q}^+$ and $q \in \mathbf{Q}$. We define a set denoted by $pE_\alpha + q$ to be the set $\{px + q \mid x \in E_\alpha\}$.

Theorem 6.5.11.6. *If E_α is an embedding in $[a, b)$ then $F_\alpha = pE_\alpha + q$ is also an embedding of α for any $p \in \mathbf{Q}^+$ and $q \in \mathbf{Q}$. Moreover $pa + q \leq F_\alpha < pb + q$.*

Proof. So first consider any $y \in F_\alpha$ so that $y = px + q$ for some $x \in E_\alpha$. Since E_α is an embedding $x \in \mathbf{Q}$ so that clearly $y = px + q \in \mathbf{Q}$ as well since $p, q \in \mathbf{Q}$. Hence since y was arbitrary we have that $F_\alpha \subseteq \mathbf{Q}$ so that (1) is satisfied.

Now consider the mapping $f : E_\alpha \rightarrow F_\alpha$ defined by $f(x) = px + q$ for $x \in E_\alpha$. Clearly $F_\alpha = \{f(x) \mid x \in E_\alpha\}$ so that f is onto. Consider then $x, y \in E_\alpha$ where $x < y$ so that we have

$$\begin{aligned} x &< y \\ px &< py && \text{(since } p > 0\text{)} \\ px + q &< py + q \\ f(x) &< f(y). \end{aligned}$$

Hence f is an isomorphism. Thus F_α is isomorphic to E_α so that clearly it is also isomorphic to α since E_α is, thereby showing (2).

Lastly for any $y \in F_\alpha$ we have $y = px + q$ for some $x \in E_\alpha$. We then have

$$\begin{aligned} a &\leq x < b \\ pa &\leq px < pb && \text{(since } p > 0\text{)} \\ pa + q &\leq px + q < pb + q \\ pa + q &\leq y < pb + q. \end{aligned}$$

This shows both (3) and the last statement. \square

For an embedding E_α in $[a, b)$ and a unit embedding U_β for ordinals α and β we define the product

$$E_\alpha \cdot U_\beta = \bigcup_{x \in E_\alpha} (\Delta x \cdot U_\beta + x),$$

where $\Delta x \in \mathbf{Q}^+$ is that guaranteed to exist by Theorem 6.5.11.4.

Theorem 6.5.11.7. *For an embedding E_α in $[a, b)$ and a unit embedding U_β the product $E_\alpha \cdot U_\beta$ is an embedding of $\beta \cdot \alpha$ in $[a, b)$.*

Proof. First consider any $y \in E_\alpha \cdot U_\beta$ so that there is an $x \in E_\alpha$ such that $y \in \Delta x \cdot U_\beta + x$. Since $y \in \Delta x \cdot U_\beta + x$ is an embedding by Theorem 6.5.11.6 it follows that $y \in \mathbf{Q}$, which shows (1) since y was arbitrary.

Now since E_α is an embedding of α there is an isomorphism $f : \alpha \rightarrow E_\alpha$. Similarly there is an isomorphism $g : \beta \rightarrow U_\beta$ since U_β is an embedding of β . Consider $\alpha \times \beta$ with lexicographic ordering \prec . Then define a mapping $h : \alpha \times \beta \rightarrow E_\alpha \cdot U_\beta$ by

$$h(\delta, \varepsilon) = \Delta f(\delta) \cdot g(\varepsilon) + f(\delta)$$

for $(\delta, \varepsilon) \in \alpha \times \beta$, noting that $\Delta f(\delta)$ is that guaranteed by Theorem 6.5.11.4 since $f(\delta) \in E_\alpha$.

First we claim that h is surjective. So consider any $z \in E_\alpha \cdot U_\beta$ so that there is an $x \in E_\alpha$ such that $z \in \Delta x \cdot U_\beta + x$. By definition then there is a $y \in U_\beta$ such that $z = \Delta x \cdot y + x$. Now let $\delta = f^{-1}(x)$ and $\varepsilon = g^{-1}(y)$ so that $x = f(\delta)$ and $y = g(\varepsilon)$, which can be done since f and g are bijections. We then have that

$$h(\delta, \varepsilon) = \Delta f(\delta) \cdot g(\varepsilon) + f(\delta) = \Delta x \cdot y + x = z,$$

which shows that h is surjective since z was arbitrary.

We also claim that h is an isomorphism and therefore also injective. So consider any $(\delta_1, \varepsilon_1)$ and $(\delta_2, \varepsilon_2)$ in $\alpha \times \beta$ where $(\delta_1, \varepsilon_1) \prec (\delta_2, \varepsilon_2)$. By the definition of lexicographic ordering we have the following:

Case: $\delta_1 < \delta_2$. Then since f is an isomorphism $f(\delta_1) < f(\delta_2)$, and also by Corollary 6.5.11.5 it follows that $f(\delta_1) < f(\delta_1) + \Delta f(\delta_1) \leq f(\delta_2)$. We also have that

$$\Delta f(\delta_1) \cdot g(\varepsilon_1) + f(\delta_1) < \Delta f(\delta_1) + f(\delta_1)$$

by Theorem 6.5.11.6 since $g(\varepsilon_1) \in U_\beta$ and U_β is a unit embedding. Also since $g(\varepsilon_2) \geq 0$ (since $g(\varepsilon_2) \in U_\beta$) and $\Delta f(\delta_2) > 0$ (by Theorem 6.5.11.4) that

$$f(\delta_2) \leq \Delta f(\delta_2) \cdot g(\varepsilon_2) + f(\delta_2).$$

Combining all this results in

$$h(\delta_1, \varepsilon_1) = \Delta f(\delta_1) \cdot g(\varepsilon_1) + f(\delta_1) < \Delta f(\delta_1) + f(\delta_1) \leq f(\delta_2) = h(\delta_2, \varepsilon_2).$$

Case: $\delta_1 = \delta_2$ and $\varepsilon_1 < \varepsilon_2$. Then obviously $f(\delta_1) = f(\delta_2)$ so that $\Delta f(\delta_1) = \Delta f(\delta_2)$ but also $g(\varepsilon_1) < g(\varepsilon_2)$ since g is an isomorphism. We then have

$$\begin{aligned} g(\varepsilon_1) &< g(\varepsilon_2) \\ \Delta f(\delta_1) \cdot g(\varepsilon_1) &< \Delta f(\delta_1) \cdot g(\varepsilon_2) && \text{(since } \Delta f(\delta_1) > 0) \\ \Delta f(\delta_1) \cdot g(\varepsilon_1) + f(\delta_1) &< \Delta f(\delta_1) \cdot g(\varepsilon_2) + f(\delta_1) \\ \Delta f(\delta_1) \cdot g(\varepsilon_1) + f(\delta_1) &< \Delta f(\delta_2) \cdot g(\varepsilon_2) + f(\delta_2) && \text{(since } \Delta f(\delta_1) = \Delta f(\delta_2) \text{ and } f(\delta_1) = f(\delta_2)) \\ h(\delta_1, \varepsilon_1) &< h(\delta_2, \varepsilon_2). \end{aligned}$$

Thus in all cases $h(\delta_1, \varepsilon_1) < h(\delta_2, \varepsilon_2)$, which shows that h is an isomorphism since $(\delta_1, \varepsilon_1)$ and $(\delta_2, \varepsilon_2)$ were arbitrary. Hence $E_\alpha \cdot U_\beta$ is isomorphic to the lexicographic ordering of $\alpha \times \beta$ and therefore also to $\beta \cdot \alpha$ by Theorem 6.5.8. This shows part (2) of the embedding definition.

Lastly consider any $z \in E_\alpha \cdot U_\beta$ so that there is an $x \in E_\alpha$ such that $z \in \Delta x \cdot U_\beta + x$. Then since $U_\beta < 1$ we have that $z < \Delta x + x \leq b$ by Theorems 6.5.11.6 and 6.5.11.4. Also since $0 \leq U_\beta$ it follows from Theorem 6.5.11.6 that $a \leq x \leq z$ since $a \leq E_\alpha$. Since z was arbitrary this shows that $a \leq E_\alpha \cdot U_\beta < b$, which shows (3). This completes the proof. \square

Theorem 6.5.11.8. *Suppose that α is a limit ordinal and that $\{\alpha_n\}$ is a sequence ($n \in \omega$) of nonzero ordinals in α . Also suppose that E_ω is an embedding of ω in $[a, b)$ and U_{α_n} is a unit embedding of α_n for each $n \in \omega$. Suppose further that $\alpha = \sup_{n \in \omega} \alpha_n$ and that $\alpha_n + \alpha_{n+1} = \alpha_{n+1}$ for every $n \in \omega$, noting that clearly $n + 1 \in \omega$ as well. Lastly, suppose that f is the isomorphism from ω to E_ω . Let*

$$A_n = \Delta f(n) \cdot U_{\alpha_n} + f(n)$$

for $n \in \omega$. Then the set

$$E_\alpha = \bigcup_{n \in \omega} A_n$$

is an embedding of α in $[a, b)$.

Proof. First consider any n and m in ω where $n \neq m$. We can assume without loss of generality $n < m$. Then $f(n) < f(m)$ since f is an isomorphism and, moreover, it follows from Corollary 6.5.11.5 that $\Delta f(n) + f(n) \leq f(m)$. Hence by Theorem 6.5.11.6 we have that

$$A_n = \Delta f(n) \cdot U_{\alpha_n} + f(n) < \Delta f(n) + f(n) \leq f(m) \leq \Delta f(m) \cdot U_{\alpha_m} + f(m) = A_m$$

since U_{α_n} and U_{α_m} are unit embeddings. Hence all the A_n are disjoint and moreover $A_0 < A_1 < A_2 < \dots$. Also clearly by Theorem 6.5.11.6 each A_n is isomorphic to α_n since U_{α_n} is.

Now for $p, q \in \mathcal{Q}$ let $[p, q) = \{x \in \mathcal{Q} \mid q \leq x < p\}$. We then claim that $E_\alpha \cap [a, f(n+1))$ for any $n \in \omega$ is isomorphic to α_n , which we shall show by induction on n . For $n = 0$ we clearly have that $a \leq A_0 \leq \Delta f(0) + f(0) \leq f(1) \leq A_m$ for any $m \geq 1$. From this it follows that $E_\alpha \cap [a, f(n+1)) = E_\alpha \cap [a, f(1)) = A_0$, which is isomorphic to $\alpha_0 = \alpha_n$ by what was shown above.

Now suppose that $E_\alpha \cap [a, f(n+1))$ is isomorphic to α_n . We clearly have $E_\alpha \cap [a, f(n+2)) = [E_\alpha \cap [a, f(n+1))] \cup [E_\alpha \cap [f(n+1), f(n+2))]$ and that $E_\alpha \cap [f(n+1), f(n+2)) = A_{n+1}$, which is isomorphic to α_{n+1} by what was shown above. Then $E_\alpha \cap [a, f(n+2))$ is the sum of $E_\alpha \cap [a, f(n+1))$ and $E_\alpha \cap [f(n+1), f(n+2)) = A_{n+1}$ so that by Theorem 6.5.3, the induction hypothesis, and the given property of $\{\alpha_n\}$ it is isomorphic to $\alpha_n + \alpha_{n+1} = \alpha_{n+1}$. This completes the inductive proof.

We also show that $E_\alpha \cap [a, f(n+1))$ is an initial segment of E_α for any $n \in \omega$. So consider any such n , any $x \in E_\alpha \cap [a, f(n+1))$, and any $y \in E_\alpha$ where $y < x$. Since $a \leq E_\alpha$ clearly $a \leq y$. We also have $y < x < f(n+1)$ (since $x \in [a, f(n+1))$). Hence $y \in [a, f(n+1))$ so that also $y \in E_\alpha \cap [a, f(n+1))$, which shows that $E_\alpha \cap [a, f(n+1))$ is an initial segment of E_α by definition.

Now we claim that E_α is a well-ordered set. So consider any nonempty subset B of E_α . Then there is some $x \in B$ and since $x \in E_\alpha$ there is an $n \in \omega$ such that $x \in A_n$. Then clearly $x \in B \cap [a, f(n+1))$ so that $B \cap [a, f(n+1))$ is a nonempty subset of $E_\alpha \cap [a, f(n+1))$. It was shown above that $E_\alpha \cap [a, f(n+1))$ is isomorphic to α_n so that it is a well-ordered set. Hence $B \cap [a, f(n+1))$ has a least element y . We claim that this is the least element of B , so consider any $z \in B$. If $z < f(n+1)$ then clearly $z \in B \cap [a, f(n+1))$ so that obviously $y \leq z$ since y is the least element of $B \cap [a, f(n+1))$. On the other hand if $z \geq f(n+1)$ then $y < f(n+1) \leq z$ (since $y \in [a, f(n+1))$) so that again $y \leq z$. Since z was arbitrary this shows that y is in fact the least element of B . Since B was an arbitrary subset of E_α this shows that E_α is well-ordered.

Since E_α is a well-ordered set it is isomorphic to some ordinal γ by Theorem 6.3.1. We then claim that $\gamma = \alpha$. Letting $C = \{\alpha_n \mid n \in \omega\}$, we show this by showing that γ is the least upper bound of C , which shows that $\gamma = \alpha$ by the least upper bound property (Lemma 6.5.11.2) since $\alpha = \sup C$ by definition. So first consider any $\alpha_n \in C$. It was shown above that α_n is isomorphic to $E_\alpha \cap [a, f(n+1))$, and this was shown to be an initial segment of E_α , which itself is isomorphic to γ . Thus it follows that $\alpha_n < \gamma$ so that $\alpha_n \leq \gamma$ is true. Since α_n was arbitrary this shows that γ is an upper bound of C .

Now consider any ordinal $\delta < \gamma$ so that $\delta \in \gamma$. Let g be the isomorphism from γ to E_α since it has been shown that they are isomorphic. Then since $g(\delta) \in E_\alpha$ so that there is an $n \in \omega$ such that $g(\delta) \in A_n$. Then also $g(\delta) \in E_\alpha \cap [a, f(n+1))$. Since $E_\alpha \cap [a, f(n+1))$ is an initial segment of E_α (just shown above) it follows that $g^{-1}[E_\alpha \cap [a, f(n+1))]$ is an initial segment of γ . Since γ is an ordinal it follows from Lemma 6.5.11.1 that $g^{-1}[E_\alpha \cap [a, f(n+1))]$ is an ordinal. Since $g^{-1} \upharpoonright E_\alpha \cap [a, f(n+1))$ is an isomorphism (since g is) and $E_\alpha \cap [a, f(n+1))$ is isomorphic to α_n (shown above), it follows that $g^{-1}[E_\alpha \cap [a, f(n+1))]$ is in fact α_n ! Then, since $g(\delta) \in E_\alpha \cap [a, f(n+1))$ we have that $\delta \in g^{-1}[E_\alpha \cap [a, f(n+1))] = \alpha_n$ so that $\delta < \alpha_n$. Hence δ is not an upper bound of C . Since δ was arbitrary this shows that γ is in fact the least upper bound of C so that $\gamma = \alpha$.

Parts (1) and (3) of the definition of an embedding are trivial to show by the same arguments as those used in the proof of Theorem 6.5.11.7. Hence E_α is an embedding of α in $[a, b)$. \square

Main Problem.

(a) Define $f : \omega \rightarrow \mathbf{Q}$ by

$$f(n) = 1 - \frac{1}{n+1} = \frac{n}{n+1}$$

for $n \in \omega$. Then let $U_\omega = \{f(n) \mid n \in \omega\}$. We claim that U_ω is a unit embedding of ω .

Proof. Clearly $U_\omega \subseteq \mathbf{Q}$ so that (1) is satisfied. To show (2) let $g : \omega \rightarrow U_\omega$ be defined by $g(n) = f(n) = 1 - 1/(n+1)$, which is clearly onto based on the definition of U_ω . We also claim that g is an isomorphism. To this end consider any $n, m \in \omega$ where $n < m$. We then have

$$\begin{aligned} n &< m \\ n+1 &< m+1 \\ \frac{n+1}{m+1} &< 1 && \text{(since } m+1 \geq 1 > 0 \text{ since } m \geq 0) \\ \frac{1}{m+1} &< \frac{1}{n+1} && \text{(since } n+1 \geq 1 > 0 \text{ since } n \geq 0) \\ -\frac{1}{m+1} &> -\frac{1}{n+1} \\ 1 - \frac{1}{m+1} &> 1 - \frac{1}{n+1} \\ g(m) &> g(n) \\ g(n) &< g(m). \end{aligned}$$

Hence g is an isomorphism so that U_ω is indeed isomorphic to ω , which shows (2).

Lastly consider any $x \in U_\omega$ so that $x = f(n) = 1 - 1/(n+1)$ for some $n \in \omega$. Then we have

$$\begin{aligned} n &\geq 0 > -1 \\ n+1 &\geq 1 > 0 \\ 1 &\geq \frac{1}{n+1} > 0 && \text{(since } n+1 \geq 1 > 0) \\ -1 &\leq -\frac{1}{n+1} < 0 \\ 0 &\leq 1 - \frac{1}{n+1} < 1 \\ 0 &\leq x < 1. \end{aligned}$$

Since x was arbitrary this shows that $0 \leq U_\omega < 1$ so that (3) holds and U_ω is a unit embedding.

Now let $E_1 = \{1\}$. Clearly this is an embedding of the ordinal 1. Moreover we have $0 \leq U_\omega < 1 \leq E_1 < 2$ so that U_ω and E_1 are disjoint. Then clearly $U_\omega \cup E_1$ is the sum of U_ω and E_1 so that it is isomorphic to $\omega + 1$ by Theorem 6.5.3 and thus an embedding of $\omega + 1$ since it is trivial to show that $0 \leq U_\omega \cup E_1 < 2$. \square

(b) Now consider the same U_ω from part (a) and let $E_\omega = 1 \cdot U_\omega + 1$ so that by Theorem 6.5.11.6 this is another embedding of ω and $1 \leq E_\omega < 2$. Hence we have that $0 \leq U_\omega < 1 \leq E_\omega < 2$ so that U_ω and E_ω are disjoint. Also clearly $E_{\omega \cdot 2} = U_\omega \cup E_\omega$ is the sum of U_ω and E_ω so that it is isomorphic to $\omega + \omega = \omega \cdot 2$ by Theorem 6.5.3. Hence since also $0 \leq E_{\omega \cdot 2} < 2$ (it is trivial to show) it is an embedding of $\omega \cdot 2$ as desired.

(c) Considering the same $E_{\omega \cdot 2}$ from part (b) let $F_\omega = 1 \cdot U_\omega + 2$ so that it is yet another an embedding of ω and $2 \leq F_\omega < 3$ by Theorem 6.5.11.6. Then by the same arguments as in part (b) it follows that $E_{\omega \cdot 2} \cup F_\omega$ is an embedding of $\omega \cdot 2 + \omega = \omega \cdot 2 + \omega \cdot 1 = \omega \cdot (2 + 1) = \omega \cdot 3$ as desired.

(d) We first aim to construct a unit embedding of ω^n for any $n \in \omega$. We do this recursively. For $n = 0$ clearly $U_{\omega^0} = \{0\}$ is a unit embedding of $\omega^0 = 1$. We have also already constructed U_ω as a unit embedding of ω in part (a). Now suppose we have constructed U_{ω^n} , a unit embedding of ω^n . Then we let $U_{\omega^{n+1}} = U_\omega \cdot U_{\omega^n}$ so that $U_{\omega^{n+1}}$ is a unit embedding of $\omega^n \cdot \omega = \omega^{n+1}$ by Theorem 6.5.11.7.

Clearly then $\{\omega^n\}$ is a sequence of ordinals in ω^ω and by definition $\omega^\omega = \sup_{n \in \omega} \omega^n$. We also have in hand the embedding U_ω and U_{ω^n} for each $n \in \omega$ as just constructed recursively. Now consider any $n \in \omega$ so that we have we have

$$\omega^n + \omega^{n+1} = \omega^n \cdot 1 + \omega^n \cdot \omega = \omega^n \cdot (1 + \omega) = \omega^n \cdot \omega = \omega^{n+1}.$$

We can therefore apply Theorem 6.5.11.8 to construct a unit embedding U_{ω^ω} of ω^ω .

(e) Here we use the operation of tetration, which we define recursively for all ordinals. So for all ordinals β we define:

1. ${}^0\beta = 1$
2. ${}^{\alpha+1}\beta = \beta^{\alpha\beta}$ for all α
3. ${}^\alpha\beta = \sup\{\gamma\beta \mid \gamma < \alpha\}$ for all limit $\alpha \neq 0$.

Thus we have ${}^1\omega = \omega$, ${}^2\omega = \omega^\omega$, ${}^3\omega = \omega^{\omega^\omega}$, etc. We then have that $\varepsilon = {}^\omega\omega = \sup_{n < \omega} {}^n\omega$ by definition.

Despite working and thinking about this problem for weeks I have been unable to come up with a way to embed ε . To be sure we can easily construct embeddings of ordinals larger than ω^ω , for example we can embed $\omega^\omega \cdot \omega^\omega = \omega^{\omega+\omega} = \omega^{\omega \cdot 2}$ by simply applying Theorem 6.5.11.7 to our just-constructed embedding of ω^ω . However, I could think of no way to get to ε easily using this approach. Ideally what we would have is a way to construct an embedding of ω^α for any ordinal α for which we already have an embedding. This would easily lead to an embedding of ε using Theorem 6.5.11.8 after constructing the embeddings for ${}^n\omega$ recursively. However, I was unable to think of how to construct this and eventually had to admit defeat and move on.

Exercise 6.5.12

Show that $(\omega \cdot 2)^2 \neq \omega^2 \cdot 2^2$.

Solution:

Proof. We have

$$\begin{aligned} (\omega \cdot 2)^2 &= (\omega \cdot 2) \cdot (\omega \cdot 2) && \text{(by Example 6.5.10a)} \\ &= [\omega \cdot (2 \cdot \omega)] \cdot 2 && \text{(by the associativity of multiplication)} \\ &= (\omega \cdot \omega) \cdot 2 && \text{(by Lemma 6.5.9.1)} \\ &= \omega^2 \cdot 2 \end{aligned}$$

whereas

$$\omega^2 \cdot 2^2 = \omega^2 \cdot 4$$

so that the two are clearly not equal by Exercise 6.5.7b since $2 \neq 4$ and $\omega^2 \neq 0$. □

Exercise 6.5.13

- (a) $\alpha^{\beta+\gamma} = \alpha^\beta \cdot \alpha^\gamma$.
 (b) $(\alpha^\beta)^\gamma = \alpha^{\beta \cdot \gamma}$.

Solution:

Lemma 6.5.13.1. *Suppose that α is an ordinal, $\beta \neq 0$ is a limit ordinal, and $\{\gamma_\delta\}$ for $\delta < \alpha + \beta$ is a non-decreasing transfinite sequence of ordinals. Then*

$$\sup_{\delta < \alpha + \beta} \gamma_\delta = \sup_{\delta < \beta} \gamma_{\alpha + \delta}.$$

Proof. First we note that $\alpha + \beta$ is a nonzero limit ordinal by Lemma 6.5.4.1. Let $A = \{\gamma_\delta \mid \delta < \alpha + \beta\}$ and

$$\gamma = \sup_{\delta < \beta} \gamma_{\alpha + \delta}.$$

We show that γ is the least upper bound of A , which shows the result by the least upper bound property (Lemma 6.5.11.2). First consider any γ_δ in A so that $\delta < \alpha + \beta$.

If $\delta < \alpha$ then $\gamma_\delta \leq \gamma_\alpha = \gamma_{\alpha+0} \leq \gamma$ since the sequence is non-decreasing and $0 < \beta$. On the other hand if $\delta \geq \alpha$ then by Lemma 6.5.5 there is a unique ordinal ε such that $\alpha + \varepsilon = \delta$. Then, since $\delta = \alpha + \varepsilon < \alpha + \beta$ it follows from Lemma 6.5.4a that $\varepsilon < \beta$. Hence by the definition of γ we have that $\gamma_\delta = \gamma_{\alpha+\varepsilon} \leq \gamma$. Thus in all cases $\gamma_\delta \leq \gamma$, which shows that γ is an upper bound of A since γ_δ was arbitrary.

Now consider any $\varepsilon < \gamma$. Then by the definition of γ it follows that there is a $\delta < \beta$ such that $\varepsilon < \gamma_{\alpha+\delta}$ since ε is not an upper bound $\{\gamma_{\alpha+\zeta} \mid \zeta < \beta\}$. Thus again by Lemma 6.5.4a we have that $\alpha + \delta < \alpha + \beta$ so that $\gamma_{\alpha+\delta} \in A$. Hence since $\varepsilon < \gamma_{\alpha+\delta}$ it cannot be that ε is an upper bound of A . Since $\varepsilon < \gamma$ was arbitrary this completes the proof that $\gamma = \sup A$ as desired. \square

Lemma 6.5.13.2. *If α is an ordinal such that $\alpha > 0$ then $0^\alpha = 0$. Otherwise if $\alpha = 0$ then $0^\alpha = 0^0 = 1$.*

Proof. First if $\alpha = 0$ then clearly $0^\alpha = 0^0 = 1$ by Definition 6.5.9a. Then if $\alpha > 0$ we show the result by transfinite induction on α . First if $\alpha = 1$ then we have

$$0^\alpha = 0^1 = 0^{0+1} = 0^0 \cdot 0 = 1 \cdot 0 = 0,$$

where we have used Definitions 6.5.9b and 6.5.6a. Now suppose that $0^\alpha = 0$ so that we have

$$0^{\alpha+1} = 0^\alpha \cdot 0 = 0,$$

where we again have used the same two definitions. Lastly suppose that α is a nonzero limit ordinal and that $0^\beta = 0$ for all $\beta < \alpha$. Then we have by Definition 6.5.9c that

$$0^\alpha = \sup_{\beta < \alpha} 0^\beta = \sup_{\beta < \alpha} 0 = 0,$$

where we have used the induction hypothesis. \square

Lemma 6.5.13.3. *If α is an ordinal then $1^\alpha = 1$.*

Proof. We show this by transfinite induction on α . First for $\alpha = 0$ we have $1^\alpha = 1^0 = 1$ by Definition 6.5.9a. Now assume that $1^\alpha = 1$ so that $1^{\alpha+1} = 1^\alpha \cdot 1 = 1^\alpha = 1$ by Definition 6.5.9b and the induction hypothesis.

Lastly suppose that α is a nonzero limit ordinal and that $1^\beta = 1$ for all $\beta < \alpha$. We then clearly have by Definition 6.5.9c that

$$1^\alpha = \sup_{\beta < \alpha} 1^\beta = \sup_{\beta < \alpha} 1 = 1.$$

This completes the inductive proof. □

Lemma 6.5.13.4. *If α , β , and γ are ordinals where $\alpha > 0$ and $\beta \leq \gamma$ then $\alpha^\beta \leq \alpha^\gamma$.*

Proof. Clearly if $\beta = \gamma$ then $\alpha^\beta = \alpha^\gamma$ so that the conclusion holds. So assume that $\beta < \gamma$.

Case: $\alpha = 1$. Then by Lemma 6.5.13.3 we have

$$\alpha^\beta = 1^\beta = 1 = 1^\gamma = \alpha^\gamma.$$

Case: $\alpha > 1$. Then it follows from Exercise 6.5.14b that $\alpha^\beta < \alpha^\gamma$.

Hence in either case $\alpha^\beta \leq \alpha^\gamma$ is true. □

(a)

Proof. We show this by transfinite induction on γ . First for $\gamma = 0$ we have

$$\begin{aligned} \alpha^{\beta+\gamma} &= \alpha^{\beta+0} \\ &= \alpha^\beta && \text{(by Definition 6.5.1a)} \\ &= \alpha^\beta \cdot 1 && \text{(by Example 6.5.7a)} \\ &= \alpha^\beta \cdot \alpha^0 && \text{(by Definition 6.5.9a)} \\ &= \alpha^\beta \cdot \alpha^\gamma. \end{aligned}$$

Now assume that $\alpha^{\beta+\gamma} = \alpha^\beta \cdot \alpha^\gamma$ so that we have

$$\begin{aligned} \alpha^{\beta+(\gamma+1)} &= \alpha^{(\beta+\gamma)+1} && \text{(by Definition 6.5.1b)} \\ &= \alpha^{\beta+\gamma} \cdot \alpha && \text{(by Definition 6.5.9b)} \\ &= (\alpha^\beta \cdot \alpha^\gamma) \cdot \alpha && \text{(by the induction hypothesis)} \\ &= \alpha^\beta \cdot (\alpha^\gamma \cdot \alpha) && \text{(by the associativity of multiplication)} \\ &= \alpha^\beta \cdot \alpha^{\gamma+1}. && \text{(by Definition 6.5.9b)} \end{aligned}$$

Lastly suppose that γ is a nonzero limit ordinal and that $\alpha^{\beta+\delta} = \alpha^\beta \cdot \alpha^\delta$ for all $\delta < \gamma$. First if $\alpha = 0$ then $\alpha^{\beta+\gamma} = 0^{\beta+\gamma} = 0$ by Lemma 6.5.13.2 since $\beta + \gamma > 0$ since $\gamma > 0$. Hence we have

$$\alpha^{\beta+\gamma} = 0 = 0^\beta \cdot 0 = 0^\beta \cdot 0^\gamma = \alpha^\beta \cdot \alpha^\gamma,$$

where we have used Definition 6.5.6a and the fact that $0^\gamma = 0$ by Lemma 6.5.13.2 since $\gamma > 0$.

On the other hand if $\alpha > 0$ then we have by Definition 6.5.9c that

$$\alpha^{\beta+\gamma} = \sup_{\delta < \beta+\gamma} \alpha^\delta.$$

It then follows from Lemma 6.5.13.4 that $\{\alpha^\delta\}$ for $\delta < \beta + \gamma$ is a non-decreasing sequence since $\alpha > 0$. Thus we can apply Lemma 6.5.13.1 so that

$$\alpha^{\beta+\gamma} = \sup_{\delta < \beta+\gamma} \alpha^\delta = \sup_{\delta < \gamma} \alpha^{\beta+\delta} = \sup_{\delta < \gamma} (\alpha^\beta \cdot \alpha^\delta)$$

by the induction hypothesis. It then follows from statement (6.6.1) in the text that

$$\alpha^{\beta+\gamma} = \sup_{\delta < \gamma} (\alpha^\beta \cdot \alpha^\delta) = \alpha^\beta \cdot \sup_{\delta < \gamma} \alpha^\delta = \alpha^\beta \cdot \alpha^\gamma$$

by Definition 6.5.9c. This completes the transfinite induction. \square

(b)

Proof. We show this by transfinite induction on γ as well. First for $\gamma = 0$ we have

$$\begin{aligned} (\alpha^\beta)^\gamma &= (\alpha^\beta)^0 = 1 && \text{(by Definition 6.5.9a)} \\ &= \alpha^0 && \text{(again by Definition 6.5.9a)} \\ &= \alpha^{\beta \cdot 0} = \alpha^{\beta \cdot \gamma}. && \text{(by Definition 6.5.6a)} \end{aligned}$$

Next suppose that $(\alpha^\beta)^\gamma = \alpha^{\beta \cdot \gamma}$ so that we have

$$\begin{aligned} (\alpha^\beta)^{\gamma+1} &= (\alpha^\beta)^\gamma \cdot \alpha^\beta && \text{(by Definition 6.5.9b)} \\ &= \alpha^{\beta \cdot \gamma} \cdot \alpha^\beta && \text{(by the induction hypothesis)} \\ &= \alpha^{\beta \cdot \gamma + \beta} && \text{(by part a)} \\ &= \alpha^{\beta \cdot (\gamma+1)}. && \text{(by Definition 6.5.6b)} \end{aligned}$$

Lastly, suppose that γ is a nonzero limit ordinal and that $(\alpha^\beta)^\delta = \alpha^{\beta \cdot \delta}$ for all $\delta < \gamma$.

Case: $\alpha = 0$. Then if $\beta = 0$ we have

$$\begin{aligned} (\alpha^\beta)^\gamma &= (0^0)^\gamma \\ &= 1^\gamma && \text{(by Definition 6.5.9a)} \\ &= 1 && \text{(by Lemma 6.5.13.3)} \\ &= 0^0 && \text{(by Definition 6.5.9a)} \\ &= 0^{0 \cdot \gamma} && \text{(by Lemma 6.5.1.1)} \\ &= \alpha^{\beta \cdot \gamma}. \end{aligned}$$

On the other hand if $\beta > 0$ then first we note that $0 = \beta \cdot 0 < \beta \cdot \gamma$ by Definition 6.5.6a and Exercise 6.5.7a since both $\beta \neq 0$ and $0 < \gamma$. We then have

$$\begin{aligned} (\alpha^\beta)^\gamma &= (0^\beta)^\gamma \\ &= 0^\gamma && \text{(by Lemma 6.5.13.2 since } \beta > 0) \\ &= 0 && \text{(by Lemma 6.5.13.2 again since } \gamma > 0) \\ &= 0^{\beta \cdot \gamma} && \text{(by Lemma 6.5.13.2 yet again since } \beta \cdot \gamma > 0 \text{ as shown above)} \\ &= \alpha^{\beta \cdot \gamma}. \end{aligned}$$

Case: $\alpha > 0$. Then we have

$$\begin{aligned} (\alpha^\beta)^\gamma &= \sup_{\delta < \gamma} (\alpha^\beta)^\delta && \text{(by Definition 6.5.9c)} \\ &= \sup_{\delta < \gamma} \alpha^{\beta \cdot \delta} && \text{(by the induction hypothesis)} \\ &= \alpha^{\sup_{\delta < \gamma} \beta \cdot \delta} && \text{(by statement (6.6.1) in the text since } \alpha > 0) \\ &= \alpha^{\beta \cdot \gamma} && \text{(by Definition 6.5.6c)} \end{aligned}$$

Since these cases are exhaustive this completes the inductive proof. \square

Exercise 6.5.14

- (a) If $\alpha \leq \beta$ then $\alpha^\gamma \leq \beta^\gamma$.
 (b) If $\alpha > 1$ and if $\beta < \gamma$, then $\alpha^\beta < \alpha^\gamma$.

Solution:

Lemma 6.5.14.1. *Suppose that γ is a nonzero limit ordinal and $\{\alpha_\nu\}$ and $\{\beta_\nu\}$ for $\nu < \gamma$ are two transfinite sequences. Also suppose that $\alpha_\nu \leq \beta_\nu$ for every $\nu < \gamma$. Then*

$$\sup_{\nu < \gamma} \alpha_\nu \leq \sup_{\nu < \gamma} \beta_\nu.$$

Proof. First let $A = \{\alpha_\nu \mid \nu < \gamma\}$ and $B = \{\beta_\nu \mid \nu < \gamma\}$ be the ranges of the sequences so that we must show that $\sup A \leq \sup B$. Now consider any $\alpha \in A$ so that $\alpha = \alpha_\nu$ for some $\nu < \gamma$. We then have that

$$\alpha = \alpha_\nu \leq \beta_\nu \leq \sup B$$

so that $\sup B$ is an upper bound of A since α was arbitrary. It then follows from the least upper bound property that $\sup A \leq \sup B$ as desired. \square

Main Problem.

- (a)

Proof. We show this by transfinite induction on γ . So first suppose that $\alpha \leq \beta$. Then for $\gamma = 0$ we clearly have

$$\alpha^\gamma = \alpha^0 = 1 = \beta^0 = \beta^\gamma,$$

where we have used Definition 6.5.9a twice. Then $\alpha^\gamma \leq \beta^\gamma$ clearly holds.

Now suppose that $\alpha^\gamma \leq \beta^\gamma$ so that we have

$$\begin{aligned} \alpha^{\gamma+1} &= \alpha^\gamma \cdot \alpha && \text{(by Definition 6.5.9b)} \\ &\leq \beta^\gamma \cdot \alpha && \text{(follows from Exercise 6.5.8b and the induction hypothesis)} \\ &\leq \beta^\gamma \cdot \beta && \text{(follows from Exercise 6.5.7 since } \alpha \leq \beta \text{)} \\ &= \beta^{\gamma+1}. && \text{(by Definition 6.5.9b)} \end{aligned}$$

Lastly suppose that γ is a nonzero limit ordinal and that $\alpha^\delta \leq \beta^\delta$ for all $\delta < \gamma$. We then have

$$\begin{aligned} \alpha^\gamma &= \sup_{\delta < \gamma} \alpha^\delta && \text{(by Definition 6.5.9c)} \\ &\leq \sup_{\delta < \gamma} \beta^\delta && \text{(by Lemma 6.5.14.1 and the induction hypothesis)} \\ &= \beta^\gamma, && \text{(by Definition 6.5.9c again)} \end{aligned}$$

which completes the transfinite induction. \square

- (b)

Proof. Suppose that $\alpha > 1$. We then show the result by transfinite induction on γ similarly to the proof of Exercise 6.5.7a. Suppose that $\beta < \delta$ implies that $\alpha^\beta < \alpha^\delta$ for all $\delta < \gamma$ and that $\beta < \gamma$.

Case: γ is a successor ordinal. Then $\gamma = \delta + 1$ for an ordinal δ and $\beta \leq \delta$ since $\beta < \gamma = \delta + 1$. We then have

$$\alpha^\beta \leq \alpha^\delta \quad \text{(by the induction hypothesis if } \beta < \delta \text{ and trivially if } \beta = \delta \text{)}$$

$$\begin{aligned}
&= \alpha^\delta \cdot 1 \\
&< \alpha^\delta \cdot \alpha && \text{(by Exercise 6.5.7a since } 1 < \alpha \text{ and } \alpha^\delta \neq 0) \\
&= \alpha^{\delta+1} && \text{(by Definition 6.5.9b)} \\
&= \alpha^\gamma .
\end{aligned}$$

Case: γ is a limit ordinal. Here we have that $\beta + 1 < \gamma$ as well since $\beta < \gamma$. We then have

$$\begin{aligned}
\alpha^\beta &< \alpha^{\beta+1} && \text{(by the induction hypothesis since } \beta < \beta + 1 < \gamma) \\
&\leq \sup_{\delta < \gamma} \alpha^\delta && \text{(since the supremum is an upper bound)} \\
&= \alpha^\gamma . && \text{(by Definition 6.5.9c)}
\end{aligned}$$

Thus in either case $\alpha^\beta < \alpha^\gamma$, thereby completing the transfinite induction. \square

Exercise 6.5.15

Find the least ξ such that

- (a) $\omega + \xi = \xi$.
- (b) $\omega \cdot \xi = \xi$, $\xi \neq 0$.
- (c) $\omega^\xi = \xi$.

[Hint for part (a): Let $\xi_0 = 0$, $\xi_{n+1} = \omega + \xi_n$, $\xi = \sup \{\xi_n \mid n \in \omega\}$.]

Solution:

Lemma 6.5.15.1. *Suppose that α and γ are a nonzero limit ordinals, that $\{\alpha_\delta\}$ is a transfinite sequence indexed by γ , and that*

$$\alpha = \sup_{\delta < \gamma} \alpha_\delta .$$

Also suppose that β_δ is another non-decreasing transfinite sequence indexed by α . Then

$$\sup_{\delta < \alpha} \beta_\delta = \sup_{\delta < \gamma} \beta_{\alpha_\delta} .$$

Proof. First, let $A = \{\alpha_\delta \mid \delta < \gamma\}$, $B = \{\beta_\delta \mid \delta < \alpha\}$, and $C = \{\beta_{\alpha_\delta} \mid \delta < \gamma\}$ so that $\alpha = \sup A$ and we must show that $\sup B = \sup C$. We show this by showing that $\sup C$ has the least upper bound property of B .

So first consider any $\beta_\delta \in B$ so that $\delta < \alpha$. It then follows that δ is not an upper bound of A so that there is a $\xi < \gamma$ such that $\delta < \alpha_\xi$. From this we have that $\beta_\delta \leq \beta_{\alpha_\xi} \leq \sup C$ since $\{\beta_\delta\}$ is a non-decreasing sequence and $\beta_{\alpha_\xi} \in C$. Since β_δ was arbitrary this shows that $\sup C$ is an upper bound of B .

Now consider any $\delta < \sup C$. Then δ is not an upper bound of C so that there is a $\xi < \gamma$ such that $\delta < \beta_{\alpha_\xi}$. And since $\xi < \gamma$ we have that $\alpha_\xi \in A$ so that $\alpha_\xi < \sup A = \alpha$ since α is a limit ordinal. Thus $\beta_{\alpha_\xi} \in B$. Since we have that $\delta < \beta_{\alpha_\xi}$ this shows that δ is not an upper bound of B . Hence since δ was arbitrary this concludes the proof that $\sup B = \sup C$ by the least upper bound property (Lemma 6.5.11.2). \square

Main Problem.

- (a) We claim that the least such ordinal here is $\xi = \omega^2$.

Proof. That this has the desired property (i.e. that $\omega + \xi = \xi$) was shown in Exercise 6.5.3b. Now we show that any ordinal $\alpha < \omega^2$ does not have this property. So consider any such α so that by Lemma 6.5.6.4 there are natural numbers n and k such that $\alpha = \omega \cdot n + k$. Thus we have

$$\omega + \alpha = \omega + \omega \cdot n + k = \omega \cdot 1 + \omega \cdot n + k = \omega \cdot (1 + n) + k = \omega \cdot (n + 1) + k$$

We then have that

$$\begin{aligned} \alpha &= \omega \cdot n + k = (\omega \cdot n + k) + 0 \\ &< (\omega \cdot n + k) + \omega && \text{(by Lemma 6.5.4a since } 0 < \omega) \\ &= \omega \cdot n + (k + \omega) && \text{(associativity of addition)} \\ &= \omega \cdot n + \omega = \omega \cdot n + \omega \cdot 1 \\ &= \omega \cdot (n + 1) && \text{(distributive law)} \\ &= \omega \cdot (n + 1) + 0 \\ &\leq \omega \cdot (n + 1) + k && \text{(by Lemma 6.5.4 since } 0 \leq k) \\ &= \omega + \alpha \end{aligned}$$

so that clearly $\omega + \alpha \neq \alpha$. Since $\alpha < \omega^2$ was arbitrary this shows our result. \square

(b) We claim that $\xi = \omega^\omega$ is the first nonzero ordinal to have this property.

Proof. First we have

$$\omega \cdot \xi = \omega \cdot \omega^\omega = \omega^1 \cdot \omega^\omega = \omega^{1+\omega} = \omega^\omega = \xi$$

where we have used Exercise 6.5.13a. Thus $\xi = \omega^2$ has the desired property.

Now consider any $0 < \alpha < \omega^\omega$. Since by Definition 6.5.9c we have that $\omega^\omega = \sup_{n < \omega} \omega^n$ it follows from the least upper bound property (Lemma 6.5.11.2) that there is an $n < \omega$ such that $\alpha < \omega^n$ since α is not an upper bound of $\{\omega^n \mid n < \omega\}$. From this it follows that the set $A = \{k \in \omega \mid \alpha < \omega^k\}$ is not empty. Since this is a set of natural numbers (which is well-ordered) it has a least element m . Note also that it has to be that $m > 0$ since were it the case that $m = 0$ then we would have $\alpha < \omega^m = \omega^0 = 1$, which implies that $\alpha = 0$, which contradicts our initial supposition that $0 < \alpha$. Thus $m \geq 1$ so that $m - 1$ is still a natural number.

Since m is the least element of A it follows that $\alpha \geq \omega^{m-1}$ since otherwise $m - 1$ would be the least element of A . Hence we have

$$\omega^{m-1} \leq \alpha < \omega^m.$$

It then follows from Exercise 6.5.8b that

$$\begin{aligned} \omega \cdot \omega^{m-1} &\leq \omega \cdot \alpha \\ \omega^1 \cdot \omega^{m-1} &\leq \omega \cdot \alpha \\ \omega^{1+m-1} &\leq \omega \cdot \alpha && \text{(by Exercise 6.5.13a)} \\ \omega^m &\leq \omega \cdot \alpha. \end{aligned}$$

Putting this together, we have that

$$\alpha < \omega^m \leq \omega \cdot \alpha$$

so that clearly $\omega \cdot \alpha \neq \alpha$. Since $1 < \alpha < \omega^\omega$ was arbitrary, this shows that ω^ω is the least such ordinal \square

(c) We claim here that $\varepsilon = {}^\omega\omega$ is the least such ordinal, where we use the notation for tetration introduced in Exercise 6.5.11e.

Proof. To show that ε has the required property we must first show that the sequence defined by $\{\omega^n\}$ for $n < \omega$ is non-decreasing even though this is somewhat obvious. We show this by induction. For $n = 0$ we have

$$\omega^n = \omega^0 = 1 < \omega = \omega^1 = \omega^{\omega^0} = \omega^1 = \omega^n$$

Now suppose that $\omega^n < \omega^{n+1}$ so that we have

$$\begin{aligned} \omega^{n+1} &= \omega^{\omega^n} \\ &< \omega^{\omega^{n+1}} && \text{(by Exercise 6.5.14b and the induction hypothesis)} \\ &= \omega^{n+2}, \end{aligned}$$

which completes the induction.

Returning to the main problem, we then have

$$\begin{aligned} \omega^\varepsilon &= \sup_{\alpha < \varepsilon} \omega^\alpha && \text{(by Definition 6.5.9c since } \varepsilon \text{ is clearly a limit ordinal)} \\ &= \sup_{n < \omega} \omega^{\omega^n} && \text{(by Lemma 6.5.15.1 since } \{\omega^n\} \text{ is non-decreasing)} \\ &= \sup_{n < \omega} \omega^{n+1} = \omega^\omega = \varepsilon \end{aligned}$$

so that ε does have the desired property.

Now we must show that it is the least such ordinal that has this property. So consider any $\alpha < \varepsilon$ then since $\varepsilon = \sup_{n < \omega} \omega^n$ it follows that there is an $n < \omega$ such that $\alpha < \omega^n$ by the least upper bound property (Lemma 6.5.11.2). It then follows that the set $A = \{k \in \omega \mid \alpha < \omega^k\}$ is not empty. Since this is a set of natural numbers it follows that it has a least element m .

Case: $m = 0$. Then $\alpha < \omega^m = \omega^0 = 1$ so that it must be that $\alpha = 0$. But then we have

$$\omega^\alpha = \omega^0 = 1 \neq 0 = \alpha$$

so that α does not have the property.

Case: $m > 0$. Then $m \geq 1$ so that $m - 1$ is still a natural number. It then follows that $\alpha \geq \omega^{m-1}$ since otherwise $m - 1$ would be the least element of A . Thus we have

$$\omega^{m-1} \leq \alpha < \omega^m.$$

It then follows from Exercise 6.5.14b that

$$\begin{aligned} \omega^{\omega^{m-1}} &\leq \omega^\alpha \\ \omega^m &\leq \omega^\alpha. \end{aligned}$$

Thus we have

$$\alpha < \omega^m \leq \omega^\alpha$$

so that clearly α does not have the property since $\omega^\alpha \neq \alpha$. Since α was arbitrary and the cases exhaustive this shows that ε is indeed the least such ordinal. \square

Exercise 6.5.16

(Characterization of Ordinal Exponentiation) Let α and β be ordinals. For $f : \beta \rightarrow \alpha$, let $s(f) = \{\xi < \beta \mid f(\xi) \neq 0\}$. Let $S(\beta, \alpha) = \{f \mid f : \beta \rightarrow \alpha \text{ and } s(f) \text{ is finite}\}$. Define $<$ on $S(\beta, \alpha)$ as follows: $f < g$ if and only if there is $\xi_0 < \beta$ such that $f(\xi_0) < g(\xi_0)$ and $f(\xi) = g(\xi)$ for all $\xi > \xi_0$. Show that $(S(\beta, \alpha), <)$ is isomorphic to $(\alpha^\beta, <)$.

Solution:

First we define summation notation for ordinals. Since ordinal addition is *not* commutative we define summation notation such that each additional term is pre-added to the previous terms, i.e. added on the left. So, for example, we have

$$\sum_{n=1}^5 \alpha_n = \alpha_5 + \alpha_4 + \alpha_3 + \alpha_2 + \alpha_1.$$

We also adopt the convention that

$$\sum_{k=n}^m \alpha_k = 0$$

any time $n > m$.

Proof. First we note that if $\beta = 0 = \emptyset$ then the only function from β to α (it could even be here that $\alpha = 0 = \emptyset$) is the vacuous function \emptyset . Hence $S(\beta, \alpha) = \{\emptyset\}$, which is clearly vacuously isomorphic to (in fact identical to) $\alpha^\beta = \alpha^0 = 1 = \{0\} = \{\emptyset\}$. Thus in the following we assume that $\beta \neq 0$, which implies that $\alpha \neq 0$ as well since functions from β to α cannot exist when $\beta \neq 0 = \emptyset$ but $\alpha = 0 = \emptyset$.

Also if $\alpha = 1 = \{0\}$ (and still $\beta \neq 0$) then there is clearly only a single function $f : \beta \rightarrow \alpha$, namely that where $f(\xi) = 0$ for all $\xi \in \beta$. Hence $S(\beta, \alpha) = \{f\}$, which is clearly trivially isomorphic to $\alpha^\beta = 1^\beta = 1 = \{0\}$. Thus in what follows we shall assume the more interesting case when $\alpha > 1$ (and $\beta > 0$).

Now we define a function $h : S(\beta, \alpha) \rightarrow \alpha^\beta$. For any $f \in S(\beta, \alpha)$ since $s(f)$ is a finite set of ordinals it follows that it is isomorphic to some natural number n . Thus its elements can be expressed as a strictly increasing sequence $\{s_k\}$ for $k < n$ where each $s_k \in \beta$ since $s(f) \subseteq \beta$. We now set

$$h(f) = \sum_{k=0}^{n-1} \alpha^{s_k} \cdot f(s_k),$$

noting that it could be the case that $n = 0$ so that $h(f) = 0$, consistent with our convention.

First we show that $h(f) \in \alpha^\beta$ so that the range of h is in fact a subset of α^β . We begin by showing by induction on m that

$$\sum_{k=0}^m \alpha^{s_k} \cdot f(s_k) < \alpha^{s_{m+1}}.$$

First for $m = 0$ we have

$$\begin{aligned} \sum_{k=0}^m \alpha^{s_k} \cdot f(s_k) &= \sum_{k=0}^0 \alpha^{s_k} \cdot f(s_k) \\ &= \alpha^{s_0} \cdot f(s_0) \\ &< \alpha^{s_0} \cdot \alpha && \text{(by Exercise 6.5.7a since } \alpha \neq 0 \text{ and } f(s_0) < \alpha) \\ &= \alpha^{s_0+1} = \alpha^{s_{m+1}} && \text{(by Definition 6.5.9b)} \end{aligned}$$

Now suppose that

$$\sum_{k=0}^m \alpha^{s_k} \cdot f(s_k) < \alpha^{s_{m+1}}.$$

It follows from Exercise 6.5.14b that

$$\alpha^{s_{m+1}} \leq \alpha^{s_{m+1}}$$

since $\{s_k\}$ is a strictly increasing sequence so that $s_m + 1 \leq s_{m+1}$. We then have

$$\begin{aligned}
\sum_{k=0}^{m+1} \alpha^{s_k} \cdot f(s_k) &= \alpha^{s_{m+1}} \cdot f(s_{m+1}) + \sum_{k=0}^m \alpha^{s_k} \cdot f(s_k) \\
&< \alpha^{s_{m+1}} \cdot f(s_{m+1}) + \alpha^{s_{m+1}} && \text{(by the induction hypothesis and Lemma 6.5.4a)} \\
&\leq \alpha^{s_{m+1}} \cdot f(s_{m+1}) + \alpha^{s_{m+1}} && \text{(follows from Lemma 6.5.4 and the above)} \\
&= \alpha^{s_{m+1}} \cdot [f(s_{m+1}) + 1] && \text{(by Definition 6.5.6b)} \\
&= \alpha^{s_{m+1}} \cdot \alpha && \text{(by Exercise 6.5.7 since } f(s_{m+1}) + 1 \leq \alpha \text{)} \\
&= \alpha^{s_{m+1}+1}, && \text{(by Definition 6.5.9b)}
\end{aligned}$$

which completes the induction. Hence we have

$$h(f) = \sum_{k=0}^{n-1} \alpha^{s_k} \cdot f(s_k) < \alpha^{s_{n-1}+1} \leq \alpha^\beta$$

by Exercise 6.5.14b since $s_{n-1} + 1 \leq \beta$ since $s_{n-1} < \beta$. Clearly then $h(f) \in \alpha^\beta$ by definition of ordinal ordering.

Now we show that h is an increasing function and therefore also injective. So consider any f and g in $S(\beta, \alpha)$ such that $f \prec g$. Then by the definition of \prec there is a $\xi_0 < \beta$ such that $f(\xi_0) < g(\xi_0)$ and $f(\xi) = g(\xi)$ for all $\xi_0 < \xi < \beta$. Now let $S = s(f) \cup s(g)$, which is clearly a finite set of ordinals since $s(f)$ and $s(g)$ are. Hence it is also isomorphic to a natural number n so that it can be expressed as a strictly increasing sequence $\{s_k\}$ for $k < n$. Moreover

$$h(f) = \sum_{k=0}^{n-1} \alpha^{s_k} \cdot f(s_k) \qquad h(g) = \sum_{k=0}^{n-1} \alpha^{s_k} \cdot g(s_k)$$

since for each term where $s_k \notin s(f)$ we have that $f(s_k) = 0$ so that the term contributes nothing to the sum by Definition 6.5.6a (and similarly for g). Also since $f(\xi_0) < g(\xi_0)$ it has to be that $0 < g(\xi_0)$ so that $\xi_0 \in s(g)$ and hence $\xi_0 \in S$. From this it follows that there is an $m < n$ such that $s_m = \xi_0$.

We then have

$$\begin{aligned}
\alpha^{s_m} \cdot f(s_m) + \sum_{k=0}^{m-1} \alpha^{s_k} \cdot f(s_k) & && \\
&< \alpha^{s_m} \cdot f(s_m) + \alpha^{s_{m-1}+1} && \text{(by Lemma 6.5.4a and the above)} \\
&\leq \alpha^{s_m} \cdot f(s_m) + \alpha^{s_m} && \text{(since } \{s_k\} \text{ is an increasing sequence)} \\
&= \alpha^{s_m} \cdot [f(s_m) + 1] && \text{(by Definition 6.5.6b)} \\
&\leq \alpha^{s_m} \cdot g(s_m) && \text{(since } f(s_m) < g(s_m) \text{)} \\
&\leq \alpha^{s_m} \cdot g(s_m) + \sum_{k=0}^{m-1} \alpha^{s_k} \cdot g(s_k). && \text{(by Lemma 6.5.4)}
\end{aligned}$$

Thus it follows yet again from Lemma 6.5.4a that

$$\begin{aligned}
h(f) &= \sum_{k=0}^{n-1} \alpha^{s_k} \cdot f(s_k) \\
&= \sum_{k=m+1}^{n-1} \alpha^{s_k} \cdot f(s_k) + \alpha^{s_m} \cdot f(s_m) + \sum_{k=0}^{m-1} \alpha^{s_k} \cdot f(s_k)
\end{aligned}$$

$$\begin{aligned}
&< \sum_{k=m+1}^{n-1} \alpha^{s_k} \cdot g(s_k) + \alpha^{s_m} \cdot g(s_m) + \sum_{k=0}^{m-1} \alpha^{s_k} \cdot g(s_k) \\
&= \sum_{k=0}^{n-1} \alpha^{s_k} \cdot g(s_k) = h(g),
\end{aligned}$$

which shows that h is indeed increasing.

Lastly we show that h is a surjective function, which then completes the proof that h is an isomorphism so that $(S(\beta, \alpha), \prec)$ is isomorphic to $(\alpha^\beta, <)$ as desired. So consider any $\gamma \in \alpha^\beta$ so that by definition $\gamma < \alpha^\beta$.

We construct a function $f : \beta \rightarrow \alpha$ recursively by first initializing $f(\xi) = 0$ for all $\xi \in \beta$. We then initialize $\rho = \gamma$ and perform the following iteratively:

If $\rho < \alpha$ then we set $f(0) = \rho$, noting that by definition $\rho \in \alpha$ and $0 \in \beta$ since $\beta > 0$. We then terminate the iteration, noting that it could be that $f(0) = \rho = 0$.

If $\rho \geq \alpha$ then $1 < \alpha \leq \rho$ so that by Lemma 6.6.2 there is a greatest ordinal σ such that $\alpha^\sigma \leq \rho$. Then clearly we have $\alpha^\sigma \leq \rho \leq \gamma < \alpha^\beta$ so that it follows from Exercise 6.514b that $\sigma < \beta$ since $\alpha > 1$ (the exercise only asserts implication in one direction but the other direction follows immediately similarly to the proof of Lemma 6.5.4a). Then since clearly $\alpha^\sigma \neq 0$ it follows from Theorem 6.6.3 that there are unique ordinals τ and ρ' such that

$$\alpha^\sigma \cdot \tau + \rho' = \rho$$

and $\rho' < \alpha^\sigma$. We claim the following about these:

1. $\tau < \alpha$. To the contrary, assume that $\tau \geq \alpha$ so that we have

$$\rho = \alpha^\sigma \cdot \tau + \rho' \geq \alpha^\sigma \cdot \tau + 0 = \alpha^\sigma \cdot \tau \geq \alpha^\sigma \cdot \alpha = \alpha^{\sigma+1}$$

by Lemma 6.5.4, Exercise 6.5.7 since $\alpha^\sigma \neq 0$, and Definition 6.5.9b. However, this contradicts the definition of σ , i.e. that it is greatest ordinal ξ such that $\rho \geq \alpha^\xi$. Hence it must be that $\tau < \alpha$.

2. $0 < \tau$. Were it the case that $\tau = 0$ then we would have

$$\rho = \alpha^\sigma \cdot \tau + \rho' = \alpha^\sigma \cdot 0 + \rho' = 0 + \rho' = \rho'.$$

However, we then would have both $\rho \geq \alpha^\sigma$ and $\rho = \rho' < \alpha^\sigma$, which is clearly a contradiction. So it must be that $\tau \neq 0$ so $\tau > 0$.

3. $\rho' < \rho$. We have

$$\rho' < \alpha^\sigma = \alpha^\sigma \cdot 1 \leq \alpha^\sigma \cdot \tau = \alpha^\sigma \cdot \tau + 0 \leq \alpha^\sigma \cdot \tau + \rho' = \rho$$

by Exercise 6.5.7 since $\alpha^\sigma \neq 0$ and Lemma 6.5.4 since $0 \leq \rho'$, noting that we also just showed that $1 \leq \tau$ since $0 < \tau$.

We therefore set $f(\sigma) = \tau$, noting that we have shown that $\sigma < \beta$ and $\tau < \alpha$ (so that $\sigma \in \beta$ and $\tau \in \alpha$). We then repeat the above, setting ρ' as the new ρ , noting that $\rho' < \rho \leq \gamma$ so that $\rho' \leq \gamma$ is still true.

Now it has to be that this construction terminates after a finite number of iterations since each ρ in the iterations forms a strictly decreasing sequence of ordinals. Hence this sequence has to be finite since otherwise the range of this sequence would be a set of ordinals with no least element. It thus follows that $f(\xi) \neq 0$ for a finite number of $\xi \in \beta$ so that $s(f)$ is finite and $f \in S(\beta, \alpha)$. The fact that $h(f) = \gamma$ follows immediately from the construction of f so that h is surjective since γ was arbitrary. \square

In conclusion we note that h maps $f \in S(\beta, \alpha)$ to its corresponding ordinal $\gamma \in \alpha^\beta$ by expanding γ in base- α with digits in the range of f . Running through the above procedures for finite α and β shows that this is the usual base expansion in base- α . However, this is much more general since α or β (or both) might be transfinite. The interesting conclusion here is that any ordinal can be expressed with a finite number of (potentially transfinite) digits with respect to any other ordinal (potentially transfinite) as a base. It would seem that Theorem 6.6.4 (the normal form) is a special case of this in which the base is ω .

§6.6 The Normal Form

Exercise 6.6.1

Show that $\omega^\varepsilon = \varepsilon$.

Solution:

This was shown in Exercise 6.5.15c.

Exercise 6.6.2

Find the first few terms of the Goodstein sequence starting at $m = 28$.

Solution:

We show the first 5 terms of the Goodstein sequences for $m = 28$:

$$\begin{aligned}
 m_0 &= m = 28 = 2^{2^2} + 2^{2+1} + 2^2 \\
 m_1 &= 3^{3^3} + 3^{3+1} + 3^3 - 1 = 3^{3^3} + 3^{3+1} + 3^2 \cdot 2 + 3 \cdot 2 + 2 \approx 7.626 \times 10^{12} \\
 m_2 &= 4^{4^4} + 4^{4+1} + 4^2 \cdot 2 + 4 \cdot 2 + 1 \approx 1.341 \times 10^{154} \\
 m_3 &= 5^{5^5} + 5^{5+1} + 5^2 \cdot 2 + 5 \cdot 2 \approx 1.911 \times 10^{2182} \\
 m_4 &= 6^{6^6} + 6^{6+1} + 6^2 \cdot 2 + 6 \cdot 2 - 1 = 6^{6^6} + 6^{6+1} + 6^2 \cdot 2 + 6 + 5 \approx 2.659 \times 10^{36305}.
 \end{aligned}$$

Chapter 7 Alephs

§7.1 Initial Ordinals

Exercise 7.1.1

If X is an infinite well-orderable set, then X has nonisomorphic well-orderings.

Solution:

Lemma 7.1.1.1. *If α is an infinite ordinal then $|\alpha| = |\alpha + 1|$, i.e. α and $\alpha + 1$ are equipotent.*

Proof. First we note that since α is infinite we have $\alpha + 1 > \alpha \geq \omega$. We then construct a bijection from $\alpha + 1$ to α . So define $f : \alpha + 1 \rightarrow \alpha$ by

$$f(\beta) = \begin{cases} \beta + 1 & \beta < \omega \\ \beta & \beta \geq \omega \text{ and } \beta \neq \alpha \\ 0 & \beta = \alpha \end{cases}$$

for $\beta \in \alpha + 1$.

First we show that f is injective. So consider any β and γ in $\alpha + 1$ where $\beta \neq \gamma$. Without loss of generality we can assume that $\beta < \gamma$. We then have the following:

Case: $\beta < \omega$. Then clearly $f(\beta) = \beta + 1 < \omega$ since $\beta < \omega$ and ω is a limit ordinal, but we also clearly have that $0 < \beta + 1 = f(\beta)$. Now, if also $\gamma < \omega$ then clearly $f(\beta) = \beta + 1 < \gamma + 1 = f(\gamma)$ since $\beta < \gamma$. If $\gamma \geq \omega$ and $\gamma \neq \alpha$ then we have $f(\beta) < \omega \leq \gamma = f(\gamma)$. Lastly if $\gamma = \alpha$ then we have $f(\gamma) = 0 < f(\beta)$.

Case: $\beta \geq \omega$ and $\beta \neq \alpha$. Here since $\beta < \gamma$ we have $\omega \leq \beta < \gamma$. Thus if also $\gamma \neq \alpha$ then clearly we have $f(\beta) = \beta < \gamma = f(\gamma)$. On the other hand if $\gamma = \alpha$ then $f(\gamma) = 0 < \omega \leq \beta = f(\beta)$.

Thus in every case we have $f(\beta) \neq f(\gamma)$, thereby showing that f is injective. We note that the case in which $\beta = \alpha$ is impossible since α is the greatest element of $\alpha + 1$ but $\gamma > \beta$ and $\gamma \in \alpha + 1$.

Next we show that f is surjective. So consider any $\beta \in \alpha$.

Case: $\beta < \omega$. If $\beta = 0$ then clearly $f(\alpha) = 0 = \beta$. On the other hand if $0 < \beta < \omega$ then β is a successor ordinal, say $\beta = \gamma + 1$, so that $\gamma < \beta < \omega$ hence clearly $\gamma \in \alpha + 1$ and $f(\gamma) = \gamma + 1 = \beta$.

Case: $\beta \geq \omega$. Then since $\beta \in \alpha$ we have $\beta < \alpha < \alpha + 1$ so that $\beta \neq \alpha$ but $\beta \in \alpha + 1$. Then clearly $f(\beta) = \beta$.

Hence in all cases there is a $\gamma \in \alpha + 1$ such that $f(\gamma) = \beta$ so that f is surjective. Therefore we have shown that f is a bijection so that by definition $\alpha + 1$ and α are equipotent. \square

Lemma 7.1.1.2. *If an infinite set A with order \prec is isomorphic to an ordinal α then it can also be re-ordered to be isomorphic to $\alpha + 1$.*

Proof. Since A is infinite and the isomorphism from A to α is a bijective function, it follows that they are equipotent so that α is also infinite. Then by Lemma 7.1.1.1 α is equipotent to $\alpha + 1$ so that A is also equipotent to $\alpha + 1$. Hence there is an $f : A \rightarrow \alpha + 1$ that is bijective. We then simply re-order A according to $\alpha + 1$, i.e. we create the following order on A :

$$R = \{(a, b) \in A \times A \mid f(a) < f(b)\}$$

so that clearly (A, R) is isomorphic to $(\alpha + 1, <)$. \square

Main Problem.

Proof. For an infinite well-orderable set X we show that X has an infinite number of non-isomorphic well-orderings. So let \prec be a well-ordering of X so that by Theorem 6.3.1 (X, \prec) is isomorphic to some ordinal α . We then show by induction that, for any natural number n , there is an ordering R_n of X such that it is isomorphic to $\alpha + n$. For $n = 0$ we have that, for $R_0 = \prec$, clearly (X, R_0) is isomorphic to $(\alpha, <)$ by what has already been established. Now suppose that there is an ordering R_n of X such that (X, R_n) is isomorphic to $(\alpha + n, <)$. Then since X is an infinite set it follows from Lemma 7.1.1.2 that there is an ordering R_{n+1} such that X is isomorphic to $(\alpha + n) + 1 = \alpha + (n + 1)$. This completes the inductive proof. We note that clearly each of these re-orderings are mutually non-isomorphic since different ordinals are not isomorphic to each other. \square

Exercise 7.1.2

If α and β are at most countable ordinals then $\alpha + \beta$, $\alpha \cdot \beta$, and α^β are at most countable. [Hint: Use the representation of ordinal operations from Theorems 5.3 and 5.8 and Exercise 5.16 in Chapter 6. Another possibility is a proof by transfinite induction.]

Solution:

First we show that $\alpha + \beta$ is at most countable.

Proof. First we define two sets:

$$W_1 = \{(0, \gamma) \mid \gamma \in \alpha\} \qquad W_2 = \{(1, \gamma) \mid \gamma \in \beta\} .$$

We also define the order $<_1$ on W_1 so that $(0, \gamma) <_1 (0, \delta)$ if and only if $\gamma < \delta$ for $(0, \gamma)$ and $(0, \delta)$ in W_1 (so that γ and δ are in α). Similarly we define the order $<_2$ on W_2 so that $(1, \gamma) <_2 (1, \delta)$ if and only if $\gamma < \delta$ for $(1, \gamma)$ and $(1, \delta)$ in W_2 (so that γ and δ are in β).

Clearly W_1 and W_2 are disjoint, $(W_1, <_1)$ is isomorphic to α , and $(W_2, <_2)$ is isomorphic to β . It then follows from Theorem 6.5.3 that the sum $(W, <)$ is isomorphic to $\alpha + \beta$.

Now, since they are isomorphic, clearly W_1 is equipotent to α and therefore is at most countable. Similarly W_2 is at most countable by virtue of being isomorphic to β . It then follows from Theorem 4.2.6 and Theorem 4.3.5 that $W = W_1 \cup W_2$ is at most countable. Then, since $(W, <)$ is isomorphic to $\alpha + \beta$, W and $\alpha + \beta$ are equipotent so that $\alpha + \beta$ must be at most countable too. \square

Next we show that $\alpha \cdot \beta$ is at most countable.

Proof. Since α and β are at most countable it follows from Exercise 4.2.2 and Theorem 4.3.7 that $\alpha \times \beta$ is at most countable. Then, since $\alpha \cdot \beta$ is isomorphic to the antilexicographic ordering of $\alpha \times \beta$ by Theorem 6.5.8, it follows that $\alpha \cdot \beta$ is equipotent to $\alpha \times \beta$ and therefore at most countable. \square

Lastly we show that α^β is at most countable.

Proof. First if we have that $\alpha = 0$ then either $\alpha^\beta = 0^\beta = 1$ (if $\beta = 0$) or $\alpha^\beta = 0^\beta = 0$ (if $\beta > 0$). Clearly both 0 and 1 are both at most countable so in the following we assume that $\alpha \neq 0$.

Now, let $\text{Seq}(\alpha \cdot \beta)$ be the set of all finite sequences of elements of $\alpha \cdot \beta$, and $S(\beta, \alpha)$ be the set as defined in Exercise 6.5.16 such that $S(\beta, \alpha)$ with the order defined there is isomorphic (and therefore equipotent) to α^β . We shall construct a function $g : S(\beta, \alpha) \rightarrow \text{Seq}(\alpha \cdot \beta)$.

So consider any $f \in S(\beta, \alpha)$ so that $f : \beta \rightarrow \alpha$ and $s(f) = \{\xi < \beta \mid f(\xi) \neq 0\}$ (as defined in the exercise) is finite. Hence there is a natural number n such that $s(f)$ can be expressed as an increasing sequence $h : n \rightarrow s(f)$. We now define another sequence $t : n \rightarrow \alpha \cdot \beta$ by

$$t(k) = \alpha \cdot h(k) + f(h(k))$$

for $k \in n$. We then set $g(f) = t$.

The first thing we show is that $g(f)$ is really a sequence whose elements are in $\alpha \cdot \beta$ for any $f \in S(\beta, \alpha)$. Hence for any such f again let h be the increasing finite sequence whose range is $s(f)$ and let $t = g(f)$. Then for any $k \in n$ we note that $h(k) < \beta$ since $h(k) \in s(f)$ so that $h(k) + 1 \leq \beta$. We also note that $f(h(k)) \in \alpha$ since $f : \beta \rightarrow \alpha$ so that $f(h(k)) < \alpha$. Thus we have by definition that

$$\begin{aligned} t(k) &= \alpha \cdot h(k) + f(h(k)) \\ &< \alpha \cdot h(k) + \alpha && \text{(by Lemma 6.5.4)} \\ &= \alpha \cdot (h(k) + 1) && \text{(by Definition 6.5.6b)} \end{aligned}$$

$$\leq \alpha \cdot \beta. \quad (\text{by Exercise 6.5.7 since } \alpha \neq 0)$$

Hence $t(k) < \alpha \cdot \beta$ so that $t(k) \in \alpha \cdot \beta$. Thus $t : n \rightarrow \alpha \cdot \beta$ and since clearly this sequence is finite it follows that $t \in \text{Seq}(\alpha \cdot \beta)$.

Now we show that g is injective. So consider f_1 and f_2 in $S(\beta, \alpha)$ such that $f_1 \neq f_2$. Let h_1, n_1 and h_2, n_2 be the increasing sequences and natural numbers for f_1 and f_2 , respectively, as defined above. Also let $t_1 = g(f_1)$ and $t_2 = g(f_2)$.

Case: $n_1 \neq n_2$. In this case the sequences are different sizes so that clearly $t_1 \neq t_2$ since t_1 and t_2 are sequences of sizes n_1 and n_2 , respectively.

Case: $n_1 = n_2$. Since $f_1 \neq f_2$ there must be a $\gamma \in \beta$ such that $f_1(\gamma) \neq f_2(\gamma)$. Without loss of generality we can assume that $f_1(\gamma) < f_2(\gamma)$.

If $f_1(\gamma) = 0$ then by definition $\gamma \notin s(f_1)$ but $\gamma \in s(f_2)$. Hence there is a $k \in n_2$ such that $h_2(k) = \gamma$. Also since $n_1 = n_2$ clearly $k \in n_1$. However, it must be that $h_1(k) \neq \gamma = h_2(k)$ since otherwise it would be that $\gamma \in s(f_1)$. Suppose that $h_1(k) < h_2(k)$ so that $h_1(k) + 1 \leq h_2(k)$ and we have

$$\begin{aligned} t_1(k) &= \alpha \cdot h_1(k) + f_1(h_1(k)) \\ &< \alpha \cdot h_1(k) + \alpha && (\text{by Lemma 6.5.4}) \\ &= \alpha \cdot (h_1(k) + 1) && (\text{by Definition 6.5.6b}) \\ &\leq \alpha \cdot h_2(k) && (\text{by Exercise 6.5.7 since } \alpha \neq 0) \\ &\leq \alpha \cdot h_2(k) + f_2(h_2(k)) \\ &= t_2(k) \end{aligned}$$

so that $t_1(k) \neq t_2(k)$ and therefore $t_1 \neq t_2$. The case in which $h_1(k) > h_2(k)$ is analogous.

On the other hand if $f_1(\gamma) \neq 0$ then $0 < f_1(\gamma) < f_2(\gamma)$ so that $\gamma \in s(f_1)$ and $\gamma \in s(f_2)$. Thus there are k_1 and k_2 in $n_1 = n_2$ such that $h_1(k_1) = h_2(k_2) = \gamma$. If $k_1 = k_2$ then we have

$$\begin{aligned} t_1(k_1) &= \alpha \cdot h_1(k_1) + f_1(h_1(k_1)) \\ &= \alpha \cdot \gamma + f_1(\gamma) \\ &\neq \alpha \cdot \gamma + f_2(\gamma) && (\text{by Lemma 6.5.4b since } f_1(\gamma) \neq f_2(\gamma)) \\ &= \alpha \cdot h_2(k_2) + f_2(h_2(k_2)) \\ &= t_2(k_2) = t_2(k_1) \end{aligned}$$

so that $t_1 \neq t_2$. If $k_1 < k_2$ then since h_1 is increasing we have $h_2(k_2) = h_1(k_1) < h_1(k_2)$ so that $h_2(k_2) + 1 \leq h_1(k_2)$ and so

$$\begin{aligned} t_2(k_2) &= \alpha \cdot h_2(k_2) + f_2(h_2(k_2)) \\ &< \alpha \cdot h_2(k_2) + \alpha && (\text{by Lemma 6.5.4}) \\ &= \alpha \cdot (h_2(k_2) + 1) && (\text{by Definition 6.5.6b}) \\ &\leq \alpha \cdot h_1(k_2) && (\text{by Exercise 6.5.7 since } \alpha \neq 0) \\ &\leq \alpha \cdot h_1(k_2) + f_1(h_1(k_2)) \\ &= t_1(k_2) \end{aligned}$$

and $t_1 \neq t_2$. The final sub-case in which $k_1 > k_2$ is analogous.

Hence in all cases and sub-cases $g(f_1) = t_1 \neq t_2 = g(f_2)$, which shows that g is injective. From this it follows from Definition 4.1.4 that $|S(\beta, \alpha)| \leq |\text{Seq}(\alpha \cdot \beta)|$. However, from what was shown above it follows that $\alpha \cdot \beta$ is at most countable since α and β are. Thus $\text{Seq}(\alpha \cdot \beta)$ is also at most countable by Exercise 4.2.4 and Theorem 4.3.10 so that $S(\beta, \alpha)$ must be at most countable since it was just shown that $|S(\beta, \alpha)| \leq |\text{Seq}(\alpha \cdot \beta)|$. Lastly, since $S(\beta, \alpha)$ is equipotent to α^β it follows that α^β is at most countable as well. \square

Exercise 7.1.3

For any set A , there is a mapping of $\mathcal{P}(A \times A)$ onto $h(A)$. [Hint: Define $f(R) =$ the ordinal isomorphic to R , if $R \subseteq A \times A$ is a well-ordering of its field; $f(R) = 0$ otherwise.]

Solution:

Lemma 7.1.3.1. *For a set A and ordinal α , if α is equipotent to a subset of A then $\alpha < h(A)$.*

Proof. Suppose that α is equipotent to $X \subseteq A$ but that $\alpha \geq h(A)$. Clearly if $\alpha = h(A)$ then $h(A)$ is equipotent to $X \subseteq A$ (since α is), which contradicts the definition of the Hartogs number. On the other hand if $\alpha > h(A)$ then let f be a bijection from α to X . Then, since $h(A) < \alpha$ we have that $h(A) \in \alpha$ and $h(A) \subseteq \alpha$ since ordinals are transitive. It then follows that $f \upharpoonright h(A)$ is a bijection from $h(A)$ to $f[h(A)] \subseteq X \subseteq A$. Hence again $h(A)$ is equipotent to a subset of A , contradicting the definition of the Hartogs number. So it has to be that $\alpha < h(A)$ as desired. \square

Main Problem.

Proof. We define a function $f : \mathcal{P}(A \times A) \rightarrow h(A)$. So for any $R \in \mathcal{P}(A \times A)$ clearly $R \subseteq A \times A$ so that R is a relation on A . If R is a well-ordering of some $X \subseteq A$ then by Theorem 6.3.1 there is a unique ordinal α such that (X, R) is isomorphic to α . We then set

$$f(R) = \begin{cases} \alpha & R \text{ is a well-ordering of some } X \subseteq A \\ 0 & R \text{ is not a well-ordering of any } X \subseteq A \end{cases}.$$

First we show that $f(R)$ really is in $h(A)$ for any $R \in \mathcal{P}(A \times A)$. So for any such R , if R is not a well-ordering of some $X \subseteq A$ then clearly $f(R) = 0$. Then, since clearly $\emptyset \subseteq A$ and \emptyset is equipotent to 0, it follows that $0 < h(A)$ by Lemma 7.1.3.1. Hence $f(R) = 0 \in h(A)$. On the other hand if R is a well-ordering of some $X \subseteq A$ then let α be the ordinal isomorphic to (X, R) so that $f(R) = \alpha$. Since this means that α is equipotent to $X \subseteq A$, it again follows from Lemma 7.1.3.1 that $\alpha < h(A)$ so that $f(R) = \alpha \in h(A)$.

To show that f is surjective consider any $\alpha \in h(A)$ so that $\alpha < h(A)$. Since by definition $h(A)$ is the *least* ordinal that is not equipotent to a subset of A it follows that α has to be equipotent to an $X \subseteq A$. Then let R be the well-ordering of X such that (X, R) is isomorphic to α . Clearly R is a relation on X and therefore also a relation on A since $X \subseteq A$. Thus $R \subseteq A \times A$ so that $R \in \mathcal{P}(A \times A)$. Clearly also $f(R) = \alpha$ and since α was arbitrary this shows that f is surjective. \square

Note that this does not mean that $h(A) \leq |\mathcal{P}(A \times A)|$ unless the Axiom of Choice is employed.

Exercise 7.1.4

$|A| < |A| + h(A)$ for all A .

Solution:

Lemma 7.1.4.1. $|h(A)| \not\leq |A|$ for any set A .

Proof. Suppose to the contrary that $|h(A)| \leq |A|$ so that there is an injective f from $h(A)$ to A . Then let $X = f[h(A)]$ so that clearly $X \subseteq A$. But then f considered as function from $h(A)$ to X is a bijection so that $h(A)$ is equipotent to a subset of A , which contradicts the definition of the Hartogs number. Hence it must be that $|h(A)| \not\leq |A|$ as desired. \square

Note that this does not imply that $h(a) > |A|$ without using the Axiom of Choice.

Main Problem.

Proof. Since we are only interested in the cardinalities of A and $h(A)$ and their cardinal sum we can assume that they are disjoint. Clearly $f : A \rightarrow A \cup h(A)$ defined by $f(x) = x$ for $x \in A$ is an injective function so that $|A| \leq |A| + h(A)$ by the definition of cardinal addition. So suppose that $|A| = |A| + h(A)$. Then there is a bijective function f from $A \cup h(A)$ into A so that $f \upharpoonright h(A)$ is an injective function from $h(A)$ to A . By definition this means that $h(A) \leq |A|$, but this contradicts Lemma 7.1.4.1. So it must be that $|A| < |A| + h(A)$ as desired. \square

Exercise 7.1.5

$|h(A)| < |\mathcal{P}(\mathcal{P}(A \times A))|$ for all A . [Hint: Prove that $|\mathcal{P}(h(A))| \leq |\mathcal{P}(\mathcal{P}(A \times A))|$ by assigning to each $X \in \mathcal{P}(h(A))$ the set of all well orderings $R \subseteq A \times A$ for which the ordinal isomorphic to R belongs to X .]

Solution:

Proof. First we show that $|\mathcal{P}(h(A))| \leq |\mathcal{P}(\mathcal{P}(A \times A))|$ by constructing an injective $f : \mathcal{P}(h(A)) \rightarrow \mathcal{P}(\mathcal{P}(A \times A))$. So consider any $X \in \mathcal{P}(h(A))$ so that $X \subseteq h(A)$. Then let Y be the set of well-orderings $R \subseteq A \times A$ (so that $R \in \mathcal{P}(A \times A)$) of subsets $B \subseteq A$ such that (B, R) is isomorphic to some $\alpha \in X$. We then set $f(X) = Y$, noting that clearly $f(X) = Y \in \mathcal{P}(\mathcal{P}(A \times A))$ since for any $R \in Y$ we have that $R \in \mathcal{P}(A \times A)$ so that $Y \subseteq \mathcal{P}(A \times A)$ hence $Y \in \mathcal{P}(\mathcal{P}(A \times A))$. Note also that $Y \neq \emptyset$ because every $\alpha \in h(A)$ is equipotent to some subset $B \subseteq A$ (by the definition of the Hartogs number) so that the well-ordering of B according to α will be in Y .

We claim that f is injective. So consider X_1 and X_2 in $\mathcal{P}(h(A))$ (so that $X_1 \subseteq h(A)$ and $X_2 \subseteq h(A)$) such that $f(X_1) = f(X_2)$. Then consider any $\alpha \in X_1$. Then since $f(X_1) \neq \emptyset$ there is a well-ordering $R \in f(X_1)$ of a subset of A that is isomorphic to α . Then since $f(X_1) = f(X_2)$ we have $R \in f(X_2)$ as well. It follows from this that $\alpha \in X_2$. Thus $X_1 \subseteq X_2$ since α was arbitrary. A similar argument shows that $X_2 \subseteq X_1$ so that we conclude that $X_1 = X_2$. This shows that f is injective.

Hence we have shown that $|\mathcal{P}(h(A))| \leq |\mathcal{P}(\mathcal{P}(A \times A))|$. We also have by Cantor's Theorem (Theorem 5.1.8 in the text) that $|h(A)| < |\mathcal{P}(h(A))|$. It therefore follows from Exercise 4.1.2a that $|h(A)| < |\mathcal{P}(\mathcal{P}(A \times A))|$ as desired. \square

Exercise 7.1.6

Let $h^*(A)$ be the least ordinal α such that there exists no function with domain A and range α . Prove:

- (a) If $\alpha \geq h^*(A)$, then there is no function with domain A and range α .
- (b) $h^*(A)$ is an initial ordinal.
- (c) $h(A) \leq h^*(A)$.
- (d) If A is well-orderable, then $h(A) = h^*(A)$.
- (e) $h^*(A)$ exists for all A .

[Hint for part (e): Show that $\alpha \in h^*(A)$ if and only if $\alpha = 0$ or α is the ordinal isomorphic to R , where R is a well-ordering of some partition of A into equivalence classes.]

Solution:

Lemma 7.1.6.1. *If A is a well-orderable set and B is any other set, there is a function from A onto B if and only if $|B| \leq |A|$.*

Proof. Suppose that R is a well-ordering of A .

(\rightarrow) Suppose that f is a function from A onto B . If A is empty then clearly B must be as well or else f could not be onto. Thus we have $|B| = |\emptyset| = 0 \leq 0 = |\emptyset| = |A|$. So we can assume that A is nonempty so that if B is empty then $|B| = |\emptyset| = 0 < |A|$. Hence we can assume that B is nonempty as well.

Then, for each $b \in B$, define the set $A_b = \{a \in A \mid f(a) = b\}$, which clearly not empty since f is onto. Then, since R is a well-ordering of A and $A_b \subseteq A$, there is a unique least element of a_b of A_b according to R . We then define $g : B \rightarrow A$ by simply setting $g(b) = a_b$ for any $b \in B$.

We then claim that g is injective. So consider b_1 and b_2 in B where $b_1 \neq b_2$. Then since $a_{b_1} \in A_{b_1}$ clearly $f(a_{b_1}) = b_1$. Similarly $f(a_{b_2}) = b_2$ so that clearly $a_{b_1} \neq a_{b_2}$ since f is a function and $b_1 \neq b_2$. Hence $g(b_1) = a_{b_1} \neq a_{b_2} = g(b_2)$, which shows that g is injective since b_1 and b_2 were arbitrary. Then by definition $|B| \leq |A|$ as desired.

(\leftarrow) Now suppose that $|B| \leq |A|$ so that there is an injective $f : B \rightarrow A$. If B is empty then clearly it has to be that $|B| = |\emptyset| = 0 \leq |A|$ regardless of A . So we can assume that B is nonempty so that there is a $b \in B$. Since f is injective, the inverse f^{-1} is a function from $\text{ran}(f)$ onto B . Now we construct a function $g : A \rightarrow B$ by setting

$$g(x) = \begin{cases} f^{-1}(x) & x \in \text{ran}(f) \\ b & x \notin \text{ran}(f) \end{cases}$$

for any $x \in A$. It should be clear that g maps A onto B since, for any $b \in B$, we have $f(b) \in \text{ran}(f)$ (hence also $f(b) \in A$) so that $g(f(b)) = f^{-1}(f(b)) = b$. \square

Main Problem.

First we note that presumably $h^*(A)$ for a set A is the least *nonzero* ordinal α such that there is no function from A onto α . The fact that $h^*(A)$ is nonzero is important since, for a non-empty set A , there is no function from A onto $\emptyset = 0$ so that 0 is actually the least ordinal such that such a function does not exist! We also make note of the fact that, even for $A = \emptyset$, the empty function $f = \emptyset$ is a vacuously a function from A onto $\emptyset = 0$ so that $h^*(A) = 1$ anyway since there is no function from $A = \emptyset$ onto $1 = \{0\}$.

(a)

Proof. Suppose to the contrary that there is a function f from A onto α . Then since $0 < h^*(A) \leq \alpha$ clearly $0 \in h^*(A)$ and $h^*(A) \subseteq \alpha$. So we define a function $g : A \rightarrow h^*(A)$ by

$$g(a) = \begin{cases} f(a) & f(a) \in h^*(A) \\ 0 & f(a) \notin h^*(A) \end{cases}.$$

for any $a \in A$. Clearly each $g(a) \in h^*(A)$ but we also claim that g is onto. To this end consider any $\beta \in h^*(A)$. Since $h^*(A) \subseteq \alpha$ we have that $\beta \in \alpha$ also. Then, since f is onto α there is an $a \in A$ such that $f(a) = \beta$. Since $f(a) = \beta \in h^*(A)$ it follows by definition that $g(a) = f(a) = \beta$. Since β was arbitrary this shows that g is onto. However, the existence of g contradicts the definition of $h^*(A)$ so that it must be that there is no such function from A onto α . \square

(b)

Proof. Suppose to the contrary that $h^*(A)$ is *not* an initial ordinal so that there is an $\alpha < h^*(A)$ such that $|\alpha| = |h^*(A)|$. Let f then be a bijection from α onto $h^*(A)$. Also since $\alpha < h^*(A)$ it follows from the definition of $h^*(A)$ that there is a function g from A onto α (since otherwise $h^*(A)$ would not be the *least* such ordinal for which such a function does not exist). But then $f \circ g$ is a function from A onto $h^*(A)$, which contradicts the definition of $h^*(A)$. Hence it has to be that $h^*(A)$ is in fact an initial ordinal. \square

(c)

Proof. Suppose to the contrary that $h(A) > h^*(A)$. Then by the definition of $h(A)$ there is a subset $X \subseteq A$ such that $h^*(A)$ is equipotent to X . So let f be a bijection from $h^*(A)$ to X . We then define a function $g : A \rightarrow h^*(A)$ by

$$g(a) = \begin{cases} f^{-1}(a) & a \in X \\ 0 & a \notin X \end{cases}$$

for any $a \in A$, noting that $0 < h^*(A)$ so that $0 \in h^*(A)$. Clearly g is into $h^*(A)$ but we also claim that it is onto. So consider any $\alpha \in h^*(A)$ and let $a = f(\alpha)$ so that $a \in X$ and therefore $a \in A$ since $X \subseteq A$. We then have $g(a) = f^{-1}(a) = f^{-1}(f(\alpha)) = \alpha$ since $a \in X$. Since α was arbitrary this shows that g is onto. However, the existence of g contradicts the definition of $h^*(A)$ so that it must be that in fact $h(A) \leq h^*(A)$ as desired. \square

(d)

Proof. We show that $h^*(A) \leq h(A)$, from which the result clearly follows since also $h^*(A) \geq h(A)$ by part (c). So suppose to the contrary that $h^*(A) > h(A)$. Then by the definition of $h^*(A)$ it follows that there is a function from A onto $h(A)$. It then follows from Lemma 7.1.6.1 that $|h(A)| \leq |A|$ since A is well-orderable. However, this contradicts Lemma 7.1.4.1 so that it must be that $h^*(A) \leq h(A)$ so that the result follows. \square

(e)

Proof. Consider any set A and let S denote the set of well-orderings of some partition of A into equivalence classes, noting that it could be that $S = \emptyset$. Since each $R \in S$ is isomorphic to a unique ordinal, let H be the set of ordinals that are isomorphic to some $R \in S$, which exists by the Axiom Schema of Replacement. Then let $\alpha = \{0\} \cup H$ and we claim that $\alpha = h^*(A)$.

First we show that α is indeed an ordinal number. Since α is a set of ordinals clearly it is well-ordered by Theorem 6.2.6d. We also must show that α is transitive, so consider any $\beta \in \alpha$. Then either $\beta = 0$ or $\beta \in H$. If $\beta = 0 = \emptyset$ then clearly $\beta \subseteq \alpha$. On the other hand if $\beta \in H$ then there is a partition P of A and a well-ordering R of P such that (P, R) is isomorphic to $(\beta, <)$. Now consider any $\gamma \in \beta$ so that $\gamma < \beta$. It then follows that γ is isomorphic to an initial segment of β and therefore also to an initial segment P' of P ordered by R . Let L be the least element of P (which is also the least element of P'), which exists since R is a well-ordering. Then let

$$L' = L \cup \left(A - \bigcup P' \right),$$

i.e. L' is the set containing the elements of L and any elements of A that are not covered in the initial segment P' . Then let

$$P'' = \{L'\} \cup (P' - \{L\}),$$

i.e. P'' is P' but with L replaced with L' . It is easy to show that P'' is a partition of A and that it is isomorphic to γ with the same ordering as R except with L replaced by L' . Hence by definition we have that $\gamma \in \alpha$. Since $\gamma \in \beta$ was arbitrary this shows that $\beta \subseteq \alpha$, and since $\beta \in \alpha$ was arbitrary this shows that α is transitive and hence an ordinal number.

Now we show that there is no function from A onto α . So suppose to the contrary that there is such a function f . We then define the set

$$E = \{(a, b) \in A \times A \mid f(a) = f(b)\} .$$

It is trivial to show that this is an equivalence relation on A so that A/E is a partition of A by Theorem 4.4.7. Moreover let g be the mapping from A/E to α defined as follows: for any $B \in A/E$ let $g(B)$ be the least element of $\{f(x) \mid x \in B\}$, noting that $\{f(x) \mid x \in B\}$ contains only a single element since B is an equivalence class where $f(x) = f(y)$ for any x and y in B . It is trivial to show that g is a bijective function so that we can well-order A/E according to the ordinal α since α is the range of g . However, it then follows by definition that $\alpha \in \alpha$, which contradicts Lemma 6.2.7. Hence it must be that there is no function from A onto α .

Lastly we show that there is a function from A onto β for every nonzero $\beta < \alpha$. So consider any such β so that $\beta \in \alpha$. Then either $\beta = 0$ or $\beta \in H$, but since β is nonzero it must be that $\beta \in H$. Then by definition there is a partition P of A and well-ordering R of P such that (P, R) is isomorphic to $(\beta, <)$. Let f then be the isomorphism from P to β . We then define the mapping $g : A \rightarrow \beta$ as follows: for any $a \in A$ there is a unique $B \in P$ such that $a \in B$ since P is a partition of A . We then set $g(a) = f(B)$. It is easy to show that g is onto.

It follows from what has been shown that indeed $\alpha = h^*(A)$. □

§7.2 Addition and Multiplication of Alephs

Exercise 7.2.1

Give a direct proof of $\aleph_\alpha + \aleph_\alpha = \aleph_\alpha$ by expressing ω_α as a disjoint union of two sets of cardinality \aleph_α .

Solution:

Lemma 7.2.1.1. *If α and β are ordinals and $\alpha > \beta$ then there is a $\gamma < \alpha$ such that $|\beta| \leq |\gamma|$.*

Proof. Clearly for $\gamma = \beta$ we have $\gamma = \beta < \alpha$ and $|\beta| = |\gamma|$ so that $|\beta| \leq |\gamma|$ is true. □

Lemma 7.2.1.2. *If a set $(A, <)$ is isomorphic to ordinal α and $\alpha > \beta$ for another ordinal β then there is an $a \in A$ such that $|\beta| \leq |X|$ for the set*

$$X = \{x \in A \mid x < a\} .$$

Proof. First let f be the isomorphism from α to A . Clearly by Lemma 7.2.1.1 there is an ordinal $\gamma < \alpha$ such that $|\beta| \leq |\gamma|$. Now let $a = f(\gamma)$ so that clearly $a \in A$, and let

$$X = \{x \in A \mid x < a\} .$$

Now, we claim that $X = f[\gamma]$. So consider any $x \in X$ so that $x < a$. It then follows that $f^{-1}(x) < f^{-1}(a) = \gamma$ since f^{-1} is an isomorphism since f is. Hence $f^{-1}(x) \in \gamma$ so that clearly $x = f(f^{-1}(x)) \in f[\gamma]$. Since x was arbitrary it follows that $X \subseteq f[\gamma]$. Now consider any $x \in f[\gamma]$ so that there is a $\delta \in \gamma$ such that $x = f(\delta)$. Then $\delta < \gamma$ so that $x = f(\delta) < f(\gamma) = a$ since f is an isomorphism so that by definition $x \in X$. Hence $f[\gamma] \subseteq X$. This shows that $X = f[\gamma]$. Then, since f is bijective, it follows that $|\beta| \leq |\gamma| = |f[\gamma]| = |X|$. □

Lemma 7.2.1.3. *For initial ordinals α and β , if $|\alpha| \leq |\beta|$, then $\alpha \leq \beta$.*

Proof. Suppose that $|\alpha| \leq |\beta|$ but that $\alpha > \beta$. Then clearly β is isomorphic (and therefore equipotent) to an initial segment of α so that $|\beta| \leq |\alpha|$. Then by the Cantor-Bernstein Theorem we have $|\alpha| = |\beta|$. However since α is an initial ordinal and $\beta < \alpha$ it cannot be that $|\alpha| = |\beta|$. Thus we have a contradiction so that it must be that $\alpha \leq \beta$ as desired. \square

Lemma 7.2.1.4. *For any ordinal α and any infinite initial ordinal Ω where $\Omega < \omega_\alpha$, there is a $\gamma < \alpha$ such that $\Omega = \omega_\gamma$.*

Proof. We show this by induction on α . For $\alpha = 0$ we have $\omega_\alpha = \omega_0 = \omega$ so that there is no infinite initial ordinal Ω such that $\Omega < \omega_\alpha = \omega$. Hence the hypothesis is vacuously true. Now suppose that, for every infinite initial ordinal $\Omega < \omega_\alpha$, there is a $\gamma < \alpha$ such that $\Omega = \omega_\gamma$. Consider any infinite initial ordinal $\Omega < \omega_{\alpha+1}$. Then $\Omega < \omega_{\alpha+1} = h(\omega_\alpha)$ so that Ω is equipotent to some subset of ω_α by the definition of the Hartogs number. From this it clearly follows that $|\Omega| \leq |\omega_\alpha|$ and hence $\Omega \leq \omega_\alpha$ by Lemma 7.2.1.3 since both Ω and ω_α are initial ordinals. If $\Omega = \omega_\alpha$ then we are finished but if $\Omega < \omega_\alpha$ then by the induction hypothesis there is a $\gamma < \alpha$ such that $\Omega = \omega_\gamma$ so that we are also finished.

Now suppose that α is a nonzero limit ordinal and that for every $\beta < \alpha$ and infinite initial ordinal $\Omega < \omega_\beta$ there is a $\gamma < \beta$ such that $\Omega = \omega_\gamma$. Consider then any infinite initial ordinal $\Omega < \omega_\alpha$. Then since $\omega_\alpha = \sup\{\omega_\beta \mid \beta < \alpha\}$ it follows that Ω is not an upper bound of $\{\omega_\beta \mid \beta < \alpha\}$ so that there is a $\beta < \alpha$ such that $\Omega < \omega_\beta$. But then by the induction hypothesis there is a $\gamma < \beta$ such that $\Omega = \omega_\gamma$. This completes the transfinite induction. \square

Lemma 7.2.1.5. *For ordinal $\alpha > 0$ and an ordinal $\beta < \omega_\alpha$ there is an ordinal $\gamma < \alpha$ such that $|\beta| \leq \aleph_\gamma$.*

Proof. First, if β is finite then clearly $\beta < \omega$ so that $\beta \in \omega$. Then $\beta \subseteq \omega$ since ω is transitive (since it is an ordinal number). Hence $|\beta| \leq |\omega| = \aleph_0$ (i.e. $\gamma = 0$ so that $\gamma < \alpha$). On the other hand if β is infinite then by Theorem 7.1.3 β is equipotent to some initial ordinal Ω . Clearly Ω is infinite since β is and clearly $\Omega < \omega_\alpha$ since $\beta < \omega_\alpha$ and ω_α is an initial ordinal. It then follows from Lemma 7.2.1.4 that there is a $\gamma < \alpha$ such that $\Omega = \omega_\gamma$. Then we have $|\beta| = |\Omega| = |\omega_\gamma| = \aleph_\gamma$ so that $|\beta| \leq \aleph_\gamma$ is true. \square

Lemma 7.2.1.6. *Every infinite initial ordinal is a limit ordinal.*

Proof. Suppose that α is an infinite initial ordinal and that it is a successor so that $\alpha = \beta + 1$. It was shown in Lemma 7.1.1.1 that $|\beta| = |\beta + 1| = |\alpha|$, but since clearly $\beta < \alpha$ this contradicts the fact that α is an initial ordinal. Hence α must be a limit ordinal. \square

Lemma 7.2.1.7. *For well ordered sets A and B either $|A| \leq |B|$ or $|B| \leq |A|$ (or both in which case $|A| = |B|$).*

Proof. By Theorem 6.1.3 we have:

Case: A and B are isomorphic. Let f be the isomorphism from A to B . Then clearly f is a bijection so that $|A| = |B|$. Also since f is injective $|A| \leq |B|$. Clearly also f^{-1} is bijective from B to A so that $|B| \leq |A|$ as well.

Case: A is isomorphic to an initial segment of B . Then if f is the isomorphism clearly f is an injective function from A to B so that $|A| \leq |B|$.

Case: B is isomorphic to an initial segment of A . Then if f is the isomorphism clearly f is an injective function from B to A so that $|B| \leq |A|$.

Since these cases are exhaustive by Theorem 6.1.3 clearly the result has been shown.

Note that this did not require the Axiom of Choice. \square

Corollary 7.2.1.8. *If A and B are well ordered sets then $|A| \not\leq |B|$ if and only if $|B| < |A|$.*

Proof. (\rightarrow) Suppose that $|A| \not\leq |B|$. Then it follows from Lemma 7.2.1.7 above that $|B| \leq |A|$. Suppose that $|B| = |A|$. Then there is a bijection f from B to A . But then clearly f^{-1} is also a bijection and therefore injective. Hence by definition $|A| \leq |B|$, a contradiction. So it cannot be that $|B| = |A|$. Hence $|B| < |A|$ by definition as desired.

(\leftarrow) We show this by proving the contrapositive. So suppose that $|A| \leq |B|$. Also suppose that $|B| \leq |A|$ so that by Lemma 7.2.1.7 above $|A| = |B|$. Thus we have shown that

$$\begin{aligned} |B| \leq |A| &\rightarrow |A| = |B| \\ |B| \not\leq |A| &\vee |A| = |B| \\ \neg(|B| \leq |A| \wedge |A| \neq |B|) & \\ &\neg(|B| < |A|), \end{aligned}$$

thereby showing the contrapositive. □

Main Problem.

The following proof is similar to the proof of Theorem 7.2.1.

Proof. Suppose that A_1 and A_2 are disjoint sets that are both equipotent to ω_α for some ordinal α . Then there are bijections f_1 and f_2 from A_1 and A_2 , respectively, to ω_α . We define a well-ordering \prec of $A = A_1 \cup A_2$ as follows: for a and b in A we let $a \prec b$ if and only if

- a and b are in A_1 and $f_1(a) < f_1(b)$, or
- a and b are in A_2 and $f_2(a) < f_2(b)$, or
- $a \in A_1$ and $b \in A_2$ and $f_1(a) \leq f_2(b)$, or
- $a \in A_2$ and $b \in A_1$ and $f_2(a) < f_1(b)$.

First we show that \prec is transitive. So consider a , b , and c in A such that $a \prec b$ and $b \prec c$.

Case: $a \in A_1$

Case: $b \in A_1$

Case: $c \in A_1$. Then $f_1(a) < f_1(b) < f_1(c)$ so that $f_1(a) < f_1(c)$ and hence $a \prec c$.

Case: $c \in A_2$. Then $f_1(a) < f_1(b) \leq f_2(c)$ so that $f_1(a) \leq f_2(c)$ is true and hence $a \prec c$.

Case: $b \in A_2$

Case: $c \in A_1$. Then $f_1(a) \leq f_2(b) < f_1(c)$ so that $f_1(a) < f_1(c)$ and hence $a \prec c$.

Case: $c \in A_2$. Then $f_1(a) \leq f_2(b) < f_2(c)$ so that $f_1(a) \leq f_2(c)$ is true and hence $a \prec c$.

Case: $a \in A_2$

Case: $b \in A_1$

Case: $c \in A_1$. Then $f_2(a) < f_1(b) < f_1(c)$ so that $f_2(a) < f_1(c)$ and hence $a \prec c$.

Case: $c \in A_2$. Then $f_2(a) < f_1(b) \leq f_2(c)$ so that $f_2(a) < f_2(c)$ and hence $a \prec c$.

Case: $b \in A_2$

Case: $c \in A_1$. Then $f_2(a) < f_2(b) < f_1(c)$ so that $f_2(a) < f_1(c)$ and hence $a \prec c$.

Case: $c \in A_2$. Then $f_2(a) < f_2(b) < f_2(c)$ so that $f_2(a) < f_2(c)$ and hence $a \prec c$.

Since all cases imply that $a \prec c$ and a, b , and c were arbitrary this shows that \prec is transitive.

Now we show that for any a and b in A , that either $a \prec b$, $a = b$, or $b \prec a$ and that only one of these is true. So consider any a and b in A . Then we have

Case: $a \in A_1$

Case: $b \in A_1$. Then clearly exactly one of the following is true: $a \prec b$ if $f_1(a) < f_1(b)$, $a = b$ if $f_1(a) = f_1(b)$ since f_1 is a bijection, and $b \prec a$ if $f_1(b) < f_1(a)$.

Case: $b \in A_2$. Clearly $a = b$ is not possible since A_1 and A_2 are disjoint. Then if $f_1(a) \leq f_2(b)$ then $a \prec b$ and if $f_1(a) > f_2(b)$ then $b \prec a$, noting that these are mutually exclusive.

Case: $a \in A_2$

Case: $b \in A_1$. Then again clearly $a = b$ is not possible since A_1 and A_2 are disjoint. Then if $f_2(a) < f_1(b)$ then $a \prec b$ and if $f_2(a) \geq f_1(b)$ then $b \prec a$, noting that these are mutually exclusive.

Case: $b \in A_2$. Then clearly exactly one of the following is true: $a \prec b$ if $f_2(a) < f_2(b)$, $a = b$ if $f_2(a) = f_2(b)$ since f_2 is a bijection, and $b \prec a$ if $f_2(b) < f_2(a)$.

Thus we have shown that \prec is a total (strict) order on A .

Now we show that \prec is also a well-ordering. So let X be a nonempty subset of A . Let $B = f_1[X \cap A_1] \cup f_2[X \cap A_2]$, noting that this is a nonempty set of ordinals. Then let B has a least element α .

Case: $\alpha \in f_1[X \cap A_1]$. Then we claim that $x = f_1^{-1}(\alpha)$ is the \prec -least element of X , noting that $x \in A_1$. Note also that clearly then $f_1(x) = f_1(f_1^{-1}(\alpha)) = \alpha$. So consider any $y \in X$.

Case: $y \in A_1$. Then $y \in X \cap A_1$ so that $f_1(y) \in f_1[X \cap A_1]$ so $f_1(y) \in B$. Thus $f_1(x) = \alpha \leq f_1(y)$ since α is the least element of B . Clearly if $f_1(x) = f_1(y)$ then $x = y$ since f_1 is bijective. On the other hand if $f_1(x) < f_1(y)$ then by definition $x \prec y$. Hence in either case we have $x \preceq y$.

Case: $y \in A_2$. Then $y \in X \cap A_2$ so that $f_2(y) \in f_2[X \cap A_2]$ so that $f_2(y) \in B$. Thus $f_1(x) = \alpha \leq f_2(y)$ so that by definition $x \prec y$. Hence again $x \preceq y$ is true.

Case: $\alpha \notin f_1[X \cap A_1]$. Then it has to be that $\alpha \in f_2[X \cap A_2]$. Then we claim that $x = f_2^{-1}(\alpha)$ is the \prec -least element of X , noting that $x \in A_2$. Note also that clearly then $f_2(x) = f_2(f_2^{-1}(\alpha)) = \alpha$. So consider any $y \in X$.

Case: $y \in A_1$. Then $y \in X \cap A_1$ so that $f_1(y) \in f_1[X \cap A_1]$ so $f_1(y) \in B$. Thus $f_2(x) = \alpha \leq f_1(y)$ since α is the least element of B . Now, it cannot be that $f_2(x) = \alpha = f_1(y)$ for then α would be in $f_1[X \cap A_1]$. So it must be that $f_2(x) = \alpha < f_1(y)$ so that by definition $x \prec y$ so that $x \preceq y$ is true.

Case: $y \in A_2$. Then $y \in X \cap A_2$ so that $f_2(y) \in f_2[X \cap A_2]$ so that $f_2(y) \in B$. Thus $f_2(x) = \alpha \leq f_2(y)$. Then if $f_2(x) = \alpha = f_2(y)$ then $x = y$ since f_2 is bijective. On the other hand if $f_2(x) = \alpha < f_2(y)$ then $x \prec y$ by definition. In either case we have $x \preceq y$.

Hence in all cases we have shown that X has a \prec -least element so that \prec is a well-ordering of A .

Now we show by transfinite induction that $\aleph_\alpha + \aleph_\alpha = \aleph_\alpha$ for all ordinals α . First it was already shown in a previous chapter that $\aleph_0 + \aleph_0 = \aleph_0$. So now consider any $\alpha > 0$ and suppose $\aleph_\gamma + \aleph_\gamma = \aleph_\gamma$ for all $\gamma < \alpha$.

Then consider two disjoint sets A_1 and A_2 that are both equipotent to ω_α and the well-ordering \prec on $A = A_1 \cup A_2$ as defined above, also again letting f_1 and f_2 be the isomorphisms from A_1 and A_2 , respectively, to ω_α . Now let a be any element of A and define

$$X = \{x \in A \mid x \prec a\}.$$

Let $X_1 = X \cap A_1$ and $X_2 = X \cap A_2$ so that clearly X_1 and X_2 are disjoint and $X = X_1 \cup X_2$. From this it follows from the definition of cardinal addition that $|X| = |X_1| + |X_2|$.

If $a \in A_1$ then define $\beta = f_1(a) \in \omega_\alpha$ so that $\beta < \omega_\alpha$. It follows from this and Lemma 7.2.1.5 that there is a $\gamma < \alpha$ such that $|\beta| \leq \aleph_\gamma$ since $\alpha > 0$, noting also that $\aleph_\gamma < \aleph_\alpha$ by the remarks following Definition 7.1.8. Now consider any $x_1 \in X_1$ so that also $x_1 \in X$. Then by definition $x_1 \prec a$ so that by the definition of \prec we have $f_1(x_1) < f_1(a) = \beta$ since $x_1 \in A_1$ and $a \in A_1$. Hence $f_1(x_1) \in \beta$ so that $f_1[X_1] \subseteq \beta$ since x_1 was arbitrary. Hence, since f_1 is bijective, we have $|X_1| = |f_1[X_1]| \leq |\beta| \leq \aleph_\gamma$. Next, consider any $x_2 \in X_2$ so that $x_2 \in X$ and hence $x_2 \prec a$. Then, again by the definition of \prec , we have that $f_2(x_2) < f_1(a) = \beta$ since $x_2 \in A_2$ and $a \in A_1$. Hence $f_2(x_2) \in \beta$ so that $f_2[X_2] \subseteq \beta$ since x_2 was arbitrary. Thus we have $|X_2| = |f_2[X_2]| \leq |\beta| \leq \aleph_\gamma$ since f_2 is bijective.

A similar argument shows that $|X_1| \leq \aleph_\gamma$ and $|X_2| \leq \aleph_\gamma$ for some $\gamma < \alpha$ in the case when $a \in A_2$. However, in this case we must set $\beta = f_2(a) + 1$, noting that $\beta \in \omega_\alpha$ since $f_2(a) \in \omega_\alpha$ and ω_α is a limit ordinal by Lemma 7.2.1.6.

Thus in all cases we have

$$\begin{aligned} |X| &= |X_1| + |X_2| \\ &\leq \aleph_\gamma + \aleph_\gamma && \text{(by property (d) of in section 5.1)} \\ &= \aleph_\gamma && \text{(by the induction hypothesis since } \gamma < \alpha) \\ &< \aleph_\alpha. \end{aligned}$$

Thus we have shown that $|X| < \aleph_\alpha = |\omega_\alpha|$ for any $a \in A$, and hence $|\omega_\alpha| \not\leq |X|$ by Corollary 7.2.1.8 since ω_α and X are both well-ordered. If δ is the ordinal isomorphic to (A, \prec) (which exists by Theorem 6.3.1 since we have shown that \prec is a well-ordering), then it follows from the contrapositive of Lemma 7.2.1.2 that $\delta \leq \omega_\alpha$ and hence $|A| = |\delta| \leq |\omega_\alpha| = \aleph_\alpha$. Thus we have

$$\aleph_\alpha + \aleph_\alpha = |A_1| + |A_2| = |A| \leq \aleph_\alpha.$$

Since obviously $0 \leq \aleph_\alpha$ it follows again from property (d) in section 5.1 that

$$\aleph_\alpha = \aleph_\alpha + 0 \leq \aleph_\alpha + \aleph_\alpha.$$

Hence, by the Cantor-Bernstein Theorem we have that $\aleph_\alpha = \aleph_\alpha + \aleph_\alpha$, which completes the inductive step. \square

Exercise 7.2.2

Give a direct proof of $n \cdot \aleph_\alpha = \aleph_\alpha$ by constructing a one-to-one mapping of ω_α onto $n \times \omega_\alpha$ (where n is a positive natural number).

Solution:

Lemma 7.2.2.1. *If α is a limit ordinal and n is a natural number then $n \cdot \alpha = \alpha$.*

Proof. First, clearly we have by definition that $n \cdot \omega = \sup\{n \cdot k \mid k < \omega\} = \omega$. Then, since α is a limit ordinal, we have from Exercise 6.5.10 that $\alpha = \omega \cdot \beta$ for some ordinal β . Thus we have

$$n \cdot \alpha = n \cdot (\omega \cdot \beta) = (n \cdot \omega) \cdot \beta = \omega \cdot \beta = \alpha.$$

\square

Main Problem.

Proof. Consider any natural number $n > 0$ and any ordinal number α . We then show that $n \cdot \aleph_\alpha = \aleph_\alpha$ by constructing a bijective $f : \omega_\alpha \rightarrow n \times \omega_\alpha$, which clearly shows the result by the definition of cardinal multiplication.

So consider any $\beta \in \omega_\alpha$. Then, since $n > 0$, we have by Theorem 6.6.3 that there is a unique ordinal γ and unique natural number $k < n$ such that

$$\beta = n \cdot \gamma + k,$$

where $k < n$ and clearly

$$\gamma = 1 \cdot \gamma \leq n \cdot \gamma = n \cdot \gamma + 0 \leq n \cdot \gamma + k = \beta$$

since $1 \leq n$ and $0 \leq k$. Hence we have $k \in n$ and $\gamma \leq \beta < \omega_\alpha$ so that $\gamma \in \omega_\alpha$. We then set $f(\beta) = (k, \gamma) \in n \times \omega_\alpha$.

First we show that f is injective. So consider β_1 and β_2 in ω_α where $f(\beta_1) = f(\beta_2)$. If we have $f(\beta_1) = (k_1, \gamma_1)$ and $f(\beta_2) = (k_2, \gamma_2)$ then clearly this means that $k_1 = k_2$ and $\gamma_1 = \gamma_2$. It then clearly follows that

$$\beta_1 = n \cdot \gamma_1 + k_1 = n \cdot \gamma_2 + k_2 = \beta_2,$$

which shows that f is injective.

Now we show that f is also surjective. So consider any $(k, \gamma) \in n \times \omega_\alpha$ so that $k \in n$ and $\gamma \in \omega_\alpha$. Then let $\beta = n \cdot \gamma + k$ so that clearly $f(\beta) = (k, \gamma)$. However, we must show that β is actually in ω_α . To see this, first we note that since $\gamma < \omega_\alpha$ is an ordinal we have $\gamma = \delta + m$ for some limit ordinal δ and natural number m by Exercise 6.5.4, where clearly $\delta \leq \gamma$. Hence we have

$$\beta = n \cdot \gamma + k = n \cdot (\delta + m) + k = (n \cdot \delta + n \cdot m) + k = n \cdot \delta + (n \cdot m + k) = \delta + (n \cdot m + k),$$

where $n \cdot \delta = \delta$ by Lemma 7.2.2.1 since δ is a limit ordinal. Then since also $\delta \leq \gamma < \omega_\alpha$ and $n \cdot m + k$ is a natural number we clearly have that $\beta = \delta + (n \cdot m + k) < \omega_\alpha$ as well since ω_α is a limit ordinal (by Theorem 7.1.9b). Hence $\beta \in \omega_\alpha$.

Thus we have shown that f is bijective so that by definition $n \cdot \aleph_\alpha = |n \times \omega_\alpha| = |\omega_\alpha| = \aleph_\alpha$ as desired. \square

Exercise 7.2.3

Show that

- (a) $\aleph_\alpha^n = \aleph_\alpha$ for all positive natural numbers n .
- (b) $|[\aleph_\alpha]^n| = \aleph_\alpha$, where $[\aleph_\alpha]^n$ is the set of all n -element subsets of \aleph_α for all $n > 0$.
- (c) $|[\aleph_\alpha]^{<\omega}| = \aleph_\alpha$, where $[\aleph_\alpha]^{<\omega}$ is the set of all finite subsets of \aleph_α .

[Hint: Use Theorem 7.2.1 and induction; for (c), proceed as in the proof of Theorem 3.10 in Chapter 4, and use $\aleph_0 \cdot \aleph_\alpha = \aleph_\alpha$.]

Solution:

(a)

Proof. For any ordinal α , we show this by induction on n , noting that we only need to show this for positive n so that $n \geq 1$ (in fact it is untrue for $n = 0$). First, for $n = 1$ we clearly have

$\aleph_\alpha^n = \aleph_\alpha^1 = \aleph_\alpha$ by what was shown in Exercise 5.1.2. Now assume that $\aleph_\alpha^n = \aleph_\alpha$. We then have

$$\begin{aligned}
\aleph_\alpha^{n+1} &= \aleph_\alpha^n \cdot \aleph_\alpha^1 && \text{(by Theorem 5.1.7a)} \\
&= \aleph_\alpha^n \cdot \aleph_\alpha && \text{(again by Exercise 5.1.2)} \\
&= \aleph_\alpha \cdot \aleph_\alpha && \text{(by the induction hypothesis)} \\
&= \aleph_\alpha. && \text{(by Theorem 7.2.1)}
\end{aligned}$$

This completes the induction step. □

(b)

Proof. For ordinal α and natural number n , first we show that $|\llbracket \aleph_\alpha \rrbracket^n| \leq \aleph_\alpha$ by constructing an injective $f : \llbracket \aleph_\alpha \rrbracket^n \rightarrow \omega_\alpha^n$. For a set $X \in \llbracket \aleph_\alpha \rrbracket^n$ we have that $|X| = n$. Thus there is a bijective g from n to X , and since clearly $X \subseteq \aleph_\alpha = \omega_\alpha$ it follows that g is a function from n to ω_α . Hence we simply set $f(X) = g$. Now consider any X_1 and X_2 in $\llbracket \aleph_\alpha \rrbracket^n$ where $X_1 \neq X_2$. Then let g_1 and g_2 be the corresponding bijections from n to X_1 and X_2 , respectively. Thus $f(X_1) = g_1$ and $f(X_2) = g_2$. Then, since clearly the range of g_1 is X_1 , the range of g_2 is X_2 , and $X_1 \neq X_2$ it follows that $f(X_1) = g_1 \neq g_2 = f(X_2)$, which shows that f is injective. Hence it follows that $|\llbracket \aleph_\alpha \rrbracket^n| \leq |\omega_\alpha^n| = |\omega_\alpha|^{|n|} = \aleph_\alpha^n = \aleph_\alpha$ by what was shown in part (a).

Now we show that also $\aleph_\alpha \leq |\llbracket \aleph_\alpha \rrbracket^n|$ by constructing an injective $f : \omega_\alpha \rightarrow \llbracket \aleph_\alpha \rrbracket^n$. So for any $\beta \in \omega_\alpha$ let $X = \{\beta + k \mid k \in n\}$, noting that clearly $X \subseteq \omega_\alpha = \aleph_\alpha$ since ω_α is a limit ordinal (since then $\beta + k \in \omega_\alpha$ for any natural number k). Also clearly $|X| = n$ so that $X \in \llbracket \aleph_\alpha \rrbracket^n$. We then set $f(\beta) = X$. Now consider any β_1 and β_2 in ω_α where $\beta_1 \neq \beta_2$ and let $X_1 = \{\beta_1 + k \mid k \in n\}$ and $X_2 = \{\beta_2 + k \mid k \in n\}$ so that $f(\beta_1) = X_1$ and $f(\beta_2) = X_2$. Since $\beta_1 \neq \beta_2$ we can assume that $\beta_1 < \beta_2$ without loss of generality. Now, clearly $\beta_1 = \beta_1 + 0$ is the least element of X_1 and $\beta_2 = \beta_2 + 0$ the least element of X_2 . Since $\beta_1 < \beta_2$ it then follows that $\beta_1 \notin X_2$, but since $\beta_1 \in X_1$ this clearly implies that $f(\beta_1) = X_1 \neq X_2 = f(\beta_2)$. This shows that f is injective so that $\aleph_\alpha = |\omega_\alpha| \leq |\llbracket \aleph_\alpha \rrbracket^n|$.

Since we have shown that both $|\llbracket \aleph_\alpha \rrbracket^n| \leq \aleph_\alpha$ and $\aleph_\alpha \leq |\llbracket \aleph_\alpha \rrbracket^n|$, it follows from the Cantor-Bernstein Theorem that $|\llbracket \aleph_\alpha \rrbracket^n| = \aleph_\alpha$, which is what we intended to show. □

(c)

Proof. For any ordinal α we show first note that clearly $\llbracket \aleph_\alpha \rrbracket^{<\omega} = \bigcup_{n < \omega} \llbracket \aleph_\alpha \rrbracket^n$. We show that $|\llbracket \aleph_\alpha \rrbracket^{<\omega}| = \aleph_\alpha$ by constructing a bijective $f : \omega \times \omega_\alpha \rightarrow \bigcup_{n < \omega} \llbracket \aleph_\alpha \rrbracket^n$. So consider any $n \in \omega$ and $\beta \in \omega_\alpha$. Now, by what was shown in part (b), we have $|\llbracket \aleph_\alpha \rrbracket^n| = \aleph_\alpha = |\omega_\alpha|$ so that there is a bijective $g_n : \omega_\alpha \rightarrow \llbracket \aleph_\alpha \rrbracket^n$, i.e. g_n is a transfinite enumeration of $\llbracket \aleph_\alpha \rrbracket^n$. We then set $f(n, \beta) = g_n(\beta)$, from which it should be clear that $f(n, \beta) \in \bigcup_{k < \omega} \llbracket \aleph_\alpha \rrbracket^k$ since $f(n, \beta) = g_n(\beta) \in \llbracket \aleph_\alpha \rrbracket^n$.

First we show that f is injective. To this end consider any (n_1, β_1) and (n_2, β_2) in $\omega \times \omega_\alpha$ where $(n_1, \beta_1) \neq (n_2, \beta_2)$. Then either $n_1 \neq n_2$ or $\beta_1 \neq \beta_2$ (or both). Let g_{n_1} and g_{n_2} be the corresponding bijections from ω_α to $\llbracket \aleph_\alpha \rrbracket^{n_1}$ and $\llbracket \aleph_\alpha \rrbracket^{n_2}$, respectively, as described above. Clearly if $n_1 \neq n_2$ then $f(n_1, \beta_1) = g_{n_1}(\beta_1) \in \llbracket \aleph_\alpha \rrbracket^{n_1}$ whereas $f(n_2, \beta_2) = g_{n_2}(\beta_2) \in \llbracket \aleph_\alpha \rrbracket^{n_2}$ so that $f(n_1, \beta_1) \neq f(n_2, \beta_2)$ since $\llbracket \aleph_\alpha \rrbracket^{n_1}$ and $\llbracket \aleph_\alpha \rrbracket^{n_2}$ are clearly disjoint (since $\llbracket \aleph_\alpha \rrbracket^{n_1}$ contains only sets with n_1 elements and $\llbracket \aleph_\alpha \rrbracket^{n_2}$ contains only sets with n_2 elements and $n_1 \neq n_2$). On the other hand, if $n_1 = n_2$ then it must be the case that $\beta_1 \neq \beta_2$. It also follows that $g_{n_1} = g_{n_2}$ since $n_1 = n_2$. Hence we have $f(n_1, \beta_1) = g_{n_1}(\beta_1) \neq g_{n_1}(\beta_2) = g_{n_2}(\beta_2) = f(n_2, \beta_2)$ since $g_{n_1} = g_{n_2}$ is injective and $\beta_1 \neq \beta_2$. Thus in any case $f(n_1, \beta_1) \neq f(n_2, \beta_2)$ so that f is injective.

Next we show that f is surjective. So consider any $X \in \bigcup_{n < \omega} \llbracket \aleph_\alpha \rrbracket^n$ so that there is an $n < \omega$ such that $X \in \llbracket \aleph_\alpha \rrbracket^n$. Then let $\beta = g_n^{-1}(X)$ (where g_n is the bijection from ω_α to $\llbracket \aleph_\alpha \rrbracket^n$ as described above). Then clearly $(n, \beta) \in \omega \times \omega_\alpha$ and we have $f(n, \beta) = g_n(\beta) = g_n(g_n^{-1}(X)) = X$. Since X was arbitrary this shows that f is surjective.

Hence f is a bijection so that $|\aleph_\alpha|^{<\omega} = |\bigcup_{n < \omega} [\aleph_\alpha]^n| = |\omega \times \omega_\alpha| = |\omega| \cdot |\omega_\alpha| = \aleph_0 \cdot \aleph_\alpha = \aleph_\alpha$ by Corollary 7.2.2 since clearly $0 \leq \alpha$. This shows the desired result. \square

Exercise 7.2.4

If α and β are ordinals and $|\alpha| \leq \aleph_\gamma$ and $|\beta| \leq \aleph_\gamma$, then $|\alpha + \beta| \leq \aleph_\gamma$, $|\alpha \cdot \beta| \leq \aleph_\gamma$, $|\alpha^\beta| \leq \aleph_\gamma$ (where $\alpha + \beta$, $\alpha \cdot \beta$, and α^β are ordinal operations).

Solution:

Lemma 7.2.4.1. *If α and β are ordinals then $|\alpha + \beta| = |\alpha| + |\beta|$.*

Proof. Suppose that A and B are disjoint sets where $|A| = |\alpha|$ and $|B| = |\beta|$ so that, by the definition of cardinal addition, $|\alpha| + |\beta| = |A \cup B|$. We then show the result by constructing a bijection f from $A \cup B$ to the ordinal $\alpha + \beta$. First, since $|A| = |\alpha|$, there is a bijective $f_A : A \rightarrow \alpha$. Similarly there is a bijective $f_B : B \rightarrow \beta$ since $|B| = |\beta|$. Now consider any $x \in A \cup B$. We then set

$$f(x) = \begin{cases} f_A(x) & x \in A \\ \alpha + f_B(x) & x \in B \end{cases},$$

noting that this is unambiguous since A and B are disjoint. If $x \in A$ then we clearly have $f(x) = f_A(x) < \alpha = \alpha + 0 \leq \alpha + \beta$ by Lemma 6.5.4 so that $f(x) \in \alpha + \beta$. On the other hand if $x \in B$ then $f(x) = \alpha + f_B(x) < \alpha + \beta$ again by Lemma 6.5.4 since $f_B(x) < \beta$, and hence again $f(x) \in \alpha + \beta$. This shows that f really is a function into $\alpha + \beta$.

Next we show that f is injective. So consider any x and y in $A \cup B$ where $x \neq y$.

Case: x and y are both in A . Then $f(x) = f_A(x) \neq f_A(y) = f(y)$ since f_A is injective and $x \neq y$.

Case: $x \in A$ and $y \in B$. Then $f(x) = f_A(x) < \alpha = \alpha + 0 \leq \alpha + f_B(y) = f(y)$ by Lemma 6.5.4 so that clearly $f(x) \neq f(y)$.

Case: $x \in B$ and $y \in A$. This is analogous to the previous case.

Case: x and y are both in B . Then $f(x) = \alpha + f_B(x) \neq \alpha + f_B(y) = f(y)$ by Lemma 6.5.4b since f_B is injective and $x \neq y$ so that $f_B(x) \neq f_B(y)$.

Hence in all cases we have $f(x) \neq f(y)$, which shows that f is injective.

Lastly, we show that f is surjective. So consider any y in $\alpha + \beta$. If $y < \alpha$ then $y \in \alpha$. Since f_A is surjective there is an $x \in A$ such that $f_A(x) = y$. Then, since $x \in A$, clearly $f(x) = f_A(x) = y$. If $y \geq \alpha$ then by Lemma 6.5.5 there is an ordinal ξ such that $\alpha + \xi = y$. It then follows that $\alpha + \xi = y < \alpha + \beta$ so that $\xi < \beta$ by Lemma 6.5.4a. Hence $\xi \in \beta$ so that there is an $x \in B$ such that $f_B(x) = \xi$ since f_B is surjective. Then, since $x \in B$, clearly we have $f(x) = \alpha + f_B(x) = \alpha + \xi = y$. Hence in both cases there is an $x \in A \cup B$ where $f(x) = y$. Since y was arbitrary, this shows that f is surjective.

Thus we have shown that f is bijective so that $|\alpha| + |\beta| = |A \cup B| = |\alpha + \beta|$. \square

Lemma 7.2.4.2. *If α and β are ordinals then $|\alpha \cdot \beta| = |\alpha| \cdot |\beta|$.*

Proof. First, if $\alpha = 0$ then

$$|\alpha \cdot \beta| = |0 \cdot \beta| = |0| = 0 = 0 \cdot |\beta| = |0| \cdot |\beta| = |\alpha| \cdot |\beta|.$$

The result also holds when $\beta = 0$. Hence going forward we can assume that α and β are nonzero and therefore nonempty.

We then show the result by constructing a bijective $f : \alpha \times \beta \rightarrow \alpha \cdot \beta$. So consider any $(\gamma, \delta) \in \alpha \times \beta$. It then follows that $\gamma < \alpha$ and $\delta < \beta$ so that $\delta + 1 \leq \beta$. We then set $f(\gamma, \delta) = \alpha \cdot \delta + \gamma$. We note that

$$f(\gamma, \delta) = \alpha \cdot \delta + \gamma < \alpha \cdot \delta + \alpha = \alpha \cdot \delta + \alpha \cdot 1 = \alpha \cdot (\delta + 1) \leq \alpha \cdot \beta$$

by Lemma 6.5.4, Exercise 6.5.2, and Exercise 6.5.7. Hence $f(\gamma, \delta) \in \alpha \cdot \beta$ so that f is into $\alpha \cdot \beta$.

Now we show that f is injective. So consider any (γ_1, δ_1) and (γ_2, δ_2) in $\alpha \times \beta$ where $(\gamma_1, \delta_1) \neq (\gamma_2, \delta_2)$. Then clearly we have $\gamma_1 < \alpha$, $\gamma_2 < \alpha$, $\delta_1 < \beta$, and $\delta_2 < \beta$. We then have

Case: $\delta_1 = \delta_2$. Then it must be that $\gamma_1 \neq \gamma_2$. We then have

$$f(\gamma_1, \delta_1) = \alpha \cdot \delta_1 + \gamma_1 \neq \alpha \cdot \delta_1 + \gamma_2 = \alpha \cdot \delta_2 + \gamma_2 = f(\gamma_2, \delta_2),$$

where we have used Lemma 6.5.4b and Exercise 6.5.7b since $\alpha \neq 0$.

Case: $\delta_1 \neq \delta_2$. Without loss of generality we can assume that $\delta_1 < \delta_2$ so that clearly $\delta_1 + 1 \leq \delta_2$. Then we have

$$f(\gamma_1, \delta_1) = \alpha \cdot \delta_1 + \gamma_1 < \alpha \cdot \delta_1 + \alpha = \alpha \cdot (\delta_1 + 1) \leq \alpha \cdot \delta_2 \leq \alpha \cdot \delta_2 + \gamma_2 = f(\gamma_2, \delta_2),$$

where we have again used Lemma 6.5.4a, Exercise 6.5.2, and Exercise 6.5.7.

Hence in all cases we have $f(\gamma_1, \delta_1) \neq f(\gamma_2, \delta_2)$, which shows that f is injective.

Lastly, we show that f is surjective. So consider any ordinal $\xi \in \alpha \cdot \beta$ so that $\xi < \alpha \cdot \beta$. Since $\alpha \neq 0$, it follows from Theorem 6.6.3 that there is a unique δ and unique $\gamma < \alpha$ such that $\xi = \alpha \cdot \delta + \gamma$. Note that we have $\delta < \beta$ since otherwise $\delta \geq \beta$ would imply that

$$\xi = \alpha \cdot \delta + \gamma \geq \alpha \cdot \delta \geq \alpha \cdot \beta$$

by Lemma 6.5.4 and Exercise 6.5.7, which is impossible since $\xi < \alpha \cdot \beta$. Hence we have that $(\gamma, \delta) \in \alpha \times \beta$, and clearly $f(\gamma, \delta) = \xi$. Since ξ was arbitrary this shows that f is surjective.

Thus we have shown that f is bijective so that by the definition of cardinal multiplication we have $|\alpha \cdot \beta| = |\alpha \times \beta| = |\alpha| \cdot |\beta|$ as desired. \square

The following two lemmas are straightforward generalizations of Theorems 4.3.9 and 4.3.10, respectively.

Lemma 7.2.4.3. *Consider a nonzero ordinals α and β . Let $\langle A_\gamma \mid \gamma < \alpha \rangle$ be a (potentially transfinite) system of at most $|\beta|$ sets, and let $\langle a_\gamma \mid \gamma < \alpha \rangle$ be a system of enumerations for $\langle A_\gamma \mid \gamma < \alpha \rangle$, i.e., for each $\gamma < \alpha$, $a_\gamma = \langle a_\gamma(\delta) \mid \delta < \beta \rangle$ is a (potentially transfinite) sequence, and $A_\gamma = \{a_\gamma(\delta) \mid \delta < \beta\}$. Then $\bigcup_{\gamma < \alpha} A_\gamma$ is at most $|\alpha| \cdot |\beta|$.*

Proof. We define a function $f : \alpha \times \beta \rightarrow \bigcup_{\gamma < \alpha} A_\gamma$ by simply setting $f(\gamma, \delta) = a_\gamma(\delta)$ for any $\gamma \in \alpha$ and $\delta \in \beta$. We show that f is onto by considering any $x \in \bigcup_{\gamma < \alpha} A_\gamma$ so that there is an $\gamma < \alpha$ such that $x \in A_\gamma$. Then, since A_γ is the range of a_γ , there is a $\delta < \beta$ such that $a_\gamma(\delta) = x$. Hence $f(\gamma, \delta) = a_\gamma(\delta) = x$ so that f is onto since x was arbitrary.

We also have that $\alpha \times \beta$ is well-orderable by Theorem 6.5.8 (for example the lexicographic ordering has order type $\beta \cdot \alpha$). It then follows from Lemma 7.1.6.1 that $\left| \bigcup_{\gamma < \alpha} A_\gamma \right| \leq |\alpha \times \beta| = |\alpha| \cdot |\beta|$ since f is onto. Hence $\bigcup_{\gamma < \alpha} A_\gamma$ is at most $|\alpha| \cdot |\beta|$ as desired. \square

Lemma 7.2.4.4. *If A is a set with cardinality \aleph_γ for some ordinal γ , then the set $\text{Seq}(A)$ of all finite sequences of elements of A also has cardinality \aleph_γ .*

Proof. Let f be a bijection from ω_γ to A . We also know from Theorem 7.2.1 that $|\omega_\gamma \times \omega_\gamma| = \aleph_\gamma \cdot \aleph_\gamma = \aleph_\gamma$, so let g be a bijection from ω_γ to $\omega_\gamma \times \omega_\gamma$. For each $n < \omega_0 = \omega$ (i.e. $n \in \mathbf{N}$) we define a transfinite enumeration $\langle a_n(\alpha) \mid \alpha < \omega_\gamma \rangle$ of A^n , where A^n of course denotes the set of all sequences of elements of A of length n . We define these enumerations recursively. Clearly for $n = 0$ we have $A^n = A^0 = \{\emptyset\}$ so that we can set

$$\begin{aligned} a_0(\alpha) &= \emptyset \text{ for any } \alpha < \omega_\gamma \\ a_1(\alpha) &= \langle f(\alpha) \rangle \text{ for any } \alpha < \omega_\gamma \end{aligned}$$

Then, having defined a_n , for any $\alpha < \omega_\gamma$, we let $g(\alpha) = (\alpha_1, \alpha_2)$ and define the sequence $a_{n+1}(\alpha)$ of length n as follows:

$$a_{n+1}(\alpha)(k) = \begin{cases} a_n(\alpha_1)(k) & k < n \\ f(\alpha_2) & k = n. \end{cases}$$

for any $k < n + 1$.

Clearly each $a_n(\alpha)$ is a sequence of length n for any $n < \omega_0$ and $\alpha < \omega_\gamma$, but we must show that a_n is in fact an enumeration by showing that it is onto A^n for all $n < \omega_0$. We show this by induction on n . Obviously this is true for the trivial $n = 0$ case and for the $n = 1$ case as well since, for any sequence $\langle a \rangle$ of length 1 where $a \in A$, we have that there is an $\alpha < \omega_\gamma$ such that $f(\alpha) = a$ since f is onto. Hence by definition $a_1(\alpha) = \langle f(\alpha) \rangle = \langle a \rangle$ so that a_1 is onto.

Now suppose that a_n is onto A^n and consider any sequence $h \in A^{n+1}$. Now, since $h \upharpoonright n$ is a sequence of length n there is an $\alpha_1 < \omega_\gamma$ such that $a_n(\alpha_1) = h \upharpoonright n$ by the induction hypothesis. We also have $h(n) \in A$ so that there is an $\alpha_2 < \omega_\gamma$ such that $f(\alpha_2) = h(n)$ since again f is onto. Now, clearly $(\alpha_1, \alpha_2) \in \omega_\gamma \times \omega_\gamma$ so that there is an $\alpha < \omega_\gamma$ such that $g(\alpha) = (\alpha_1, \alpha_2)$ since g is onto. Now consider any $k < n + 1$. If $k < n$ then clearly we have $a_{n+1}(\alpha)(k) = a_n(\alpha_1)(k) = h(k)$ since $a_n(\alpha_1) = h \upharpoonright n$. On the other hand, if $k = n$, then by definition $a_{n+1}(\alpha)(k) = f(\alpha_2) = h(n) = h(k)$. Thus $a_{n+1}(\alpha) = h$ since k was arbitrary and the cases are exhaustive. This shows that a_{n+1} is onto since h was arbitrary, hence it is an enumeration.

Clearly we have that $\text{Seq}(A) = \bigcup_{n < \omega_0} A^n$ and, since we also have an transfinite enumeration (indexed by ω_γ) of each A^n it follows from Lemma 7.2.4.3 that

$$|\text{Seq}(A)| = \left| \bigcup_{n < \omega_0} A^n \right| \leq |\omega_0| \cdot |\omega_\gamma| = \aleph_0 \cdot \aleph_\gamma = \aleph_\gamma,$$

where we have utilized Corollary 7.2.2 since $0 \leq \gamma$. Also clearly $\aleph_\gamma \leq |\text{Seq}(A)|$ since, for example, the function $f : \omega_\gamma \rightarrow \text{Seq}(A)$ defined by $f(\alpha) = \langle \alpha \rangle$ (for any $\alpha < \omega_\gamma$) is injective. Thus by the Cantor-Bernstein Theorem we have $|\text{Seq}(A)| = \aleph_\gamma$ as desired. \square

Corollary 7.2.4.5. *If A is a nonempty finite set then the set $\text{Seq}(A)$ of all finite sequences of elements of A is countable.*

Proof. First we note that clearly the set $B \cup \omega$ is countable (since B is finite and ω is countable) so that $\text{Seq}(A \cup \omega)$ is also countable by Lemma 7.2.4.4. Now let f be a function from $\text{Seq}(A)$ to $\text{Seq}(A \cup \omega)$ defined by the identity $f(g) = g$ for any sequence $g \in \text{Seq}(A)$. Note that $A \subseteq A \cup \omega$ so that any sequence of elements of A is also a sequence with elements in $A \cup \omega$. Clearly f is injective so that $|\text{Seq}(A)| \leq |\text{Seq}(A \cup \omega)| = \aleph_0$.

Since A is nonempty there is an $a \in A$. For any $n < \omega$ consider the finite sequence $g_n(k) = a$ for any $k < n$, which is clearly a sequence of length n with elements in A . Consider then the function f from ω to $\text{Seq}(A)$ defined $f(n) = g_n$ for $n < \omega$. Clearly this is an injective function since, for any n_1 and n_2 in ω where $n_1 \neq n_2$, we have that $f(n_1) = g_{n_1}$ and $f(n_2) = g_{n_2}$ are sequences of different lengths so cannot be equal. Thus we have $\aleph_0 = |\omega| \leq |\text{Seq}(A)|$ as well so that $|\text{Seq}(A)| = \aleph_0$ by the Cantor-Bernstein Theorem. \square

The following is a corollary to Exercise 7.2.3c.

Corollary 7.2.4.6. *Suppose that A is a set such that $|A| = \aleph_\alpha$ for some ordinal α . Let $[A]^{<\omega}$ denote the set of all finite subsets of A . Then $|[A]^{<\omega}| = \aleph_\alpha$.*

Proof. First let $[\aleph_\alpha]^{<\omega}$ again denote the set of finite subsets of \aleph_α . Since $|A| = \aleph_\alpha$ there is a bijective $f : A \rightarrow \aleph_\alpha$. We then define $g : [A]^{<\omega} \rightarrow [\aleph_\alpha]^{<\omega}$ by letting $g(B) = f[B]$ for any $B \in [A]^{<\omega}$, noting that clearly $g(B) \subseteq \aleph_\alpha$ and $g(B)$ is finite so that $g(B) \in [\aleph_\alpha]^{<\omega}$. We now show that g is bijective.

First consider any B_1 and B_2 in $[A]^{<\omega}$ where $g(B_1) = g(B_2)$. Hence $f[B_1] = g(B_1) = g(B_2) = f[B_2]$. So consider any $x \in B_1$ so that $f(x) \in f[B_1]$. Then also $f(x) \in f[B_2]$ so that there is a $y \in B_2$ such that $f(y) = f(x)$. But since f is injective it has to be that $y = x$ so that $x = y \in B_2$. Hence $B_1 \subseteq B_2$ since x was arbitrary. A similar argument shows that $B_2 \subseteq B_1$ as well so that $B_1 = B_2$. This shows that g is injective.

Now consider any $C \in [\aleph_\alpha]^{<\omega}$ and let $B = f^{-1}[C]$, noting that f^{-1} is a bijective function since f is. We claim then that $g(B) = C$. So consider any $y \in g(B) = f[B]$ so that there is a $x \in B$ such that $f(x) = y$. Then we have $x \in f^{-1}[C]$ by the definition of B so that there is a $z \in C$ such that $f^{-1}(z) = x$. Hence $z = f(f^{-1}(z)) = f(x) = y$ so that $y = z \in C$. Thus $g(B) \subseteq C$ since y was arbitrary. Now consider any $y \in C$ so that clearly $x = f^{-1}(y) \in f^{-1}[C] = B$ and also $f(x) = y$. Hence clearly $y = f(x) \in f[B] = g(B)$ so that $C \subseteq g(B)$ since y was arbitrary. This shows that $g(B) = C$ so that g is surjective since C was arbitrary.

We have just shown that g is bijective so that $|[A]^{<\omega}| = |[\aleph_\alpha]^{<\omega}| = \aleph_\alpha$ by Exercise 7.2.3c. \square

Lemma 7.2.4.7. *If α and β are ordinals where at least one is infinite then $|\alpha^\beta| \leq \max(|\alpha|, |\beta|)$.*

Proof. To show this we reference the representation of ordinal exponentiation discussed in Exercise 6.5.16. In that exercise we showed that the set $S(\beta, \alpha) = \{f \mid f : \beta \rightarrow \alpha \text{ and } s(f) \text{ is finite}\}$, where $s(f) = \{\xi < \beta \mid f(\xi) \neq 0\}$ for any $f : \beta \rightarrow \alpha$, can be ordered to be isomorphic to α^β . From this it clearly follows that $|S(\beta, \alpha)| = |\alpha^\beta|$.

We now construct an injective function f from $S(\beta, \alpha)$ to $[\beta]^{<\omega} \times \text{Seq}(\alpha)$, where $[\beta]^{<\omega}$ is the set of all finite subsets of β and $\text{Seq}(\alpha)$ is the set of all finite sequences of elements of α . So consider any $g \in S(\beta, \alpha)$ so that $g : \beta \rightarrow \alpha$ and $s(g)$ is a finite subset of β . Also clearly $s(g)$ is a finite set of ordinals so that there is a unique isomorphism h from some natural number n to $s(g)$. Clearly then $g \circ h$ is a finite sequence from n to α . Thus we have that $s(g) \in [\beta]^{<\omega}$ and $g \circ h \in \text{Seq}(\alpha)$, so we set $f(g) = (s(g), g \circ h)$.

To see that this mapping is injective consider g_1 and g_2 in $S(\beta, \alpha)$ where $g_1 \neq g_2$. Then there is some $\xi < \beta$ where $g_1(\xi) \neq g_2(\xi)$. Let h_1 and h_2 be the isomorphisms from natural numbers n_1 and n_2 to $s(g_1)$ and $s(g_2)$, respectively, as described above. If $s(g_1) \neq s(g_2)$ then clearly $f(g_1) = (s(g_1), g_1 \circ h_1) \neq (s(g_2), g_2 \circ h_2) = f(g_2)$. So assume that $s(g_1) = s(g_2)$, from which it follows that $n_1 = n_2$ and $h_1 = h_2$. Since $h_1 = h_2$ are bijections there is a $k \in n_1 = n_2$ such that $h_1(k) = h_2(k) = \xi$, noting that it has to be that $\xi \in s(g_1) = s(g_2)$ since otherwise we would have $g_1(\xi) = 0 = g_2(\xi)$. We then have $(g_1 \circ h_1)(k) = g_1(h_1(k)) = g_1(\xi) \neq g_2(\xi) = g_2(h_2(k)) = (g_2 \circ h_2)(k)$ so that $g_1 \circ h_1 \neq g_2 \circ h_2$. Thus once again $f(g_1) = (s(g_1), g_1 \circ h_1) \neq (s(g_2), g_2 \circ h_2) = f(g_2)$, which shows that f is injective.

So since f is injective it follows that $|\alpha^\beta| = |S(\beta, \alpha)| \leq |[\beta]^{<\omega} \times \text{Seq}(\alpha)| = |[\beta]^{<\omega}| \cdot |\text{Seq}(\alpha)|$. Suppose first that $|\alpha| \leq |\beta|$ so it has to be that β is infinite so that $\max(|\alpha|, |\beta|) = |\beta| = \aleph_\gamma$ for some ordinal γ . Thus by Lemma 7.2.4.6 we have $|[\beta]^{<\omega}| = \aleph_\gamma$. If $\alpha = 0 = \emptyset$ then clearly $\text{Seq}(\alpha) = \{\emptyset\}$ so that $|\text{Seq}(\alpha)| = 1$. If α is finite but nonzero then it is nonempty so that $|\text{Seq}(\alpha)| = \aleph_0$ by Corollary 7.2.4.5. Lastly, if α is infinite then $|\alpha| = \aleph_\delta$ for some $\delta \leq \gamma$ since $\aleph_\delta = |\alpha| \leq |\beta| = \aleph_\gamma$. Hence $|\text{Seq}(\alpha)| = \aleph_\gamma$.

by Lemma 7.2.4.4. Thus in all three cases $\kappa = |\text{Seq}(\alpha)|$ is either a natural number or \aleph_δ for some $\delta \leq \gamma$. It then follows from Corollary 7.2.2 that

$$|\alpha^\beta| \leq |[\beta]^{<\omega}| \cdot |\text{Seq}(\alpha)| = \aleph_\gamma \cdot \kappa = \kappa \cdot \aleph_\gamma = \aleph_\gamma = \max(|\alpha|, |\beta|).$$

On the other hand if $|\beta| \leq |\alpha|$ then it must be that α is infinite so that $\max(|\alpha|, |\beta|) = |\alpha| = \aleph_\gamma$ for some ordinal γ . Thus by Lemma 7.2.4.4 we have $|\text{Seq}(\alpha)| = \aleph_\gamma$ as well. If β is finite then clearly every subset of β is finite so that $[\beta]^{<\omega} = \mathcal{P}(\beta)$ is finite by Theorem 4.2.8. Hence $|[\beta]^{<\omega}| = n$ for some natural number n . If β is infinite then $|\beta| = \aleph_\delta$ for some $\delta \leq \gamma$ since $\aleph_\delta = |\beta| \leq |\alpha| = \aleph_\gamma$. We then have that $|[\beta]^{<\omega}| = \aleph_\delta$ as well by Lemma 7.2.4.6. Hence in either case $\kappa = |[\beta]^{<\omega}|$ is a natural number or \aleph_δ for some $\delta \leq \gamma$. Hence we have

$$|\alpha^\beta| \leq |[\beta]^{<\omega}| \cdot |\text{Seq}(\alpha)| = \kappa \cdot \aleph_\gamma = \aleph_\gamma = \max(|\alpha|, |\beta|)$$

again by Corollary 7.2.2. Thus in all cases we have shown that $|\alpha^\beta| \leq \max(|\alpha|, |\beta|)$ as desired. \square

Main Problem.

Proof. That $|\alpha + \beta| \leq \aleph_\gamma$ follows almost immediately from Lemma 7.2.4.1. We have that $|\alpha + \beta| = |\alpha| + |\beta| \leq \aleph_\gamma + \aleph_\gamma = \aleph_\gamma$, where we have also used property (c) of cardinal numbers after Lemma 5.1.2, and Corollary 7.2.3.

Similarly, $|\alpha \cdot \beta| \leq \aleph_\gamma$ follows from Lemma 7.2.4.2. We have that $|\alpha \cdot \beta| = |\alpha| \cdot |\beta| \leq \aleph_\gamma \cdot \aleph_\gamma = \aleph_\gamma$, where we have used property (i) of cardinal numbers following Lemma 5.1.4, and Theorem 7.2.1.

The analogous lemma for ordinal exponentiation (i.e. that $|\alpha^\beta| = |\alpha|^{|\beta|}$ for ordinals α and β) is evidently not true. As a counterexample consider $\alpha = 2$ and $\beta = \omega$. We then have that $|\alpha^\beta| = |2^\omega| = |\omega| = \aleph_0$ is countable whereas we know that $|\alpha|^{|\beta|} = |2|^{|\omega|} = 2^{\aleph_0}$ is uncountable.

However, the somewhat analogous Lemma 7.2.4.7 will help us show the desired result. First, if both α and β are finite then clearly α^β is also finite so that clearly $|\alpha^\beta| \leq \aleph_\gamma$. On the other hand, if at least one of α or β is infinite, then we have that $|\alpha^\beta| \leq \max(|\alpha|, |\beta|) \leq \aleph_\gamma$ by Lemma 7.2.4.7 as desired, noting that clearly $\max(|\alpha|, |\beta|) \leq \aleph_\gamma$ since both $|\alpha| \leq \aleph_\gamma$ and $|\beta| \leq \aleph_\gamma$. \square

Exercise 7.2.5

If X is the image of ω_α by some function f , then $|X| \leq \aleph_\alpha$. [Hint: Construct a one-to-one mapping g of X into ω_α by letting $g(x) =$ the least element of the inverse image of $\{x\}$ by f .]

Solution:

Proof. Clearly f is a function from ω_α onto its image X so that it follows from Lemma 7.1.6.1 that $|X| \leq |\omega_\alpha| = \aleph_\alpha$ as desired. \square

Note that the proof of Lemma 7.1.6.1 uses exactly the technique given in the hint to argue its conclusion.

Exercise 7.2.6

If X is a subset of ω_α such that $|X| < \aleph_\alpha$, then $|\omega_\alpha - X| = \aleph_\alpha$.

Solution:

Lemma 7.2.6.1. *If κ and λ are cardinal numbers and $\kappa \leq \lambda < \aleph_\alpha$ for some ordinal α , then $\kappa + \lambda < \aleph_\alpha$.*

Proof. We have

$$\begin{aligned}\kappa + \lambda &= \lambda + \kappa && \text{(by the commutativity of cardinal addition)} \\ &\leq \lambda + \lambda && \text{(by property (d) in section 5.1)} \\ &= \lambda \cdot 1 + \lambda \cdot 1 \\ &= \lambda \cdot (1 + 1) && \text{(by property (g) in section 5.1)} \\ &= \lambda \cdot 2.\end{aligned}$$

If λ is finite then clearly $\lambda \cdot 2$ is finite so that $\kappa + \lambda \leq \lambda \cdot 2 < \aleph_\alpha$. On the other hand if λ is infinite then $\lambda = \aleph_\beta$ where $\beta < \alpha$ since $\aleph_\beta = \lambda < \aleph_\alpha$. Thus we have $\kappa + \lambda \leq \lambda \cdot 2 = 2 \cdot \lambda = 2 \cdot \aleph_\beta = \aleph_\beta < \aleph_\alpha$ by Corollary 7.2.2. Hence in either case $\kappa + \lambda < \aleph_\alpha$ as desired. \square

Main Problem.

Proof. First, we clearly have that $\omega_\alpha - X$ and X are disjoint and $(\omega_\alpha - X) \cup X = \omega_\alpha$ so that, by the definition of cardinal addition, we have

$$|\omega_\alpha - X| + |X| = |(\omega_\alpha - X) \cup X| = |\omega_\alpha| = \aleph_\alpha.$$

Now suppose that $|\omega_\alpha - X| < \aleph_\alpha$. Since $\omega_\alpha - X \subseteq \omega_\alpha$ and $X \subseteq \omega_\alpha$ are both sets of ordinals, they are clearly both well ordered by $<$. Thus $|\omega_\alpha - X| \leq |X|$ or $|X| \leq |\omega_\alpha - X|$ by Lemma 7.2.1.7. Since we also have $|\omega_\alpha - X| < \aleph_\alpha$ and $|X| < \aleph_\alpha$, in either case it follows from Lemma 7.2.6.1 that

$$\aleph_\alpha = |\omega_\alpha - X| + |X| < \aleph_\alpha,$$

which is clearly a contradiction. Thus it must be that $|\omega_\alpha - X| \not< \aleph_\alpha$ so that $\aleph_\alpha \leq |\omega_\alpha - X|$ by Corollary 7.2.1.8. Since $\omega_\alpha - X \subseteq \omega_\alpha$ we clearly also have that $|\omega_\alpha - X| \leq |\omega_\alpha| = \aleph_\alpha$. Hence $|\omega_\alpha - X| = \aleph_\alpha$ by the Cantor-Bernstein Theorem. \square

Chapter 8 The Axiom of Choice

As in the text, asterisks indicate exercises for which the Axiom of Choice is needed.

§8.1 The Axiom of Choice and its Equivalents

Exercise 8.1.1

Prove: If a set A can be linearly ordered, then every system of finite subsets of A has a choice function. (It does not follow from the Zermelo-Fraenkel axioms that every set can be linearly ordered.)

Solution:

Lemma 8.1.1.1. *Any linear ordering of a finite set is a well-ordering.*

Proof. We show this by strong induction on the cardinality of the set. So consider natural number n and suppose that all linear ordered sets of cardinality $k < n$ are well-orderings. Also suppose that (A, \preccurlyeq) is any linearly ordered set with $|A| = n$. Consider any nonempty $B \subseteq A$ so that there is a $b \in B$. Then clearly $C = B - \{b\}$ is also a finite set with $C \subset B \subseteq A$ so that $|C| < n$. Clearly also C is linearly ordered by \preccurlyeq so that, by the induction hypothesis, C is well-ordered by \preccurlyeq . Now, if $C = \emptyset$, then it follows that $B = \{b\}$, which clearly has least element b . On the other hand, if $C \neq \emptyset$, then it has a least element c since it is well-ordered. Since \preccurlyeq is a *linear* ordering, it has to be that either $c \preccurlyeq b$ or $b \preccurlyeq c$.

Case: $c \preccurlyeq b$. Then consider any $x \in B$. If $x = b$ then obviously $c \preccurlyeq b = x$. If $x \neq b$ then $x \in C = B - \{b\}$ so that again $c \preccurlyeq x$ since c is the least element of C . This shows that c is the least element of B since x was arbitrary.

Case: $b \preccurlyeq c$. Then consider any $x \in B$. If $x = b$ then obviously $b \preccurlyeq b = x$. If $x \neq b$ then $x \in C = B - \{b\}$ so that $b \preccurlyeq c \preccurlyeq x$ since c is the least element of C . This shows that b is the least element of B since x was arbitrary.

Hence in all cases we have that B has a \preccurlyeq -least element. This shows that \preccurlyeq is a well-ordering of A since $B \subseteq A$ was arbitrary. This completes the inductive proof. \square

Main Problem.

Proof. Suppose that A is a set that can be linearly ordered and suppose that \preccurlyeq is a such a linear ordering. Suppose also that S is a system of finite subsets of A . Then clearly any $B \in S$ is finite and linearly ordered by \preccurlyeq . We then have that \preccurlyeq is a well-ordering of B by Lemma 8.1.1.1. So we then set

$$f(B) = \begin{cases} \text{the least element of } B \text{ according to } \preccurlyeq & B \neq \emptyset \\ \emptyset & B = \emptyset \end{cases}$$

for any $B \in S$. Clearly then f is a choice function for S . \square

Exercise 8.1.2

If A can be well-ordered, then $\mathcal{P}(A)$ can be linearly ordered. [Hint: Let $<$ be a well-ordering of A ; for $X, Y \subseteq A$ define $X \prec Y$ if and only if the $<$ -least element of $X \Delta Y$ belongs to X .]

Solution:

Proof. Suppose that $<$ is a well-ordering of A . Then, following the hint, defined the relation $X \prec Y$ if and only if the $<$ -least element of $X \Delta Y$ is in X for any X and Y in $\mathcal{P}(A)$. Note that, for any $x \in X \Delta Y = (X - Y) \cup (Y - X)$, we clearly have that $x \in X$ or $x \in Y$ so that $x \in A$ since $X \subseteq A$ and $Y \subseteq A$. Hence $X \Delta Y \subseteq A$ so that it is also well-ordered by $<$.

First we show that \prec is a (strict) order on $\mathcal{P}(A)$. Hence we must show that it is asymmetric and transitive. So consider any X and Y in $\mathcal{P}(A)$ where $X \prec Y$. Then by definition the $<$ -least element x in $X \Delta Y$ is in X . Suppose also that $Y \prec X$ so that, since $Y \Delta X = X \Delta Y$, x is also in Y . Then clearly x can be neither in $X - Y$ nor $Y - X$, but then it cannot be that $x \in (X - Y) \cup (Y - X) = X \Delta Y$. This is a contradiction since x was defined to be in $X \Delta Y$. Hence it cannot be that $Y \prec X$ as well, which shows that \prec is asymmetric since X and Y were arbitrary.

To see that \prec is transitive, consider any $X, Y, Z \in \mathcal{P}(A)$ where $X \prec Y$ and $Y \prec Z$. Then the least element x of $X \Delta Y$ is in X and the least element y of $Y \Delta Z$ is in Y . Thus it has to be that $x \in X - Y$ and $y \in Y - Z$ so that $x \in X$, $x \notin Y$, $y \in Y$, and $y \notin Z$. Note that, in particular, this

means that $x \neq y$. Since clearly $\{x, y\} \subseteq A$, it follows that it has a $<$ -least element a . Thus either $a = x$ or $a = y$. For each case we show that

1. $a \in X$
2. $a \in X \triangle Z$
3. a is a lower bound of $X \triangle Z$

Case: $a = x$. Then clearly $a = x \leq y$ so that $a < y$ since $a = x \neq y$.

1. Clearly $a \in X$ since $a = x$.
2. Suppose that $a \in Z$. Then, since $a = x \notin Y$, we have that $a \in Z - Y$ so that $a \in Y \triangle Z$. Hence $y \leq a$ since y is the least element of $Y \triangle Z$, but this contradicts the fact that $y > a$. So it must be that in fact $a \notin Z$ so that $a \in X - Z$. Thus $a \in X \triangle Z$.
3. Now consider any $z \in X \triangle Z$.

Case: $z \in X - Z$. Then, if $z \in Y$, we have that $z \in Y - Z$ so that $z \in Y \triangle Z$. It then follows that $a = x \leq y \leq z$ since y is the least element of $Y \triangle Z$. On the other hand, if $z \notin Y$, then we have $z \in X - Y$ so that $z \in X \triangle Y$. Hence $a = x \leq z$ since x is the least element of $X \triangle Y$.

Case: $z \in Z - X$. Then, if $z \in Y$, we have $x \in Y - X$ so $z \in X \triangle Y$. Then $a = x \leq z$ since x is the least element of $X \triangle Y$. On the other hand, if $z \notin Y$, then we have $z \in Z - Y$ so that $z \in Y \triangle Z$. Then, as before, $a = x \leq y \leq z$ since y is the least element of $Y \triangle Z$.

Hence in all cases $a \leq z$ so that a is a lower bound since z was arbitrary.

Case: $a = y$. Then clearly $a = y \leq x$ so that $a < x$ since $a = y \neq x$.

1. Suppose that $a \notin X$. Then since $a = y \in Y$ we have that $a \in Y - X$ so that also $a \in X \triangle Y$. But then $x \leq a$ since x is the least element of $X \triangle Y$, which contradicts the fact that $x > a$. Hence it has to be that $a \in X$.
2. We already know that $a = y \notin Z$ so that $a \in X - Z$ since we just showed that $a \in X$. Hence $a \in X \triangle Z$.
3. Consider any $z \in X \triangle Z$.

Case: $z \in X - Z$. If also $z \in Y$ then clearly $z \in Y - Z$ so that $z \in Y \triangle Z$. It then follows that $a = y \leq z$ since y is the least element of $Y \triangle Z$. On the other hand, if $z \notin Y$, then clearly $z \in X - Y$ so that $z \in X \triangle Y$. Hence $a = y \leq x \leq z$ since x is the least element of $X \triangle Y$.

Case: $z \in Z - X$. Then, if $z \in Y$, clearly $z \in Y - X$ so that $z \in X \triangle Y$. We then have that $a = y \leq x \leq z$ since x is the least element of $X \triangle Y$. On the other hand, if $z \notin Y$, then $z \in Z - Y$ so that $z \in Y \triangle Z$. Then clearly $a = y \leq z$ since y is the least element of $Y \triangle Z$.

Hence in all cases $a \leq z$ so that a is a lower bound since z was arbitrary.

Thus in all cases we have that a is the least element of $X \triangle Z$ (since it is in $X \triangle Z$ and also is a lower bound) and $a \in X$. By definition, this shows that $X \prec Z$ so that \prec is transitive. This also shows that \prec is a (strict) order.

Lastly, we show that \prec is a linear ordering. So consider any $X, Y \in \mathcal{P}(A)$. Assume that $X \neq Y$ so that $X - Y \neq \emptyset$ or $Y - X \neq \emptyset$ (or both). From this it follows that $X \triangle Y \neq \emptyset$. Since also clearly $X \triangle Y \subseteq A$, it has a least element a . If $a \in X - Y$ then $a \in X$ so that $X \prec Y$. Similarly, if $a \in Y - X$ then $a \in Y$ so that $Y \prec X$. Hence we have shown that either $X = Y$, $X \prec Y$, or $Y \prec X$ so that \prec is in fact linear since X and Y were arbitrary.

This completes the proof since we have shown that \prec is a linear ordering of $\mathcal{P}(A)$. □

Exercise 8.1.3*

Let (A, \leq) be an ordered set in which every chain has an upper bound. Then for every $a \in A$, there is a \leq -maximal element x of A such that $a \leq x$.

Solution:

Lemma 8.1.3.1. *For any set A , there is a $b \notin A$.*

Proof. Let $X = \{\alpha \in A \mid \alpha \text{ is an ordinal number}\}$. Then by Theorem 6.2.6e there is an ordinal α such that $\alpha \notin X$. It also has to be that $\alpha \notin A$ since, if it were, then α would be in X since it is an ordinal number, which would be a contradiction. \square

Main Problem.

The proof of this is similar to the proof of Zorn's Lemma from the Axiom of Choice (part of Theorem 8.1.13 in the text).

Proof. First, by Lemma 8.1.3.1, there is a $b \notin A$. Also, by the Axiom of Choice, there is a choice function g on $\mathcal{P}(A)$. Now consider any $a \in A$. We then define a transfinite sequence $\langle a_\alpha \mid \alpha < h(A) \rangle$ by transfinite recursion as follows. Set $a_0 = a$. Then, having constructed the sequence $\langle a_\xi \mid \xi < \alpha \rangle$ for $0 < \alpha < h(A)$, we define the set $A_\alpha = \{x \in A \mid a_\xi < x \text{ for all } \xi < \alpha\}$. We then set

$$a_\alpha = \begin{cases} g(A_\alpha) & \text{if } a_\xi \neq b \text{ for all } \xi < \alpha \text{ and } A_\alpha \neq \emptyset \\ b & \text{otherwise.} \end{cases}$$

We claim that there is an $\alpha < h(A)$ such that $a_\alpha = b$. To see this, suppose to the contrary that $a_\alpha \neq b$ for all $\alpha < h(A)$ so that it has to be that each $a_\alpha \in A$. Consider now any $\alpha < h(A)$ and $\beta < h(A)$ where $\alpha \neq \beta$. Without loss of generality we can assume that $\alpha < \beta$. Clearly then, by definition, we have that $a_\beta \in A_\beta$ so that $a_\xi < a_\beta$ for all $\xi < \beta$. But since $\alpha < \beta$, we have that $a_\alpha < a_\beta$ so that $a_\alpha \neq a_\beta$. Since α and β were arbitrary, this shows that the sequence is an injective function from $h(A)$ to A . However, this would mean that $h(A)$ is equipotent to some subset of A , which contradicts the definition of the Hartogs number. Hence it has to be that $a_\alpha = b$ for some $\alpha < h(A)$.

So let $\lambda < h(A)$ be the least ordinal such that $a_\lambda = b$ and let $C = \{a_\xi \mid \xi < \lambda\}$. We claim that C is a chain in (A, \leq) . So consider any a_α and a_β in C so that $\alpha < \lambda$ and $\beta < \lambda$. Without loss of generality we can assume that $\alpha \leq \beta$. If $\alpha = \beta$ then obviously $a_\alpha = a_\beta$ so that $a_\alpha \leq a_\beta$ clearly holds. If $\alpha < \beta$ then, by what was shown above, we have that $a_\alpha < a_\beta$ so that $a_\alpha \leq a_\beta$ again holds. Hence, in every case, a_α and a_β are comparable in \leq , which shows that C is a chain since a_α and a_β were arbitrary.

Thus, since C is a chain of A , it has an upper bound $c \in A$. We claim that c is also a maximal element of A . To show this, suppose that there is an $x \in A$ such that $c < x$. Now consider any $\xi < \lambda$. Then, since c is an upper bound of C , we have that $a_\xi \leq c < x$ so that $a_\xi < x$ since orders are transitive. It then follows from the definition of A_λ that $x \in A_\lambda$ so that $A_\lambda \neq \emptyset$. Also note that, by the definition of λ , we have that $a_\xi \neq b$ for any $\xi < \lambda$. Thus, by the recursive definition of the sequence, it follows that $a_\lambda = g(A_\lambda) \neq b$, which contradicts the definition of λ (as the least ordinal such that $a_\lambda = b$). So it has to be that there is no such element x , which shows that c is in fact a maximal element of A .

Now, it has to be that $0 \neq \lambda$ since $a_0 = a \neq b = a_\lambda$. It then follows that $0 < \lambda$ since λ is an ordinal. Hence $a = a_0 \in C$ by the definition of C . Then, since c is an upper bound of C , we have that c is a maximal element of A where $a \leq c$. Since a was arbitrary this shows the desired result. \square

Exercise 8.1.4

Prove that Zorn's Lemma is equivalent to the statement: For all (A, \leq) , the set of all chains of (A, \leq) has an \subseteq -maximal element.

Solution:

Proof. (\rightarrow) First, suppose that Zorn's Lemma is true, and let C be the set of all chains of (A, \leq) . First, it is trivial to show that \subseteq is a partial order on C , i.e. that it is reflexive, antisymmetric, and transitive. Let $B \subseteq C$ be any \subseteq -chain, and let $U = \bigcup B$.

First we claim that that $U \in C$, which requires that we show that U is a chain of (A, \leq) . So consider any $x, y \in U = \bigcup B$ so that there are sets $X, Y \in B$ such that $x \in X$ and $y \in Y$. Since B is a \subseteq -chain it follows that either $X \subseteq Y$ or $Y \subseteq X$. In the case of $X \subseteq Y$ then clearly both x and y are in Y (since $x \in X$ and $X \subseteq Y$). Then, since $Y \in C$ (since $Y \in B$ and $B \subseteq C$), we have that Y is a \leq -chain. Hence x and y are comparable in \leq . The case in which $Y \subseteq X$ is analogous. Since $x, y \in U$ were arbitrary, this shows that U is a \leq -chain so that $U \in C$.

We also claim that U is an upper bound (with respect to \subseteq) of B . To show this, consider any $X \in B$ and any $x \in X$. Then clearly $x \in \bigcup B = U$. Hence $X \subseteq U$ since x was arbitrary. Since also $X \in B$ was arbitrary, this shows that U is an upper bound of B .

Thus, since B was an arbitrary \subseteq -chain, this shows that every chain of (C, \subseteq) has an upper bound. It then follows from Zorn's Lemma that C has a \subseteq -maximal element as desired.

(\leftarrow) Suppose that the set of all chains of (A, \leq) has a \subseteq -maximal element for any (A, \leq) . So consider any such ordered set (A, \leq) where every chain has a upper bound. Let C be the set of all chains of (A, \leq) so that C has a \subseteq -maximal element M by our initial supposition. Then, since $M \in C$, it is a chain so it has an upper bound $a \in A$. We claim that a is a maximal element of (A, \leq) .

To show this, assume to the contrary that there is a $b \in A$ such that $a < b$, and let $M' = M \cup \{b\}$. Consider any $x, y \in M'$. If $x, y \in M$ then clearly x and y are comparable in \leq since M is a chain. On the other hand, if $x \in M$ but $y = b$, then $x \leq a$ since a is an upper bound of M . We also have that $a < b = y$ so that $x < y$ since orders are transitive. The case in which $y \in M$ but $x = b$ similarly leads to $y < x$. Lastly, if $x = y = b$ then clearly $x \leq y$ is true. Hence in all cases x and y are comparable in \leq , which shows that M' is a chain since x and y were arbitrary. Therefore $M' \in C$.

Now, it has to be that $b \notin M$ since, if it were, a could not be an upper bound of M since $a < b$ (and therefore it cannot be that $b \leq a$ since the strict ordering is asymmetric). So, since $b \notin M$ it follows that $M \subset M \cup \{b\} = M'$, which contradicts the fact that M is a \subseteq -maximal element of C since also $M' \in C$. So it has to be that there is no such b where $a < b$, which shows that a is in fact a maximal element of (A, \leq) . This proves Zorn's Lemma since (A, \leq) was arbitrary. \square

Exercise 8.1.5

Prove that Zorn's Lemma is equivalent to the statement: If A is a system of sets such that, for each $B \subseteq A$ which is linearly ordered by \subseteq , $\bigcup B \in A$, then A has an \subseteq -maximal element.

Solution:

Proof. (\rightarrow) Suppose Zorn's Lemma and let A be a system of sets where $\bigcup B \in A$ for any B that is linearly ordered by \subseteq . We know that \subseteq is a partial order on A . So let B be any \subseteq -chain of A . Then we know that $\bigcup B \in A$, and we also claim that $\bigcup B$ is an upper bound of B . To see this, consider any $X \in B$ and any $x \in X$, so that clearly $x \in \bigcup B$. Hence $X \subseteq \bigcup B$ since x was arbitrary. This

shows that $\bigcup B$ is an upper bound of B since X was arbitrary. Since B was an arbitrary chain, this shows that (A, \subseteq) is an ordered set where every chain has an upper bound. Thus by Zorn's Lemma there is a \subseteq -maximal element of A as desired.

(\leftarrow) Now suppose that A has a \subseteq -maximal element for any system of sets A such that $\bigcup B \in A$ for any $B \subseteq A$ where B is linearly ordered by \subseteq . Consider any ordered set (A, \leq) and let C be the set of all chains of A . Let B be any subset of C that is linearly ordered by \subseteq , and consider any x and y in $\bigcup B$. Then there are X and Y in B such that $x \in X$ and $y \in Y$. Since B is linearly ordered by \subseteq we have that either $X \subseteq Y$ or $Y \subseteq X$. In the former case we have $x \in X \subseteq Y$ so that both x and y are in Y . Hence x and y are comparable in \leq since $Y \in B \subseteq C$ so that Y is a \leq -chain. A similar argument shows that x and y are comparable if $Y \subseteq X$. Since x and y were arbitrary this shows that $\bigcup B$ is a \leq -chain so that $\bigcup B \in C$.

Thus C is a system of sets that meet the criteria of the initial supposition since B was arbitrary. Hence C has a \subseteq -maximal element. Since (A, \leq) was an arbitrary ordered set and we have shown that the set of all chains of (A, \leq) has a \subseteq -maximal element, Zorn's Lemma follows from Exercise 8.1.4. \square

Exercise 8.1.6

A system of sets A has *finite character* if $X \in A$ if and only if every finite subset of X belongs to A . Prove that Zorn's Lemma is equivalent to the following (Tukey's Lemma): Every system of sets of finite character has an \subseteq -maximal element. [Hint: Use Exercise 8.1.5.]

Solution:

Proof. (\rightarrow) Suppose Zorn's Lemma and let A be an arbitrary system of sets of finite character. Suppose that B is any subset of A that is linearly ordered by \subseteq and let C be any finite subset of $\bigcup B$. Now, for each $x \in C$ there is a set $X_x \in B$ such that $x \in X_x$, since $x \in C \subseteq \bigcup B$. Clearly the set $D = \{X_x \mid x \in C\}$ is a subset of B so that D is also linearly ordered by \subseteq . Also clearly D is finite since C is. Hence D has a \subseteq -greatest element X . Note that the Axiom of Choice is not needed in selecting the set X_x for each $x \in C$ since we are only making a finite number of choices. So consider any $x \in C$ so that $x \in X_x \subseteq X$. Hence $x \in X$ so that $C \subseteq X$ since x was arbitrary. We also have that $X \in B \subseteq A$ so $X \in A$. Therefore C is a finite subset of X , which is an element of A , so that C is also in A since A has finite character. Since C was an arbitrary finite subset of $\bigcup B$ and $C \in A$ it follows that $\bigcup B \in A$. Hence, since B was an arbitrary linearly ordered (by \subseteq) subset of A , we have by Exercise 8.1.5 and Zorn's Lemma that A has a \subseteq -maximal element as desired.

(\leftarrow) Now suppose that every system of sets of finite character has a \subseteq -maximal element. Let (A, \leq) be any ordered set and let C be the set of all chains of (A, \leq) . Now suppose that $X \in C$ and let Y be any finite subset of X . Clearly since $X \in C$, it is linearly ordered by \leq so that Y is as well since $Y \subseteq X$. Hence $Y \in C$. Now let X' be any set such that every finite subset of X' is in C . Consider any $x, y \in X'$. Then $\{x, y\}$ is clearly a finite subset of X' so that it is in C and therefore a \leq -chain. Hence x and y are comparable in \leq , which shows that X' itself is a \leq -chain since x and y were arbitrary. Hence $X' \in C$. Thus we have just shown that $X \in C$ if and only if every finite subset of X is in C so that C has finite character by definition. Therefore, by the initial supposition, C has a \subseteq maximal element. Since again C is the set of all chains of the arbitrary (A, \leq) , Zorn's Lemma follows from Exercise 8.1.4. \square

Exercise 8.1.7*

Let E be a binary relation on a set A . Show that there exists a function $f : A \rightarrow A$ such that for all $x \in A$, $(x, f(x)) \in E$ if and only if there is some $y \in A$ such that $(x, y) \in E$.

Solution:

Proof. If $A = \emptyset$ then clearly it must be that $E = \emptyset$ since $E \subseteq A \times A = \emptyset \times \emptyset = \emptyset$. Hence $f = \emptyset$ is vacuously such a function. So assume that $A \neq \emptyset$ so that there is an $a \in A$. For any $x \in A$ define the set $Y_x = \{y \in A \mid (x, y) \in E\}$, noting that this could certainly be empty if x is not in the domain of E . Clearly $S = \{Y_x \mid x \in A\}$ is a system of sets, and so has a choice function g by the Axiom of Choice. We then define a function $f : A \rightarrow A$ by

$$f(x) = \begin{cases} g(Y_x) & Y_x \neq \emptyset \\ a & Y_x = \emptyset \end{cases}$$

for all $x \in A$. We claim that f meets the required criteria, so let x be some element of A .

(\rightarrow) Suppose that $(x, f(x)) \in E$. Then clearly for $y = f(x)$ we have that $(x, y) = (x, f(x)) \in E$. We note that, if $Y_x = \emptyset$, then $y = f(x) = a \in A$, and if $Y_x \neq \emptyset$ then $y = f(x) = g(Y_x) \in Y_x$ since g is a choice function so that again $y \in A$ since clearly $Y_x \subseteq A$.

(\leftarrow) Now suppose that there is a $y \in A$ such that $(x, y) \in E$. Then clearly by definition we have $y \in Y_x$ so that $Y_x \neq \emptyset$. Thus $f(x) = g(Y_x) \in Y_x$ since g is a choice function. We therefore have $(x, f(x)) \in E$ as desired, again by the definition of Y_x . \square

Exercise 8.1.8*

Prove that every uncountable set has a subset of cardinality \aleph_1 .

Solution:

This proof is similar to that of Theorem 8.1.4.

Proof. Let A be an uncountable set. By the Well-Ordering Principle (which is equivalent to the Axiom of Choice by Theorem 8.1.13) A can be well ordered, and so can be arranged in a bijective transfinite sequence $\langle a_\alpha \mid \alpha < \Omega \rangle$ for some ordinal Ω . Since A is then equipotent to Ω it has to be that $\omega_1 \leq \Omega$ since otherwise Ω (and therefore A) would be countable or finite. So then clearly the range of the transfinite sequence $\langle a_\alpha \mid \alpha < \omega_1 \rangle$ is a subset of A with cardinality \aleph_1 . \square

Exercise 8.1.9*

Every infinite set is equipotent to some of its proper subsets. Equivalently, Dedekind finite sets are precisely the finite sets.

Solution:

Proof. By Theorem 8.1.4, any infinite set has a countable subset so that such a set is Dedekind infinite by Exercise 5.1.10. Therefore any infinite set is equipotent to a proper subset of itself by the definition of Dedekind infinite as desired. In fact, any countable set (and by extension any infinite set) is equipotent to an infinite number of its proper subsets. To see this we note that the mapping $f(k) = k + n$ is a bijection from \mathbf{N} to $\mathbf{N} - n$, which is clearly a proper subset of \mathbf{N} , for any natural number n .

Of course the contrapositive of this is that, if a set is Dedekind finite (i.e. *not* Dedekind infinite), then the set is finite (i.e. *not* finite). \square

Exercise 8.1.10*

Let $(A, <)$ be a linearly ordered set. A sequence $\langle a_n \mid n \in \omega \rangle$ of elements of A is *decreasing* if $a_{n+1} < a_n$ for all $n \in \omega$. Prove that $(A, <)$ is a well-ordering if and only if there exists no infinite decreasing sequence in A .

Solution:

Proof. (\rightarrow) We show the contrapositive of this implication. So suppose that there is a decreasing sequence $\langle a_n \mid n \in \omega \rangle$ in A , and let X be the range of the sequence so that clearly $X \subseteq A$ and $X \neq \emptyset$. Now consider any $x \in X$ so that there is a $n \in \omega$ such that $x = a_n$. We then have that $x = a_n > a_{n+1}$ since the sequence is decreasing, noting that clearly $a_{n+1} \in X$. Since x was arbitrary this shows that X has no least element (since we have shown that $\forall x \in X \exists y \in X (x > y)$) and this is logically equivalent to $\neg \exists x \in X \forall y \in X (x \leq y)$, noting that $\neg(x > y)$ is equivalent to $x \leq y$ since the ordering is linear). Thus, since there is a nonempty subset of A that has no least element, it follows that $(A, <)$ is not a well-ordering.

(\leftarrow) We show the contrapositive of this implication as well. So suppose that $(A, <)$ is *not* a well-ordering. Then there exists a nonempty subset X of A such that X has no least element. By the Axiom of Choice the set $\mathcal{P}(X)$ has a choice function g . For any $x \in X$ we define the set $X_x = \{y \in X \mid y < x\}$.

First we claim that $X_x \neq \emptyset$ for any $x \in X$. Suppose to the contrary that there is some $x \in X$ such that $X_x = \emptyset$. Consider any other $y \in X$. Then it cannot be that $y < x$ for then $y \in X_x$ so that $X_x \neq \emptyset$. So, since the ordering is linear, it has to be that $y \geq x$. But, since y was arbitrary, this would mean that x is the least element of X , which we know cannot be since X has no least element! Therefore it has to be that indeed $X_x \neq \emptyset$ for any $x \in X$.

Now we construct a sequence $\langle a_n \mid n \in \omega \rangle$ by recursion:

$$\begin{aligned} a_0 &= g(X) \\ a_{n+1} &= g(X_{a_n}), \end{aligned}$$

noting that the recursive step is always valid since X_{a_n} is never empty, as was just shown. It is easy to see that this is a decreasing sequence. Take any $n \in \omega$ so that we have that $a_{n+1} = g(X_{a_n})$. Hence $a_{n+1} \in X_{a_n}$ since g is a choice function. By the definition of X_{a_n} it then follows that $a_{n+1} < a_n$, which shows that the sequence is decreasing since n was arbitrary. Hence we have constructed an infinite decreasing sequence in A . \square

Exercise 8.1.11*

Prove the following distributive laws (see Exercise 3.13 in Chapter 2).

$$\begin{aligned} \bigcap_{t \in T} \left(\bigcup_{s \in S} A_{t,s} \right) &= \bigcup_{f \in S^T} \left(\bigcap_{t \in T} A_{t,f(t)} \right) \\ \bigcup_{t \in T} \left(\bigcap_{s \in S} A_{t,s} \right) &= \bigcap_{f \in S^T} \left(\bigcup_{t \in T} A_{t,f(t)} \right) \end{aligned}$$

Solution:

First we show that

$$\bigcap_{t \in T} \left(\bigcup_{s \in S} A_{t,s} \right) = \bigcup_{f \in S^T} \left(\bigcap_{t \in T} A_{t,f(t)} \right).$$

Proof. First let

$$L = \bigcap_{t \in T} \left(\bigcup_{s \in S} A_{t,s} \right) \qquad R = \bigcup_{f \in S^T} \left(\bigcap_{t \in T} A_{t,f(t)} \right)$$

so that we must show that $L = R$.

(\subseteq) Consider any $x \in L$. For any $t \in T$, define the set $S_t = \{s \in S \mid x \in A_{t,s}\}$. Since $x \in L$ we have that $x \in \bigcup_{s \in S} A_{t,s}$ for all $t \in T$. Hence, for all $t \in T$, there is an $s \in S$ such that $x \in A_{t,s}$. From this it follows that $S_t \neq \emptyset$ for any $t \in T$. Now, by the Axiom of Choice, the system of sets $\{S_t \mid t \in T\}$ has a choice function g . We then define a function $f(t) = g(S_t)$ for any $t \in T$, noting that this is defined for all $t \in T$ since S_t is nonempty. Clearly then we have, for any such $t \in T$, that $f(t) = g(S_t) \in S_t \subseteq S$ so that $f(t) \in S$. Hence f is a function from T into S so that $f \in S^T$.

Now consider any specific $t \in T$ so that $f(t) \in S_t$ and hence $x \in A_{t,f(t)}$ by the definition of S_t . Thus since t was arbitrary this shows that $x \in \bigcap_{t \in T} A_{t,f(t)}$. Moreover, since $f \in S^T$ we clearly have that $x \in \bigcup_{f \in S^T} \left(\bigcap_{t \in T} A_{t,f(t)} \right) = R$. Thus, since x was arbitrary, this shows that $L \subseteq R$.

(\supseteq) Now consider any $x \in R$ so that there is an $f \in S^T$ such that $x \in \bigcap_{t \in T} A_{t,f(t)}$. Thus we have that $x \in A_{t,f(t)}$ for all $t \in T$. So consider any $t \in T$ and let $s = f(t)$, noting that $s = f(t) \in S$ since $f \in S^T$. Thus we have that $x \in A_{t,f(t)} = A_{t,s}$. Since we have shown that there is an $s \in S$ such that $x \in A_{t,s}$, it follows that $x \in \bigcup_{s \in S} A_{t,s}$. Since $t \in T$ was arbitrary we have $x \in \bigcap_{t \in T} \left(\bigcup_{s \in S} A_{t,s} \right) = L$. Hence $R \subseteq L$ since x was arbitrary. \square

Now we show that

$$\bigcup_{t \in T} \left(\bigcap_{s \in S} A_{t,s} \right) = \bigcap_{f \in S^T} \left(\bigcup_{t \in T} A_{t,f(t)} \right).$$

Proof. First let

$$L = \bigcup_{t \in T} \left(\bigcap_{s \in S} A_{t,s} \right) \qquad R = \bigcap_{f \in S^T} \left(\bigcup_{t \in T} A_{t,f(t)} \right)$$

so that we must show that $L = R$.

(\subseteq) Consider any $x \in L$ so that there is a $t \in T$ such that $x \in \bigcap_{s \in S} A_{t,s}$. Hence we have that $x \in A_{t,s}$ for all $s \in S$. Now consider any $f \in S^T$. Then we have that $f(t) \in S$ so that $x \in A_{t,f(t)}$. Therefore there is a $t \in T$ such that $x \in A_{t,f(t)}$ so that $x \in \bigcup_{t \in T} A_{t,f(t)}$. Moreover, since f was arbitrary, we have that $x \in \bigcap_{f \in S^T} \left(\bigcup_{t \in T} A_{t,f(t)} \right) = R$. This shows that $L \subseteq R$ since x was arbitrary.

(\supseteq) We show this by contrapositive. So suppose that $x \notin L$. Hence we have

$$\begin{aligned} x &\notin \bigcup_{t \in T} \left(\bigcap_{s \in S} A_{t,s} \right) \\ &\neg \exists t \in T \left(x \in \bigcap_{s \in S} A_{t,s} \right) \\ &\neg \exists t \in T \forall s \in S (x \in A_{t,s}) \end{aligned}$$

$$\forall t \in T \exists s \in S (x \notin A_{t,s}) .$$

Now, let $S_t = \{s \in S \mid x \notin A_{t,s}\}$ for each $t \in T$, noting that $S_t \neq \emptyset$ by the above for any $t \in T$. Then, by the Axiom of Choice, the system of sets $\{S_t \mid t \in T\}$ has a choice function g . We then set $f(t) = g(S_t)$ for any $t \in T$. Then clearly $f(t) = g(S_t) \in S_t \subseteq S$ for any $t \in T$ so that $f(t) \in S$ since g is a choice function. It then follows that f is a function from T into S so that $f \in S^T$. Now consider any $t \in T$ so that $f(t) = g(S_t) \in S_t$. Then, by the definition of S_t , we have that $x \notin A_{t,f(t)}$. Since $t \in T$ was arbitrary, we have thus shown that

$$\begin{aligned} & \exists f \in S^T \forall t \in T (x \notin A_{t,f(t)}) \\ & \neg \forall f \in S^T \exists t \in T (x \in A_{t,f(t)}) \\ & \neg \forall f \in S^T \left(x \in \bigcup_{t \in T} A_{t,f(t)} \right) \\ & \neg \left[x \in \bigcap_{f \in S^T} \left(\bigcup_{t \in T} A_{t,f(t)} \right) \right] \\ & x \notin \bigcap_{f \in S^T} \left(\bigcup_{t \in T} A_{t,f(t)} \right) \\ & x \notin R . \end{aligned}$$

Since x was arbitrary, this shows by contrapositive that $x \in R$ implies $x \in L$ so that $R \subseteq L$. \square

Exercise 8.1.12*

Prove that for every ordering \preceq on A , there is a linear ordering \leq on A such that $a \preceq b$ implies $a \leq b$ for all $a, b \in A$ (i.e., every partial ordering can be extended to a linear ordering).

Solution:

Proof. Let (A, \preceq) be a partially ordered set. Define the set

$$P = \{R \subseteq A \times A \mid R \text{ is a partial order of } A \text{ and } \preceq \subseteq R\} .$$

We know that \subseteq partially orders P . Now consider any \subseteq -chain C of P . If $C = \emptyset$ then, since clearly we have that $\preceq \in P$, we have that $\emptyset \subseteq \preceq$ so that \preceq is an upper bound of C . On the other hand if $C \neq \emptyset$ then there is an $R \in C$, noting that $C \subseteq P$ so that $R \in P$ as well. Then, by the definition of P , R is a partial order on A such that $\preceq \subseteq R$. Let $U = \bigcup C$. We first show that U is a partial order on A .

So consider any $a \in A$. Then $(a, a) \in R$ since R is a partial order of A (and therefore reflexive). Hence clearly then $(a, a) \in \bigcup C = U$ since $R \in C$ so that U is reflexive since a was arbitrary.

Now consider any x and y in A where $(x, y) \in U$ and $(y, x) \in U$. Then, since $U = \bigcup C$, there are $S \in C$ and $T \in C$ such that $(x, y) \in S$ and $(y, x) \in T$. Since C is a \subseteq -chain, we have that either $S \subseteq T$ or $T \subseteq S$. In the former case then we have that both $(x, y) \in T$ (since $(x, y) \in S$ and $S \subseteq T$) and $(y, x) \in T$ so that $x = y$ since T is a partial order of A (since $T \in C$ and $C \subseteq P$) and is therefore antisymmetric. The case in which $T \subseteq S$ is analogous. This shows that U is antisymmetric.

Lastly, consider x, y , and z in A such that $(x, y) \in U$ and $(y, z) \in U$. Then, again, we have that there are $S \in C$ and $T \in C$ such that $(x, y) \in S$ and $(y, z) \in T$. We also again have that $S \subseteq T$ or

$T \subseteq S$ since C is a \subseteq -chain. In the former case we have that $(x, y) \in T$ (since $(x, y) \in S$ and $S \subseteq T$) and $(y, z) \in T$ so that $(x, z) \in T$ since T is a partial order on A (again since $T \in C$ and $C \subseteq P$) and therefore transitive. Hence, since $T \in C$, clearly we have $(x, z) \in \bigcup C = U$ so that U is transitive. The case in which $T \subseteq S$ is again analogous.

Therefore we have shown that U is reflexive, antisymmetric, and transitive, and is therefore a partial order on A by definition. Now consider any $(x, y) \in \preceq$. Then $(x, y) \in R$ since $\preceq \subseteq R$. Hence $(x, y) \in \bigcup C = U$ since $R \in C$. It then follows that $\preceq \subseteq U$ since (x, y) was arbitrary. Thus U is a partial order on A and $\preceq \subseteq U$ so that by definition $U \in P$.

Now consider any $S \in C$ and any $(x, y) \in S$. Then clearly $(x, y) \in \bigcup C = U$ so that $S \subseteq U$ since (x, y) was arbitrary. Since S was arbitrary this shows that U is in fact an upper bound of C with respect to \subseteq .

Since the chain C was arbitrary, this shows that all \subseteq -chains of P have an upper bound so that the conditions of Zorn's Lemma are satisfied. We can therefore conclude that P has a \subseteq -maximal element \leq .

We claim that \leq is a linear ordering of A . So assume to the contrary that \leq is not linear so that there are a and b in A such that $(a, b) \notin \leq$ and $(b, a) \notin \leq$. Then define the relations

$$R' = \{(x, y) \in A \times A \mid x \leq a \text{ and } b \leq y\}$$

and $R = \leq \cup R'$. First, notice that $(a, a) \in \leq$ and $(b, b) \in \leq$ since \leq is reflexive so that by definition $(a, b) \in R'$ and hence $(a, b) \in R$. We claim that $R \in P$ so that we must first show that R is an order on A .

Consider any $x \in A$ so that clearly $(x, x) \in \leq$ since \leq is an ordering of A and is therefore reflexive. Hence clearly $(x, x) \in \leq \cup R' = R$ so that R is reflexive.

Now suppose any x and y in A where $(x, y) \in R$ and $(y, x) \in R$. Then clearly $(x, y) \in \leq$ or $(x, y) \in R'$ and similarly $(y, x) \in \leq$ or $(y, x) \in R'$

Case: $(x, y) \in \leq$. If $(y, x) \in \leq$ then clearly $x = y$ since \leq is an order on A and therefore is antisymmetric. The sub-case in which $(y, x) \in R'$ cannot be since, if it were, then we would have $y \leq a$ and $b \leq x$. Hence we would have that $b \leq x \leq y \leq a$ so that $b \leq a$ by transitivity, which we know is not possible by our definition of a and b .

Case: $(x, y) \in R'$. Then $x \leq a$ and $b \leq y$. Here again the sub-case in which $(y, x) \in \leq$ is impossible since we would then have $b \leq y \leq x \leq a$ so that $b \leq a$ by transitivity. Evidently the sub-case in which $(y, x) \in R'$ is also impossible since then we would have $y \leq a$ and $b \leq x$ so that $b \leq x \leq a$ as well as $b \leq y \leq a$ so that $b \leq a$ and $a \leq b$, which we know is not possible. Hence this entire case is not possible.

Thus, in the only valid case, we have that $x = y$ so that R is antisymmetric.

Now suppose $x, y,$ and z in A where $(x, y) \in R$ and $(y, z) \in R$. Then we have that $(x, y) \in \leq$ or $(x, y) \in R'$ and similarly $(y, z) \in \leq$ or $(y, z) \in R'$.

Case: $(x, y) \in \leq$. If $(y, z) \in \leq$ also then clearly we have $(x, z) \in \leq$ since \leq is an order and therefore transitive. If, on the other hand, $(y, z) \in R'$ then we have $y \leq a$ and $b \leq z$. Hence we have that $x \leq y \leq a$ so that $x \leq a$ by transitivity. We thus have $x \leq a$ and $b \leq z$ so that $(x, z) \in R'$ by definition.

Case: $(x, y) \in R'$. Then we have $x \leq a$ and $b \leq y$. If $(y, z) \in \leq$ then we have $b \leq y \leq z$ so that $b \leq z$ by transitivity. Hence $x \leq a$ and $b \leq z$ so that $(x, z) \in R'$ by definition. On the other hand, if $(y, z) \in R'$, then we have $y \leq a$ and $b \leq z$. Hence $b \leq y \leq a$ so that $b \leq a$ by transitivity, which we know is not true. Therefore this sub-case is impossible.

Hence in all valid cases we have that $(x, z) \in \leq$ or $(x, z) \in R'$ so that clearly $(x, z) \in R = \leq \cup R'$, thereby showing that R is transitive.

Therefore we have shown that R is an ordering of A . We also clearly have that $\preceq \subseteq R$ since $\preceq \subseteq \leq$ (since $\leq \in P$) and $\leq \subseteq R$ (since $R = \leq \cup R'$). Thus indeed $R \in P$.

Now, since $R = \leq \cup R'$ we clearly have that $\leq \subseteq R$. We also know that $(a, b) \in R$ but $(a, b) \notin \leq$ so that $R \neq \leq$, and hence $\leq \subset R$. However, this contradicts the fact that \leq is a maximal element of P so that it must be that \leq is in fact linear!

Lastly, consider any a and b in A where $a \preceq b$. Then $(a, b) \in \preceq$ so that $(a, b) \in \leq$ because $\preceq \subseteq \leq$ since $\leq \in P$. Thus $a \leq b$ so that \leq is the \preceq -extended linear ordering of A that we seek. \square

Exercise 8.1.13*

(Principle of Dependent Choices) If R is a binary relation on $M \neq \emptyset$ such that for each $x \in M$ there is a $y \in M$ for which xRy , then there is a sequence $\langle x_n \mid n \in \omega \rangle$ such that $x_n R x_{n+1}$ holds for all $n \in \omega$.

Solution:

Proof. Suppose such a relation R on $M \neq \emptyset$. Define the set $X_x = \{y \in M \mid xRy\}$ for each $x \in M$. Then, by the given property of R , clearly $X_x \neq \emptyset$ for any $x \in M$. Then, by the Axiom of Choice, the system of sets $\{X_x \mid x \in M\}$ has a choice function g . We also have that there is an $a \in M$ since $M \neq \emptyset$. We then define a sequence recursively as follows:

$$\begin{aligned} x_0 &= a \\ x_{n+1} &= g(X_{x_n}), \end{aligned}$$

noting that this is well defined since each X_n is nonempty.

To show that the sequence $\langle x_n \mid n \in \omega \rangle$ has the desired property, consider any $n \in \omega$. Then by the recursive definition we have that $x_{n+1} = g(X_{x_n})$ so that $x_{n+1} \in X_{x_n}$ since g is a choice function and X_{x_n} is nonempty. Then, from the definition of X_{x_n} , it follows that $x_n R x_{n+1}$. This shows the desired result since n was arbitrary. \square

Exercise 8.1.14

Assuming only the Principle of Dependent Choices, prove that every countable system of sets has a choice function (the Axiom of Countable Choice).

Solution:

Proof. Suppose that S is a countable system of sets and let $T = \{X \in S \mid X \neq \emptyset\}$. Then it suffices to show that there is a choice function h for T . To see this suppose that there is such a choice function h and define a function g on S by

$$g(X) = \begin{cases} \emptyset & X = \emptyset \\ h(X) & X \neq \emptyset \end{cases},$$

for any $X \in S$, noting that clearly $X \in T$ if $X \neq \emptyset$ so that X is in the domain of h . Then, for any nonempty $X \in S$, we have that $g(X) = h(X) \in X$ since h is a choice function. Hence g is a choice function on S .

Now we construct a choice function h for T . First note that clearly either T is finite or countable since $T \subseteq S$ and S is countable. If T is finite then it has a choice function h by Theorem 8.1.2, so

we shall assume that T is also countable. It then follows that T can be written as a sequence of sets $\langle X_n \mid n \in \omega \rangle$.

Now define the binary relation R on $\bigcup T$ by

$$R = \left\{ (x, y) \in \bigcup T \times \bigcup T \mid x \in X_n \text{ and } y \in X_{n+1} \text{ for some } n \in \omega \right\}.$$

First, since $X_0 \in T$, it is nonempty so that there is an $x \in X_0$. Then clearly $x \in \bigcup T$ so that $\bigcup T$ is nonempty. Now consider any $x \in \bigcup T$ so that there is an $n \in \omega$ such that $x \in X_n$. We then have that $X_{n+1} \neq \emptyset$ since $X_{n+1} \in T$ so that there is a $y \in X_{n+1}$, noting that clearly also $y \in \bigcup T$. It then follows from the definition of R that xRy . Since x was arbitrary this shows that the relation R on $\bigcup T$ meets the conditions of the Principle of Dependent Choices. Thus we can conclude that there is a sequence $\langle x_n \mid n \in \omega \rangle$ of elements of $\bigcup T$ where $x_n R x_{n+1}$ holds for any $n \in \omega$.

Now, since we have that $x_0 \in \bigcup T$, there is an $m \in \omega$ such that $x_0 \in X_m$. We claim that $x_n \in X_{m+n}$ for all $n \in \omega$, which we show by induction on n . First, for $n = 0$, we already know that $x_n = x_0 \in X_m = X_{m+0} = X_{m+n}$. Now suppose that $x_n \in X_{m+n}$. By the property of the sequence $\langle x_k \rangle_{k \in \omega}$ we know that $x_n R x_{n+1}$ so that, by the definition of R , we have that $x_{n+1} \in X_{(m+n)+1} = X_{m+(n+1)}$ since clearly $m+n \in \omega$ and $x_n \in X_{m+n}$. This completes the induction.

Now consider the set $T' = \{X_n \mid n \in m\}$, which is clearly a finite system of nonempty sets. Hence by Theorem 8.1.2 it has a choice function h' . Now we define a function $h : T \rightarrow \bigcup T$. For any $X \in T$ we know that there is an $n \in \omega$ such that $X = X_n$. So we set

$$h(X) = h(X_n) = \begin{cases} h'(X_n) & n < m \\ x_{n-m} & n \geq m \end{cases}.$$

We claim that h is a choice function on T . So consider any $X \in T$ (which automatically means that $X \neq \emptyset$) so that again $X = X_n$ for some $n \in \omega$. If $n < m$ then we have that $h(X) = h'(X_n) \in X_n = X$ since h' is a choice function. On the other hand, if $n \geq m$, then we note that $n - m \in \omega$ and we have $h(X) = x_{n-m} \in X_{m+(n-m)} = X_n = X$ by what was shown above. Since X was arbitrary this shows that h is a choice function on T . Hence, as shown above this means that there is a choice function g on S as desired. \square

Exercise 8.1.15

If every set is equipotent to an ordinal number, then the Axiom of Choice holds.

Solution:

Proof. Let A be any set and α be an ordinal equipotent to A . Then there is a bijection f from A to α . We can then simply order A according to f , that is order it by the relation

$$\preceq = \{(x, y) \in A \times A \mid f(x) \leq f(y)\}$$

Clearly then (A, \preceq) is isomorphic to (α, \leq) so that it is a well-ordering. Hence A can be well-ordered. Since A was an arbitrary set, this shows the Well-Ordering Principle, which is equivalent to the Axiom of Choice by Theorem 8.1.13. \square

Exercise 8.1.16

If for any sets A and B either $|A| \leq |B|$ or $|B| \leq |A|$, then the Axiom of Choice holds. [Hint: Compare A and $B = h(A)$.]

Solution:

Proof. Consider any set A . Then we have that either $|A| \leq |h(A)|$ or $|h(A)| \leq |A|$ (where $h(A)$ denotes the Hartogs number of A). Now, it cannot be that $|h(A)| \leq |A|$. For, if it were, then there would be an injection f from $h(A)$ into A . Then clearly $\text{ran } f \subseteq A$ and f is a bijection from $h(A)$ to $\text{ran } f$. Thus $h(A)$ is equipotent to $\text{ran } f \subseteq A$, which violates the definition of the Hartogs number.

Therefore it must be that $|A| \leq |h(A)|$. Hence there is an injection g from A into $h(A)$. Since $h(A)$ is an ordinal and $\text{ran } g \subseteq h(A)$, it follows that $\text{ran } g$ is a set of ordinals, which is well-ordered by Theorem 6.2.6d. So, ordering A according to g (considered as a bijection from A to $\text{ran } g$) results in a well-ordering of A . Since A was an arbitrary set, this shows the Well-Ordering Principle, from which the Axiom of Choice follows by Theorem 8.1.13. \square

Exercise 8.1.17*

If B is an infinite set and A is a subset of B such that $|A| < |B|$, then $|B - A| = |B|$.

Solution:

Proof. Since B is infinite, it follows Theorem 8.1.5 that there is a unique ordinal α such that $|B| = \aleph_\alpha$, noting that this requires the Axiom of Choice. Thus there is a bijection f from B onto ω_α . Clearly then, since $A \subseteq B$, we have that $f[A] \subseteq \omega_\alpha$. Moreover, clearly f is a bijection from A to $f[A]$ so that $|f[A]| = |A| < |B| = \aleph_\alpha$. It therefore follows from Exercise 7.2.6 that $|\omega_\alpha - f[A]| = \aleph_\alpha$. We now claim that $f[B - A] = \omega_\alpha - f[A]$.

(\subseteq) So consider any $y \in f[B - A]$ so that there is an $x \in B - A$ such that $y = f(x)$. Since clearly $x \in B$, we have that $y = f(x) \in \omega_\alpha$ since f maps B onto ω_α . Now suppose that also $y \in f[A]$ so that there is a $z \in A$ where $y = f(z)$. Then we have that $y = f(x) = f(z)$ so that $x = z$ since f is injective. Hence we have $x = z \in A$ but also $z = x \notin A$ since $z = x \in B - A$, which is a contradiction. So it must be that $y \notin f[A]$ so that indeed $y \in \omega_\alpha - f[A]$. Therefore $f[B - A] \subseteq \omega_\alpha - f[A]$ since y was arbitrary.

(\supseteq) Now consider any $y \in \omega_\alpha - f[A]$ so that $y \in \omega_\alpha$ and $y \notin f[A]$. Since f is onto ω_α , there is an $x \in B$ such that $y = f(x)$. Moreover, since $y \notin f[A]$ we can be sure that $x \notin A$. Therefore $x \in B - A$. Since $y = f(x)$ it therefore follows that $y \in f[B - A]$ so that $\omega_\alpha - f[A] \subseteq f[B - A]$ since y was arbitrary.

Thus we have shown that $f[B - A] = \omega_\alpha - f[A]$ so that clearly f is a bijection from $B - A$ onto $\omega_\alpha - f[A]$. Hence we have that $|B - A| = |\omega_\alpha - f[A]| = \aleph_\alpha = |B|$ as desired. \square

§8.2 The Axiom of Choice in Mathematics**Exercise 8.2.1**

Without using the Axiom of Choice, prove that the two definitions of closure points are equivalent if A is an open set. [Hint: X_n is open, so $X_n \cap \mathbf{Q} \neq \emptyset$, and \mathbf{Q} can be well-ordered.]

Solution:

Lemma 8.2.1.1. *If A and B are open subsets of \mathbf{R} then $A \cap B$ is open.*

Proof. First note that, if $A \cap B = \emptyset$, then this is vacuously open. Otherwise, consider any $x \in A \cap B$. Then, since A is open and $x \in A$, there is a real $\delta_1 > 0$ such that $|y - x| < \delta_1$ implies that $y \in A$ for all $y \in \mathbf{R}$. Similarly, $x \in B$ and B is open so that there is a $\delta_2 > 0$ where $|y - x| < \delta_2$ implies $y \in B$ for all $y \in \mathbf{R}$. Let $\delta = \min\{\delta_1, \delta_2\}$ so that $\delta \leq \delta_1$ and $\delta \leq \delta_2$. Then consider any $y \in \mathbf{R}$ where $|y - x| < \delta$. Then we have $|y - x| < \delta \leq \delta_1$ so that $y \in A$. Similarly, $|y - x| < \delta \leq \delta_2$ so that $y \in B$ as well. Hence $y \in A \cap B$. Since y was arbitrary and $\delta > 0$, this shows that $A \cap B$ is open. \square

Lemma 8.2.1.2. *If A is a nonempty open subset of \mathbf{R} then $A \cap \mathbf{Q} \neq \emptyset$.*

Proof. Suppose that $A \subseteq \mathbf{R}$ is nonempty and open. Then there is an $x \in A$ and, since A is open, there is a real $\delta > 0$ such that $(x - \delta, x + \delta) \subseteq A$. Now, since $\delta > 0$, we clearly have

$$\begin{aligned} -\delta &< 0 < \delta \\ x - \delta &< x + \delta. \end{aligned}$$

Since \mathbf{Q} is dense in \mathbf{R} with respect to order, there is a $q \in \mathbf{Q}$ such that $x - \delta < q < x + \delta$. Then we have $q \in (x - \delta, x + \delta)$ so that also $q \in A$ since $(x - \delta, x + \delta) \subseteq A$. Thus $q \in \mathbf{Q}$ and $q \in A$ so that $q \in A \cap \mathbf{Q}$ as desired. \square

Main Problem.

Proof. A proof of this equivalence is presented in the text as Theorem 8.2.1. Recall that, in the proof that (b) implies (a), $a \in \mathbf{R}$ is the closure point of $A \subseteq \mathbf{R}$ and $X_n = \{x \in A \mid |x - a| < 1/n\}$, and we know from (b) that each X_n is nonempty. Note that we assume that $X_0 = A \cap (-\infty, \infty) = A \cap \mathbf{R} = A$. Per the remarks after this proof, it suffices to show that the system of nonempty sets $\{X_n\}_{n \in \mathbf{N}}$ has a choice function when A is open.

First we show that each X_n is an open set. Clearly $X_0 = A$ is open, so consider any natural $n > 0$ and let $I_n = (a - 1/n, a + 1/n)$. We claim that $X_n = A \cap I_n$. To this end we have

$$\begin{aligned} x \in X_n &\leftrightarrow x \in A \wedge |x - a| < 1/n \\ &\leftrightarrow x \in A \wedge -1/n < x - a < 1/n \\ &\leftrightarrow x \in A \wedge a - 1/n < x < a + 1/n \\ &\leftrightarrow x \in A \wedge x \in I_n \\ &\leftrightarrow x \in A \cap I_n. \end{aligned}$$

for any real x , which of course shows that $X_n = A \cap I_n$. Then, since A is open and I_n is clearly an open interval, it follows from Lemma 8.2.1.1 that X_n is open as well.

Now, since each X_n is open and nonempty, it follows that $X_n \cap \mathbf{Q} \neq \emptyset$ from Lemma 8.2.1.2. Then, since \mathbf{Q} is countable, it can clearly be well-ordered. So choose a well-ordering of \mathbf{Q} . Since $X_n \cap \mathbf{Q}$ is clearly a nonempty subset of \mathbf{Q} , it then has a least element x_n according to our well-ordering. We then define a function g on $\{X_n\}_{n \in \mathbf{N}}$ by $g(X_n) = x_n$, which is clearly a choice function. \square

Exercise 8.2.2

Prove that every continuous additive function f is equal to f_a for some $a \in \mathbf{R}$.

Solution:

Proof. Consider any arbitrary continuous additive function $f : \mathbf{R} \rightarrow \mathbf{R}$. Then, by what was shown in the text, there is a real a such that $f(q) = f_a(q) = a \cdot q$ for all $q \in \mathbf{Q}$; in particular $a = f(1)$.

Now suppose to the contrary that $f \neq f_a$ so that there is an $x \in \mathbf{R}$ where $f(x) \neq f_a(x)$. So let $\varepsilon = |f(x) - f_a(x)|/2$, noting that clearly $\varepsilon > 0$ since $f(x) \neq f_a(x)$. Since f is continuous there is a real $\delta_1 > 0$ such that $|f(y) - f(x)| < \varepsilon$ for all $y \in \mathbf{R}$ where $|y - x| < \delta_1$. Also clearly f_a is also continuous so that there is a real $\delta_2 > 0$ where $|f_a(y) - f_a(x)| < \varepsilon$ for all $y \in \mathbf{R}$ where $|y - x| < \delta_2$. So let $\delta = \min\{\delta_1, \delta_2\}$. Then, since $\delta > 0$ it follows that $x - \delta < x + \delta$ so that there is a $q \in \mathbf{Q}$ where $x - \delta < q < x + \delta$ since \mathbf{Q} is order dense in \mathbf{R} . It then clearly follows that $|q - x| < \delta$ so that $|q - x| < \delta \leq \delta_1$ and $|q - x| < \delta \leq \delta_2$. Therefore $|f(q) - f(x)| < \varepsilon$ and $|f_a(q) - f_a(x)| < \varepsilon$.

We then have

$$\begin{aligned} |f(x) - f_a(x)| &\leq |f(x) - f(q)| + |f(q) - f_a(x)| \\ &\leq |f(x) - f(q)| + |f(q) - f_a(q)| + |f_a(q) - f_a(x)| \\ &< \varepsilon + 0 + \varepsilon = 2\varepsilon = |f(x) - f_a(x)|, \end{aligned}$$

which is a contradiction, noting that $|f(q) - f_a(q)| = 0$ since $f(q) = f_a(q)$ since $q \in \mathbf{Q}$. So it must be that in fact $f = f_a$ as desired. \square

This proof is similar to that of Theorem 10.3.11 later in the text. That theorem is certainly more general, and this can be easily proved from it. In particular it was shown in the text that, for an arbitrary additive and continuous $f : \mathbf{R} \rightarrow \mathbf{R}$, $f(q) = f_a(q)$ for all $q \in \mathbf{Q}$ for some $a \in \mathbf{R}$ so that $f \upharpoonright \mathbf{Q} = f_a \upharpoonright \mathbf{Q}$. Since f and f_a are both continuous and \mathbf{Q} is order dense in \mathbf{R} , it follows from Theorem 10.3.11 that $f = f_a$.

Exercise 8.2.3

Assume that μ has properties 0)-ii). Prove properties iv) and v). Also prove:

- vi) $\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$.
- vii) $\mu(\bigcup_{n=0}^{\infty} A_n) \leq \sum_{n=0}^{\infty} \mu(A_n)$.

Solution:

First, for reference, we assume the following properties of μ :

- 0) $\mu([a, b]) = b - a$ for any a and b in \mathbf{R} where $a < b$.
- i) $\mu(\emptyset) = 0$ and $\mu(\mathbf{R}) = \infty$.
- ii) If $\{A_n\}_{n=0}^{\infty}$ is a collection of mutually disjoint subsets of \mathbf{R} , then

$$\mu\left(\bigcup_{n=0}^{\infty} A_n\right) = \sum_{n=0}^{\infty} \mu(A_n).$$

First we show

- iv) If $A \cap B = \emptyset$ then $\mu(A \cup B) = \mu(A) + \mu(B)$.

Proof. Assume that $A \cap B = \emptyset$ and define $A_0 = A$, $A_1 = B$, and $A_n = \emptyset$ for all natural $n > 1$. Then clearly each of the sets in $\{A_n\}_{n=0}^{\infty}$ are mutually disjoint. It is also trivial to show that $\bigcup_{n=0}^{\infty} A_n = A \cup B$. We then have by property ii) that

$$\mu(A \cup B) = \mu\left(\bigcup_{n=0}^{\infty} A_n\right) = \sum_{n=0}^{\infty} \mu(A_n) = \mu(A_0) + \mu(A_1) + \sum_{n=2}^{\infty} \mu(A_n)$$

$$\begin{aligned}
&= \mu(A) + \mu(B) + \sum_{n=2}^{\infty} \mu(\emptyset) = \mu(A) + \mu(B) + \sum_{n=2}^{\infty} 0 \\
&= \mu(A) + \mu(B),
\end{aligned}$$

noting that we have also used property i) according to which $\mu(\emptyset) = 0$. This shows the desired result. \square

Lemma 8.2.3.1. For a measure μ , if $A \subseteq B \subseteq \mathbf{R}$ then $\mu(B - A) = \mu(B) - \mu(A)$.

Proof. Clearly A and $B - A$ are disjoint sets such that $A \cup (B - A) = B$ so that $\mu(A) + \mu(B - A) = \mu(B)$ by property iv). The result then clearly follows by subtracting $\mu(A)$ from both sides. \square

Next we show

v) If $A \subseteq B$ then $\mu(A) \leq \mu(B)$.

Proof. Suppose that $A \subseteq B$ so that $\mu(A) + \mu(B - A) = \mu(B)$ by Lemma 8.2.3.1. Since μ is a function into $[0, \infty) \cup \{\infty\}$, it follows that

$$\begin{aligned}
0 &\leq \mu(B - A) \\
\mu(A) &\leq \mu(B - A) + \mu(A) = \mu(A) + \mu(B - A) \\
\mu(A) &\leq \mu(B)
\end{aligned}$$

as desired. \square

Now we show

vi) $\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$.

Proof. Let $C = A \cap B$, $A' = A - C$, and $B' = B - C$. It is then trivial to show that A' , B' , and C are mutually disjoint sets such that $A' \cup B' \cup C = A \cup B$. We then have by a straightforward extension of property iv) that

$$\begin{aligned}
\mu(A') + \mu(B') + \mu(C) &= \mu(A \cup B) \\
\mu(A - C) + \mu(B - C) + \mu(C) &= \mu(A \cup B) \\
\mu(A) - \mu(C) + \mu(B) - \mu(C) + \mu(C) &= \mu(A \cup B) \\
\mu(A) + \mu(B) - \mu(C) &= \mu(A \cup B) \\
\mu(A) + \mu(B) - \mu(A \cap B) &= \mu(A \cup B)
\end{aligned}$$

as desired. Note that we have also used Lemma 8.2.3.1 since clearly $C \subseteq A$ and $C \subseteq B$ so that $\mu(A - C) = \mu(A) - \mu(C)$ and $\mu(B - C) = \mu(B) - \mu(C)$. \square

Lastly we show

vii) $\mu(\bigcup_{n=0}^{\infty} A_n) \leq \sum_{n=0}^{\infty} \mu(A_n)$.

Proof. Supposing that we have a system of sets $\{A_n\}_{n=0}^{\infty}$, first we define a sequence of corresponding sets recursively:

$$\begin{aligned}
A'_0 &= A_0 \\
A'_{n+1} &= A_{n+1} - \bigcup_{k=0}^n A'_k
\end{aligned}$$

Note that it is clear that $A'_n \subseteq A_n$ for any $n \in \mathbf{N}$ so that $\mu(A'_n) \leq \mu(A_n)$ by property v).

We now show that each of these sets are mutually disjoint. So consider any natural m and n where $m \neq n$. Without loss of generality, we can then assume that $m < n$. Suppose that A'_m and A'_n are *not* disjoint so that there is an $x \in A'_m \cap A'_n$. Thus $x \in A'_n = A_n - \bigcup_{k=0}^{n-1} A'_k$ so that $x \notin \bigcup_{k=0}^{n-1} A'_k$. However since also $x \in A'_m$ and $0 \leq m \leq n-1$, we also can conclude that $x \in \bigcup_{k=0}^{n-1} A'_k$. Since this is a contradiction, it must be that A'_m and A'_n are in fact disjoint, which shows mutual disjointedness since m and n were arbitrary.

Next we show that $\bigcup_{n=0}^{\infty} A'_n = \bigcup_{n=0}^{\infty} A_n$. The \subseteq direction is clear since, for any $x \in \bigcup_{n=0}^{\infty} A'_n$, there a natural n where $x \in A'_n$. Since $A'_n \subseteq A_n$ it follows that $x \in A_n$ so that clearly $x \in \bigcup_{n=0}^{\infty} A_n$. Now consider any $x \in \bigcup_{n=0}^{\infty} A_n$ so that there is a natural n where $x \in A_n$. Clearly if $n = 0$ then $x \in A_0 = A'_0$. So assume that $n > 0$ so that $\bigcup_{k=0}^{n-1} A'_k$ is defined. If $x \in \bigcup_{k=0}^{n-1} A'_k$ then there is a $0 \leq k \leq n-1$ where $x \in A'_k$. On the other hand, if $x \notin \bigcup_{k=0}^{n-1} A'_k$ then clearly $x \in A_n - \bigcup_{k=0}^{n-1} A'_k = A'_n$. Thus in all cases there is a natural k such that $x \in A'_k$ so that $x \in \bigcup_{n=0}^{\infty} A'_n$ as desired.

We therefore have

$$\begin{aligned} \mu\left(\bigcup_{n=0}^{\infty} A_n\right) &= \mu\left(\bigcup_{n=0}^{\infty} A'_n\right) \\ &= \sum_{n=0}^{\infty} \mu(A'_n) \quad (\text{by property ii) since the sets } \{A'_n\}_{n=0}^{\infty} \text{ are mutually disjoint}) \\ &\leq \sum_{n=0}^{\infty} \mu(A_n) \quad (\text{since } 0 \leq \mu(A'_n) \leq \mu(A_n) \text{ for all natural } n) \end{aligned}$$

as desired. □

Exercise 8.2.4*

Let $\mathfrak{S} = \{X \subseteq S \mid |X| \leq \aleph_0 \text{ or } |S - X| \leq \aleph_0\}$. Prove that \mathfrak{S} is a σ -algebra.

Solution:

Lemma 8.2.4.1. *If A , B , and C are sets then $A - (B - C) = (A - B) \cup (A \cap C)$.*

Proof. For any x we have

$$\begin{aligned} x \in A - (B - C) &\leftrightarrow x \in A \wedge x \notin B - C \\ &\leftrightarrow x \in A \wedge \neg(x \in B \wedge x \notin C) \\ &\leftrightarrow x \in A \wedge (x \notin B \vee x \in C) \\ &\leftrightarrow (x \in A \wedge x \notin B) \vee (x \in A \wedge x \in C) \\ &\leftrightarrow X \in A - B \vee x \in A \cap C \\ &\leftrightarrow x \in (A - B) \cup (A \cap C). \end{aligned}$$

□

Lemma 8.2.4.2. *If $A \subseteq B$ and S is another set, then $S - B \subseteq S - A$.*

Proof. Consider any $x \in S - B$ so that $x \in S$ and $x \notin B$. Then it has to be that also $x \notin A$ since otherwise it would not be that $A \subseteq B$. Hence $x \in S - A$, which shows the result since x was arbitrary. □

Main Problem.

Proof. We must show that the above definition of \mathfrak{G} satisfies the three parts of the definition of a σ -algebra:

- (a) $\emptyset \in \mathfrak{G}$ and $S \in \mathfrak{G}$.
- (b) If $X \in \mathfrak{G}$ then $S - X \in \mathfrak{G}$.
- (c) If $X_n \in \mathfrak{G}$ for all n , then $\bigcup_{n=0}^{\infty} X_n \in \mathfrak{G}$ and $\bigcap_{n=0}^{\infty} X_n \in \mathfrak{G}$.

For (a) clearly $|\emptyset| = 0 \leq \aleph_0$ so that $\emptyset \in \mathfrak{G}$, and $S - S = \emptyset$ so that $|S - S| = |\emptyset| = 0 \leq \aleph_0$. Hence $S \in \mathfrak{G}$ as well.

For (b) suppose that $X \subseteq S$ and $X \in \mathfrak{G}$. Then either $|X| \leq \aleph_0$ or $|S - X| \leq \aleph_0$. If $|X| \leq \aleph_0$ then by Lemma 8.2.4.1 we have

$$S - (S - X) = (S - S) \cup (S \cap X) = \emptyset \cup (S \cap X) = S \cap X = X$$

since $X \subseteq S$. Therefore $|S - (S - X)| = |X| \leq \aleph_0$ so that $S - X \in \mathfrak{G}$. On the other hand, if $|S - X| \leq \aleph_0$, then obviously $S - X \in \mathfrak{G}$ by definition.

Lastly, regarding (c), suppose that $\{X_n\}_{n=0}^{\infty}$ is a system of sets where each X_n is in \mathfrak{G} . Thus $|X_n| \leq \aleph_0$ or $|S - X_n| \leq \aleph_0$ for each natural n .

Now we show that $\bigcup_{n=0}^{\infty} X_n \in \mathfrak{G}$. First, if $|X_n| \leq \aleph_0$ for all natural n , then it follows from Theorem 8.1.7 that $|\bigcup_{n=0}^{\infty} X_n| \leq \aleph_0$, which of course uses the Axiom of Choice. If, on the other hand, there is a natural m such that $|X_m| \not\leq \aleph_0$, then it has to be that $|S - X_m| \leq \aleph_0$ since $X_m \in \mathfrak{G}$. Then, since clearly $X_m \subseteq \bigcup_{n=0}^{\infty} X_n$, it follows from Lemma 8.2.4.2 that $S - \bigcup_{n=0}^{\infty} X_n \subseteq S - X_m$ so that clearly $|S - \bigcup_{n=0}^{\infty} X_n| \leq |S - X_m| \leq \aleph_0$. Thus in all cases we have that either $|\bigcup_{n=0}^{\infty} X_n| \leq \aleph_0$ or $|S - \bigcup_{n=0}^{\infty} X_n| \leq \aleph_0$ so that $\bigcup_{n=0}^{\infty} X_n \in \mathfrak{G}$.

Lastly we show that $\bigcap_{n=0}^{\infty} X_n \in \mathfrak{G}$ as well. If it is the case $|S - X_n| \leq \aleph_0$ for all natural n , then clearly we have that $\bigcup_{n=0}^{\infty} (S - X_n) \leq \aleph_0$, again by Theorem 8.1.7. It also follows from Exercise 2.3.11 that $S - \bigcap_{n=0}^{\infty} X_n = \bigcup_{n=0}^{\infty} (S - X_n)$ so that we have $|S - \bigcap_{n=0}^{\infty} X_n| = |\bigcup_{n=0}^{\infty} (S - X_n)| \leq \aleph_0$. Now, on the other hand, if there is a natural m such that $|S - X_m| \not\leq \aleph_0$, then it has to be that $|X_m| \leq \aleph_0$ since $X_m \in \mathfrak{G}$. Since clearly $\bigcap_{n=0}^{\infty} X_n \subseteq X_m$, we then have $|\bigcap_{n=0}^{\infty} X_n| \leq |X_m| \leq \aleph_0$. Hence in either case we have that $|\bigcap_{n=0}^{\infty} X_n| \leq \aleph_0$ or $|S - \bigcap_{n=0}^{\infty} X_n| \leq \aleph_0$ so that $\bigcap_{n=0}^{\infty} X_n \in \mathfrak{G}$ by definition.

We have therefore shown parts (a), (b), and (c) so that \mathfrak{G} is a σ -algebra as desired. \square

Exercise 8.2.5

Let \mathfrak{C} be any collection of subsets of S . Let $\mathfrak{G} = \bigcap \{\mathfrak{T} \mid \mathfrak{C} \subseteq \mathfrak{T} \text{ and } \mathfrak{T} \text{ is a } \sigma\text{-algebra of subsets of } S\}$. Prove that \mathfrak{G} is a σ -algebra (it is called the σ -algebra generated by \mathfrak{C}).

Solution:

Proof. First, let $T = \{\mathfrak{T} \mid \mathfrak{C} \subseteq \mathfrak{T} \text{ and } \mathfrak{T} \text{ is a } \sigma\text{-algebra of subsets of } S\}$ so that $\mathfrak{G} = \bigcap T$. Then we must show that \mathfrak{G} meets the definition of a σ -algebra:

- (a) $\emptyset \in \mathfrak{G}$ and $S \in \mathfrak{G}$.
- (b) If $X \in \mathfrak{G}$ then $S - X \in \mathfrak{G}$.
- (c) If $X_n \in \mathfrak{G}$ for all n , then $\bigcup_{n=0}^{\infty} X_n \in \mathfrak{G}$ and $\bigcap_{n=0}^{\infty} X_n \in \mathfrak{G}$.

Regarding (a), consider any $\mathfrak{T} \in T$. Since \mathfrak{T} is then a σ -algebra it follows that both $\emptyset \in \mathfrak{T}$ and $S \in \mathfrak{T}$ by (a). Then, since $\mathfrak{T} \in T$ was arbitrary, it follows that both \emptyset and S are in $\bigcap T = \mathfrak{G}$.

For (b) suppose that $X \in \mathfrak{G} = \bigcap T$ so that $X \in \mathfrak{T}$ for all $\mathfrak{T} \in T$. So consider any such $\mathfrak{T} \in T$ so that clearly $X \in \mathfrak{T}$. Then, since \mathfrak{T} is then a σ -algebra, it follows that $S - X \in \mathfrak{T}$ by (b). Since $\mathfrak{T} \in T$ was arbitrary, we have that $S - X \in \bigcap T = \mathfrak{G}$.

Lastly, for part (c) of the definition, suppose that $X_n \in \mathfrak{G} = \bigcap T$ for all $n \in \mathbf{N}$. Let \mathfrak{T} be any element of T so that $X_n \in \mathfrak{T}$ for all natural n . Since \mathfrak{T} is a σ -algebra, it then follows from (c) that both $\bigcup_{n=0}^{\infty} X_n$ and $\bigcap_{n=0}^{\infty} X_n$ are in \mathfrak{T} . Since $\mathfrak{T} \in T$ was arbitrary we have that $\bigcup_{n=0}^{\infty} X_n$ and $\bigcap_{n=0}^{\infty} X_n$ are in $\bigcap T = \mathfrak{G}$.

Hence we have shown all three parts of the definition so that \mathfrak{G} is indeed a σ -algebra. \square

Exercise 8.2.6

Fix $a \in S$ and define μ on $\mathcal{P}(S)$ by: $\mu(A) = 1$ if $a \in A$, $\mu(A) = 0$ if $a \notin A$. Show that μ is a σ -additive measure on S .

Solution:

Proof. Let $\mathfrak{G} = \mathcal{P}(S)$, which we know is the largest σ -algebra of subsets of S . We must show that μ as defined above satisfies the properties of σ -additive measure on S :

- i) $\mu(\emptyset) = 0$, $\mu(S) > 0$.
- ii) If $\{X_n\}_{n=0}^{\infty}$ is a collection of mutually disjoint sets from \mathfrak{G} , then

$$\mu\left(\bigcup_{n=0}^{\infty} X_n\right) = \sum_{n=0}^{\infty} \mu(X_n).$$

To show i), we clearly have that $a \notin \emptyset$ so that by definition $\mu(\emptyset) = 0$. Also, clearly $a \in S$ so that $\mu(S) = 1 > 0$.

Regarding ii), suppose that $\{X_n\}_{n=0}^{\infty}$ is a collection of mutually disjoint sets in $\mathfrak{G} = \mathcal{P}(S)$.

Case: $a \in \bigcup_{n=0}^{\infty} X_n$. Then by definition $\mu(\bigcup_{n=0}^{\infty} X_n) = 1$. There is also an $n \in \mathbf{N}$ such that $a \in X_n$, and since the sets $\{X_k\}_{k=0}^{\infty}$ are mutually disjoint, it follows that $a \notin X_m$ for any natural $m \neq n$ (since otherwise X_n and X_m would not be disjoint). Thus we have $\mu(X_n) = 1$ while $\mu(X_m) = 0$ for all natural $m \neq n$. Hence

$$\sum_{k=0}^{\infty} \mu(X_k) = \sum_{k \in \mathbf{N}} \mu(X_k) = \sum_{k=0}^{n-1} \mu(X_k) + \mu(X_n) + \sum_{k=n+1}^{\infty} \mu(X_k) = \sum_{k=0}^{n-1} 0 + 1 + \sum_{k=n+1}^{\infty} 0 = 1.$$

Thus clearly $\mu(\bigcup_{n=0}^{\infty} X_n) = \sum_{n=0}^{\infty} \mu(X_n) = 1$.

Case: $a \notin \bigcup_{n=0}^{\infty} X_n$. Then $\mu(\bigcup_{n=0}^{\infty} X_n) = 0$ by definition. It also follows that $a \notin X_n$ for every natural X_n so that $\mu(X_n) = 0$. Hence clearly

$$\mu\left(\bigcup_{n=0}^{\infty} X_n\right) = 0 = \sum_{n=0}^{\infty} 0 = \sum_{n=0}^{\infty} \mu(X_n).$$

Thus ii) is shown in both cases so that μ is indeed a σ -additive measure on S since we also showed i). \square

Exercise 8.2.7

For $A \subseteq S$ let $\mu(A) = 0$ if $A = \emptyset$, $\mu(A) = \infty$ otherwise. Show that μ is a σ -additive measure on S .

Solution:

Proof. Let $\mathfrak{G} = \mathcal{P}(S)$, which we know is the largest σ -algebra of subsets of S . We must show that μ as defined above satisfies the properties of σ -additive measure on S :

- i) $\mu(\emptyset) = 0$, $\mu(S) > 0$.
- ii) If $\{X_n\}_{n=0}^{\infty}$ is a collection of mutually disjoint sets from \mathfrak{G} , then

$$\mu\left(\bigcup_{n=0}^{\infty} X_n\right) = \sum_{n=0}^{\infty} \mu(X_n).$$

For i) we clearly have $\mu(\emptyset) = 0$ by definition and $\mu(S) = \infty > 0$ since S is nonempty, which follows from the fact that $\mathcal{P}(S)$ is a σ -algebra of subsets of S .

Regarding ii), suppose that $\{X_n\}_{n=0}^{\infty}$ is a collection of disjoint sets in $\mathfrak{G} = \mathcal{P}(S)$.

Case: $\bigcup_{n=0}^{\infty} X_n = \emptyset$. Then by definition $\mu\left(\bigcup_{n=0}^{\infty} X_n\right) = 0$, and it also has to be that $X_n = \emptyset$ for all $n \in \mathbf{N}$ so that $\mu(X_n) = 0$. Therefore

$$\sum_{n=0}^{\infty} \mu(X_n) = \sum_{n=0}^{\infty} 0 = 0 = \mu\left(\bigcup_{n=0}^{\infty} X_n\right).$$

Case: $\bigcup_{n=0}^{\infty} X_n \neq \emptyset$. Then by definition $\mu\left(\bigcup_{n=0}^{\infty} X_n\right) = \infty$. It also follows that $X_n \neq \emptyset$ for at least one $n \in \mathbf{N}$ so that $\mu(X_n) = \infty$. We then have

$$\begin{aligned} \sum_{k=0}^{\infty} \mu(X_k) &= \sum_{k=0}^{n-1} \mu(X_k) + \mu(X_n) + \sum_{k=n+1}^{\infty} \mu(X_k) = \sum_{k=0}^{n-1} \mu(X_k) + \infty + \sum_{k=n+1}^{\infty} \mu(X_k) \\ &= \infty = \mu\left(\bigcup_{n=0}^{\infty} X_n\right) \end{aligned}$$

since each $\mu(X_m) \in \{0, \infty\}$ for all natural $m \neq n$.

Thus ii) is shown in both cases so that μ is indeed a σ -additive measure on S as desired. \square

Exercise 8.2.8

For $A \subseteq S$ let $\mu(A) = |A|$ if A is finite, $\mu(A) = \infty$ if A is infinite. μ is a σ -additive measure on S ; it is called the *counting measure* on S .

Solution:

Proof. Let $\mathfrak{G} = \mathcal{P}(S)$, which we know is the largest σ -algebra of subsets of S . We must show that μ as defined above satisfies the properties of σ -additive measure on S :

- i) $\mu(\emptyset) = 0$, $\mu(S) > 0$.

ii) If $\{X_n\}_{n=0}^{\infty}$ is a collection of mutually disjoint sets from \mathfrak{S} , then

$$\mu\left(\bigcup_{n=0}^{\infty} X_n\right) = \sum_{n=0}^{\infty} \mu(X_n).$$

For i) we have that $\mu(\emptyset) = |\emptyset| = 0$ since \emptyset is finite. If S is finite then $\mu(S) = |S| > 0$ since S is nonempty. If S is infinite then $\mu(S) = \infty > 0$ as well so that in either case $\mu(S) > 0$ as desired.

To show ii) suppose that $\{X_n\}_{n=0}^{\infty}$ is a collection of disjoint sets in $\mathfrak{S} = \mathcal{P}(S)$.

Case: There is an $m \in \mathbf{N}$ where X_m is infinite. Then obviously $\mu(X_m) = \infty$ by definition. We also clearly have that $\bigcup_{n=0}^{\infty} X_n$ is infinite since $X_m \subseteq \bigcup_{n=0}^{\infty} X_n$, and hence $\mu(\bigcup_{n=0}^{\infty} X_n) = \infty$. Then

$$\sum_{n=0}^{\infty} \mu(X_n) = \sum_{n=0}^{m-1} \mu(X_n) + \mu(X_m) + \sum_{n=m+1}^{\infty} \mu(X_n) = \sum_{n=0}^{m-1} \mu(X_n) + \infty + \sum_{n=m+1}^{\infty} \mu(X_n) = \infty$$

since $\mu(X_n) \in \mathbf{N} \cup \{\infty\}$ for every $n \neq m$. Therefore $\mu(\bigcup_{n=0}^{\infty} X_n) = \sum_{n=0}^{\infty} \mu(X_n) = \infty$ as desired.

Case: X_n is finite for every $n \in \mathbf{N}$. Clearly then $\mu(X_n) = |X_n|$ for every natural n . First, if there is a natural N such that $X_n = \emptyset$ for all $n > N$, then clearly $\bigcup_{n=0}^{\infty} X_n = \bigcup_{n=0}^N X_n$, i.e. the union is finite. It then follows that $\bigcup_{n=0}^N X_n$ is finite by Theorem 4.2.7 so that $\mu\left(\bigcup_{n=0}^N X_n\right) = \left|\bigcup_{n=0}^N X_n\right|$.

Then, since the sets $\{X_n\}_{n=0}^N$ are mutually disjoint, we have

$$\mu\left(\bigcup_{n=0}^{\infty} X_n\right) = \mu\left(\bigcup_{n=0}^N X_n\right) = \left|\bigcup_{n=0}^N X_n\right| = \sum_{n=0}^N |X_n| = \sum_{n=0}^N \mu(X_n) = \sum_{n=0}^{\infty} \mu(X_n)$$

by the definition of cardinal addition, noting that clearly the last step follows from the fact that $\mu(X_n) = |\emptyset| = 0$ for all $n > N$.

On the other hand, if there is no such N , then it follows that, for every $N \in \mathbf{N}$, there is a natural $n > N$ such that $X_n \neq \emptyset$. At this point we need two facts from real analysis, supposing that $\langle a_n \rangle_{n=0}^{\infty}$ is a real sequence:

1. By definition, the sequence converges to a real a if, for every real $\varepsilon > 0$, there is an $N \in \mathbf{N}$ such that $|a_n - a| < \varepsilon$ for every natural $n \geq N$.
2. If the infinite series $\sum_{n=0}^{\infty} a_n$ converges (to a finite value) then the sequence itself must converge to zero.

We shall show that $\sum_{n=0}^{\infty} \mu(X_n)$ diverges by showing that the sequence $\langle \mu(X_n) \rangle_{n=0}^{\infty}$ does *not* converge to zero (i.e. the contrapositive of 2). So let $\varepsilon = 1/2$, noting that clearly $\varepsilon = 1/2 > 0$. Then consider any natural N so that there is a natural $n > N$ such that $X_n \neq \emptyset$. It then follows that, since X_n is finite but nonempty, $|\mu(X_n) - 0| = |\mu(X_n)| = ||X_n|| = |X_n| \geq 1 \geq 1/2 = \varepsilon$. Therefore we have shown

$$\begin{aligned} & \exists \varepsilon > 0 \forall N \in \mathbf{N} \exists n \geq N (|\mu(X_n) - 0| \geq \varepsilon) \\ & \neg \forall \varepsilon > 0 \exists N \in \mathbf{N} \forall n \geq N (|\mu(X_n) - 0| < \varepsilon), \end{aligned}$$

which shows by definition that $\langle \mu(X_n) \rangle_{n=0}^{\infty}$ does not converge to zero. Hence $\sum_{n=0}^{\infty} \mu(X_n)$ diverges so that by convention $\sum_{n=0}^{\infty} \mu(X_n) = \infty$.

Lastly, we show that $\bigcup_{n=0}^{\infty} X_n$ must be infinite. Suppose to the contrary that $\bigcup_{n=0}^{\infty} X_n$ is finite. We then construct a function $f : \bigcup_{n=0}^{\infty} X_n \rightarrow \mathbf{N}$ as follows: for each $x \in \bigcup_{n=0}^{\infty} X_n$ there is a unique

$m \in \mathbf{N}$ where $x \in X_m$. Clearly such an m exists since $x \in \bigcup_{n=0}^{\infty} X_n$, and it is unique because the sets $\{X_n\}_{n=0}^{\infty}$ are mutually disjoint (if it was not unique then there would be distinct n and m where $x \in X_n$ and $x \in X_m$ so that $X_n \cap X_m \neq \emptyset$). We then simply set $f(x) = m$.

It then follows from Theorem 4.2.5 that $\text{ran}(f)$ is finite since $\text{dom}(f) = \bigcup_{n=0}^{\infty} X_n$ is. Since $\text{ran}(f)$ is then a finite set of natural numbers, it has greatest natural number N . But we know that there is an $m > N$ such that $X_m \neq \emptyset$ so that there is an $x \in X_m$. It then follows that clearly $x \in \bigcup_{n=0}^{\infty} X_n$ and that $f(x) = m$. However, then m would be in $\text{ran}(f)$ so that $m \leq N$ since N is the greatest element of $\text{ran}(f)$. But we already know that $m > N$, which is a contradiction. So it has to be that $\bigcup_{n=0}^{\infty} X_n$ is in fact infinite as desired.

We therefore have $\mu(\bigcup_{n=0}^{\infty} X_n) = \infty = \sum_{n=0}^{\infty} \mu(X_n)$ and hence ii) has been shown in every case and sub-case so that μ is indeed a σ -additive measure on S by definition. \square

Chapter 9 Arithmetic of Cardinal Numbers

§9.1 Infinite Sums and Products of Cardinal Numbers

Exercise 9.1.1

If J_i ($i \in I$) are mutually disjoint sets and $J = \bigcup_{i \in I} J_i$, and if κ_j ($j \in J$) are cardinals, then

$$\sum_{i \in I} \left(\sum_{j \in J_i} \kappa_j \right) = \sum_{j \in J} \kappa_j$$

(associativity of \sum)

Solution:

Proof. Suppose that $\langle A_j \mid j \in J \rangle$ are mutually disjoint sets where $|A_j| = \kappa_j$ for every $j \in J$. Then, by definition, we have that

$$\sum_{j \in J} \kappa_j = \left| \bigcup_{j \in J} A_j \right|. \quad (1)$$

Now let $S = \{J_i \mid i \in I\}$, from which it is trivial to show that $\bigcup S = J$. It follows from Exercise 2.3.10 that

$$\bigcup_{j \in J} A_j = \bigcup_{a \in \bigcup S} A_a = \bigcup_{C \in S} \left(\bigcup_{a \in C} A_a \right) = \bigcup_{i \in I} \left(\bigcup_{j \in J_i} A_j \right). \quad (2)$$

We claim that the sets $\langle \bigcup_{j \in J_i} A_j \mid i \in I \rangle$ are mutually disjoint. So consider any i_1 and i_2 in I where $i_1 \neq i_2$, and suppose that $\bigcup_{j \in J_{i_1}} A_j$ and $\bigcup_{j \in J_{i_2}} A_j$ are *not* disjoint so that there is an x where $x \in \bigcup_{j \in J_{i_1}} A_j$ and $x \in \bigcup_{j \in J_{i_2}} A_j$. Then there is a $j_1 \in J_{i_1}$ where $x \in A_{j_1}$ and a $j_2 \in J_{i_2}$ where $x \in A_{j_2}$. Now, since $\langle J_i \mid i \in I \rangle$ are mutually disjoint and $i_1 \neq i_2$, it follows that J_{i_1} and J_{i_2} are disjoint. Therefore it has to be that $j_1 \neq j_2$ (since $j_1 \in J_{i_1}$ and $j_2 \in J_{i_2}$). But then A_{j_1} and A_{j_2} are not disjoint (since x is in both) despite the fact that $j_1 \neq j_2$, which contradicts the fact that

$\langle A_j \mid j \in J \rangle$ are mutually disjoint. So it must be that in that $\bigcup_{j \in J_{i_1}} A_j$ and $\bigcup_{j \in J_{i_2}} A_j$ are disjoint, which proves the result since i_1 and i_2 were arbitrary.

Since $\langle \bigcup_{j \in J_i} A_j \mid i \in I \rangle$ have been shown to be mutually disjoint, it follows by definition that

$$\sum_{i \in I} \left| \bigcup_{j \in J_i} A_j \right| = \left| \bigcup_{i \in I} \left(\bigcup_{j \in J_i} A_j \right) \right|. \quad (3)$$

Lastly, we also clearly have that $\langle A_j \mid j \in J_i \rangle$ are mutually disjoint for any $i \in I$ (since $\langle A_j \mid j \in J \rangle$ are mutually disjoint) so that

$$\sum_{j \in J_i} \kappa_j = \left| \bigcup_{j \in J_i} A_j \right|. \quad (4)$$

Putting this all together, we have

$$\sum_{j \in J} \kappa_j = \left| \bigcup_{j \in J} A_j \right| \quad (\text{by (1)})$$

$$= \left| \bigcup_{i \in I} \left(\bigcup_{j \in J_i} A_j \right) \right| \quad (\text{by (2)})$$

$$= \sum_{i \in I} \left| \bigcup_{j \in J_i} A_j \right| \quad (\text{by (3)})$$

$$= \sum_{i \in I} \left(\sum_{j \in J_i} \kappa_j \right) \quad (\text{by (4)})$$

as desired. □

Exercise 9.1.2

If $\kappa_i \leq \lambda_i$ for all $i \in I$ then $\sum_{i \in I} \kappa_i \leq \sum_{i \in I} \lambda_i$.

Solution:

Proof. Suppose that $\langle A_i \mid i \in I \rangle$ are mutually disjoint sets such that $|A_i| = \kappa_i$ for all $i \in I$. Similarly, suppose that $\langle B_i \mid i \in I \rangle$ are mutually disjoint sets such that $|B_i| = \lambda_i$ for all $i \in I$. It then follows by definition that

$$\sum_{i \in I} \kappa_i = \left| \bigcup_{i \in I} A_i \right| \qquad \sum_{i \in I} \lambda_i = \left| \bigcup_{i \in I} B_i \right|.$$

Now, we have $|A_i| = \kappa_i \leq \lambda_i = |B_i|$ so that there is an injective function $f_i : A_i \rightarrow B_i$ for all $i \in I$. With the help of the Axiom of Choice, we can choose one of these functions for each $i \in I$ and form the system of functions $\{f_i\}_{i \in I}$.

We claim that $\{f_i\}_{i \in I}$ is a compatible system of functions. To see this, consider any i_1 and i_2 in I . If $i_1 = i_2$ then consider any $x \in \text{dom}(f_{i_1}) \cap \text{dom}(f_{i_2}) = A_{i_1} \cap A_{i_2} = A_{i_1} \cap A_{i_1} = A_{i_1}$. Then clearly $f_{i_1}(x) = f_{i_2}(x)$ since $i_1 = i_2$ and $f_{i_1} = f_{i_2}$ is a function. On the other hand, if $i_1 \neq i_2$, then we have that $\text{dom}(f_{i_1}) \cap \text{dom}(f_{i_2}) = A_{i_1} \cap A_{i_2} = \emptyset$ since $\langle A_i \mid i \in I \rangle$ are mutually disjoint and $i_1 \neq i_2$. Hence it is vacuously true that $f_{i_1}(x) = f_{i_2}(x)$ for all $x \in \text{dom}(f_{i_1}) \cap \text{dom}(f_{i_2})$ since there is no such x . Since i_1 and i_2 were arbitrary, this shows that $\{f_i\}_{i \in I}$ is a compatible system (see Definition 2.3.10). It then follows from Theorem 2.3.12 that $f = \bigcup_{i \in I} f_i$ is a function with domain $\bigcup_{i \in I} \text{dom}(f_i) = \bigcup_{i \in I} A_i$.

Though perhaps it may seem obvious, we show formally that $f(x) = f_i(x)$ for any $x \in A_i$ (for any $i \in I$). So consider any such $i \in I$ and $x \in A_i$. Then $(x, f(x)) \in f = \bigcup_{i \in I} f_i$ so that there is a $j \in I$ where $(x, f(x)) \in f_j$. Suppose for a moment that $j \neq i$ so that $x \in \text{dom}(f_j) = A_j$. Since $x \in A_i$ and $x \in A_j$ but $i \neq j$, this contradicts the fact that $\langle A_i \mid i \in I \rangle$ are mutually disjoint. Hence it must be that $j = i$ so that $(x, f(x)) \in f_j = f_i$. From this of course it follows that $f_i(x) = f(x)$ as desired.

We also claim that $f(x) \in \bigcup_{i \in I} B_i$ for any $x \in \bigcup_{i \in I} A_i$ so that $\bigcup_{i \in I} B_i$ can be the codomain of f . This is easy to show: consider any $x \in \bigcup_{i \in I} A_i$ so that there is an $i \in I$ where $x \in A_i$. It then follows that $f(x) = f_i(x) \in B_i$ since f_i is a function from A_i to B_i . Therefore we clearly have $f(x) \in \bigcup_{i \in I} B_i$. This shows the result since x was arbitrary.

We also claim that f is injective. So consider any x_1 and x_2 in $\bigcup_{i \in I} A_i$ where $x_1 \neq x_2$. Then there are i_1 and i_2 such that $x_1 \in A_{i_1}$ and $x_2 \in A_{i_2}$. If $i_1 = i_2$ then $f(x_1) = f_{i_1}(x_1) \neq f_{i_1}(x_2) = f_{i_2}(x_2) = f(x_2)$ since $f_{i_1} = f_{i_2}$ is injective. If $i_1 \neq i_2$ then $f(x_1) = f_{i_1}(x_1) \in B_{i_1}$ whereas $f(x_2) = f_{i_2}(x_2) \in B_{i_2}$. Since $\{B_i \mid i \in I\}$ are mutually disjoint and $i_1 \neq i_2$ it follows that $f(x_1) \neq f(x_2)$. Therefore f is an injective function from $\bigcup_{i \in I} A_i$ to $\bigcup_{i \in I} B_i$ so that

$$\sum_{i \in I} \kappa_i = \left| \bigcup_{i \in I} A_i \right| \leq \left| \bigcup_{i \in I} B_i \right| = \sum_{i \in I} \lambda_i$$

as desired. □

Exercise 9.1.3

Find some cardinals κ_n, λ_n ($n \in \mathbf{N}$) such that $\kappa_n < \lambda_n$ for all n , but $\sum_{n=0}^{\infty} \kappa_n = \sum_{n=0}^{\infty} \lambda_n$.

Solution:

Let $\kappa_n = 1$ and $\lambda_n = 2$ for all $n \in \mathbf{N}$. We claim that these satisfy the required properties.

Proof. Clearly we have $\kappa_n = 1 < 2 = \lambda_n$ for all $n \in \mathbf{N}$. It then follows from Exercise 9.1.4 (and also the more general Theorem 9.1.3) that

$$\sum_{n=0}^{\infty} \kappa_n = \sum_{n=0}^{\infty} 1 = \sum_{n \in \mathbf{N}} 1 = \sum_{n < \aleph_0} 1 = \aleph_0 \cdot 1 = \aleph_0 = \aleph_0 \cdot 2 = \sum_{n < \aleph_0} 2 = \sum_{n \in \mathbf{N}} 2 = \sum_{n=0}^{\infty} 2 = \sum_{n=0}^{\infty} \lambda_n$$

as desired. □

Exercise 9.1.4

Prove that $\kappa + \kappa + \cdots$ (λ times) $= \lambda \cdot \kappa$.

Solution:

Proof. First, if $\lambda = 0$ then, by convention, we have

$$\kappa + \kappa + \cdots (\lambda \text{ times}) = \kappa + \kappa + \cdots (0 \text{ times}) = 0 = 0 \cdot \kappa = \lambda \cdot \kappa$$

regardless of what κ is. So assume in what follows that $\lambda > 0$ so $\lambda \geq 1$.

For each $\alpha < \lambda$, define $A_\alpha = \{(\alpha, \beta) \mid \beta \in \kappa\}$. Consider then any $\alpha_1 < \lambda$ and $\alpha_2 < \lambda$ where $\alpha_1 \neq \alpha_2$. Suppose that both $(x, y) \in A_{\alpha_1}$ and $(x, y) \in A_{\alpha_2}$. It then follows that $x = \alpha_1$ and $x = \alpha_2$ so that $x = \alpha_1 = \alpha_2$, which contradicts our assumption that $\alpha_1 \neq \alpha_2$! So it must be that no such (x, y) exists so that A_{α_1} and A_{α_2} are disjoint. Since α_1 and α_2 were arbitrary, this shows that $\langle A_\alpha \mid \alpha < \lambda \rangle$ are mutually disjoint sets. We also clearly have that $|A_\alpha| = \kappa$ for each $\alpha < \lambda$. It therefore follows from the definition of cardinal summation that

$$\kappa + \kappa + \cdots (\lambda \text{ times}) = \sum_{\alpha < \lambda} \kappa = \left| \bigcup_{\alpha < \lambda} A_\alpha \right|.$$

Now we show that $\bigcup_{\alpha < \lambda} A_\alpha = \lambda \times \kappa$. First consider any $(x, y) \in \bigcup_{\alpha < \lambda} A_\alpha$ so that there is an $\alpha < \lambda$ where $(x, y) \in A_\alpha$. We then have that $x = \alpha$ and $y \in \kappa$. Therefore $x = \alpha < \lambda$ so that $x \in \lambda$ by the definition of $<$ for ordinal numbers. Hence $x \in \lambda$ and $y \in \kappa$ so that $(x, y) \in \lambda \times \kappa$, which shows that $\bigcup_{\alpha < \lambda} A_\alpha \subseteq \lambda \times \kappa$ since (x, y) was arbitrary. Now consider any $(x, y) \in \lambda \times \kappa$ so that $x \in \lambda$ and $y \in \kappa$. Let $\alpha = x \in \lambda$ so that $\alpha < \lambda$. Hence $(x, y) = (\alpha, y)$ for $\alpha < \lambda$ and $y \in \kappa$, which shows that $(x, y) \in A_\alpha$ so that clearly $(x, y) \in \bigcup_{\alpha < \lambda} A_\alpha$. This shows that $\lambda \times \kappa \subseteq \bigcup_{\alpha < \lambda} A_\alpha$ since again (x, y) was arbitrary. Thus $\bigcup_{\alpha < \lambda} A_\alpha = \lambda \times \kappa$ as desired.

Putting all this together, we have

$$\kappa + \kappa + \cdots (\lambda \text{ times}) = \sum_{\alpha < \lambda} \kappa = \left| \bigcup_{\alpha < \lambda} A_\alpha \right| = |\lambda \times \kappa| = \lambda \cdot \kappa$$

by the definition of cardinal multiplication since obviously $|\lambda| = \lambda$ and $|\kappa| = \kappa$. □

Exercise 9.1.5

Prove the *distributive* law:

$$\lambda \cdot \left(\sum_{i \in I} \kappa_i \right) = \sum_{i \in I} (\lambda \cdot \kappa_i).$$

Solution:

Proof. Suppose that $\langle A_i \mid i \in I \rangle$ are mutually disjoint sets such that $|A_i| = \kappa_i$ for all $i \in I$. Then by definition $\sum_{i \in I} \kappa_i = \left| \bigcup_{i \in I} A_i \right|$. Also suppose that B is a set such that $|B| = \lambda$.

We claim first that $B \times \bigcup_{i \in I} A_i = \bigcup_{i \in I} (B \times A_i)$. This is easy to show since, for any x and y , we have

$$\begin{aligned} (x, y) \in B \times \bigcup_{i \in I} A_i &\leftrightarrow x \in B \wedge \exists i \in I (y \in A_i) \\ &\leftrightarrow \exists i \in I (x \in B \wedge y \in A_i) \end{aligned}$$

$$\leftrightarrow \exists i \in I ((x, y) \in B \times A_i)$$

$$\leftrightarrow (x, y) \in \bigcup_{i \in I} (B \times A_i).$$

We then have

$$\lambda \cdot \sum_{i \in I} \kappa_i = \lambda \cdot \left| \bigcup_{i \in I} A_i \right| \quad (\text{by the definition of cardinal summation})$$

$$= \left| B \times \bigcup_{i \in I} A_i \right| \quad (\text{by the definition of cardinal multiplication})$$

$$= \left| \bigcup_{i \in I} (B \times A_i) \right| \quad (\text{by what was just shown above})$$

$$= \sum_{i \in I} |B \times A_i| \quad (\text{by the definition of cardinal summation})$$

$$= \sum_{i \in I} (\lambda \cdot \kappa_i) \quad (\text{by the definition of cardinal multiplication})$$

as desired. We note that $|\bigcup_{i \in I} (B \times A_i)| = \sum_{i \in I} |B \times A_i|$ works since the sets $\langle B \times A_i \mid i \in I \rangle$ are mutually disjoint. This is easy to see by considering i and j in I where $i \neq j$. Then, if $(x, y) \in B \times A_i$ and also $(x, y) \in B \times A_j$, it follows that $y \in A_i$ and $y \in A_j$, which cannot be since $i \neq j$ and $\langle A_i \mid i \in I \rangle$ are mutually disjoint. Hence it must be that there is no such ordered pair (x, y) so that $B \times A_i$ and $B \times A_j$ are disjoint, which proves the result since i and j were arbitrary. \square

Exercise 9.1.6

$$\left| \bigcup_{i \in I} A_i \right| \leq \sum_{i \in I} |A_i|.$$

Solution:

Proof. First let $\langle B_i \mid i \in I \rangle$ be mutually disjoint sets where $|B_i| = |A_i|$ for every $i \in I$. It then follows that $\sum_{i \in I} |A_i| = |\bigcup_{i \in I} B_i|$ by definition. For each $i \in I$ we can choose a bijection f_i from A_i to B_i by the Axiom of Choice since $|A_i| = |B_i|$. We construct a function $f : \bigcup_{i \in I} A_i \rightarrow \bigcup_{i \in I} B_i$. For each $x \in \bigcup_{i \in I} A_i$ we have that $x \in A_j$ for some $j \in I$. We choose one such j , which requires the Axiom of Choice, and set $f(x) = f_j(x)$. Clearly $f(x) = f_j(x) \in B_j$ so that then $f(x) \in \bigcup_{i \in I} B_i$, which shows that $\bigcup_{i \in I} B_i$ can be the codomain for f .

We show that f is injective. So consider any x_1 and x_2 in $\bigcup_{i \in I} A_i$ where $x_1 \neq x_2$. Then we have chosen unique j_1 and j_2 where $f(x_1) = f_{j_1}(x_1)$ and $f(x_2) = f_{j_2}(x_2)$. If $j_1 = j_2$ then we have $f(x_1) = f_{j_1}(x_1) \neq f_{j_1}(x_2) = f_{j_2}(x_2) = f(x_2)$ since $f_{j_1} = f_{j_2}$ is injective and $x_1 \neq x_2$. If $j_1 \neq j_2$ then $f(x_1) = f_{j_1}(x_1) \in B_{j_1}$ whereas $f(x_2) = f_{j_2}(x_2) \in B_{j_2}$. Since $j_1 \neq j_2$ and $\langle B_i \mid i \in I \rangle$ are mutually disjoint, it follows that $f(x_1) \neq f(x_2)$. Since this is true in both cases and x_1 and x_2 were arbitrary, it shows that f is injective.

Since f is injective we, we have

$$\left| \bigcup_{i \in I} A_i \right| \leq \left| \bigcup_{i \in I} B_i \right| = \sum_{i \in I} |A_i|$$

as desired. \square

Exercise 9.1.7

If J_i ($i \in I$) are mutually disjoint sets and $J = \bigcup_{i \in I} J_i$, and if κ_j ($j \in J$) are cardinals, then

$$\prod_{i \in I} \left(\prod_{j \in J_i} \kappa_j \right) = \prod_{j \in J} \kappa_j$$

(associativity of \prod).

Solution:

Proof. Suppose that $\langle A_j \mid j \in J \rangle$ are sets where $|A_j| = \kappa_j$ for each $j \in J$. It then follows that

$$\prod_{j \in J} \kappa_j = \left| \prod_{j \in J} A_j \right|. \quad (5)$$

Now, for any $i \in I$, set $B_i = \prod_{j \in J_i} A_j$ so that

$$\prod_{j \in J_i} \kappa_j = \left| \prod_{j \in J_i} A_j \right| = |B_i| \quad (6)$$

by definition.

Now we construct a bijection f from $\prod_{j \in J} A_j$ to $\prod_{i \in I} B_i$. So consider any $a \in \prod_{j \in J} A_j$ so that $a = \langle a_j \mid j \in J \rangle$ where $a_j \in A_j$ for every $j \in J$. Now, for each $i \in I$, set $a'_i = \langle a_j \mid j \in J_i \rangle$, noting that clearly $j \in J = \bigcup_{i \in I} J_i$ for each $j \in J_i$ so that a_j has been defined as in the range of a . We then have that $a_j \in A_j$ for $j \in J_i$ so that $a'_i \in \prod_{j \in J_i} A_j = B_i$. So set $b = \langle a'_i \mid i \in I \rangle$ so that clearly $b \in \prod_{i \in I} B_i$, and set $f(a) = b$. Since $f(a) = b \in \prod_{i \in I} B_i$ for any $a \in \prod_{j \in J} A_j$, we have that f is indeed a function from $\prod_{j \in J} A_j$ into $\prod_{i \in I} B_i$.

We claim first that f is injective. So consider any α and β in $\prod_{j \in J} A_j$ where $\alpha \neq \beta$. It then follows that $\alpha = \langle \alpha_j \mid j \in J \rangle$ and $\beta = \langle \beta_j \mid j \in J \rangle$ where both α_j and β_j are in A_j for any $j \in J$. Now, since $\alpha \neq \beta$ it follows that there is a $j_0 \in J$ such that $\alpha_{j_0} \neq \beta_{j_0}$. Also there is an $i_0 \in I$ such that $j_0 \in J_{i_0}$ since $j_0 \in J = \bigcup_{i \in I} J_i$. Now let $\alpha'_i = \langle \alpha_j \mid j \in J_i \rangle$ and $\beta'_i = \langle \beta_j \mid j \in J_i \rangle$ for $i \in I$. We then have that $\alpha'_{i_0} \neq \beta'_{i_0}$ since $j_0 \in J_{i_0}$ and $\alpha_{j_0} \neq \beta_{j_0}$. Clearly then $f(\alpha) = \langle \alpha'_i \mid i \in I \rangle$ and $f(\beta) = \langle \beta'_i \mid i \in I \rangle$ by definition so that $f(\alpha) \neq f(\beta)$ since $i_0 \in I$ and $\alpha'_{i_0} \neq \beta'_{i_0}$. This shows that f is injective since α and β were arbitrary.

We also claim that f is onto. Consider any $b \in \prod_{i \in I} B_i$ so that $b = \langle b_i \mid i \in I \rangle$ where each $b_i \in B_i$ for $i \in I$. So, for any $i \in I$, we have that $b_i \in B_i = \prod_{j \in J_i} A_j$ so that $b_i = \langle a_{ij} \mid j \in J_i \rangle$ where each $a_{ij} \in A_j$ for $j \in J_i$. Now we construct a function g . So consider any $j_0 \in J$ so that there is a unique $i_0 \in I$ such that $j_0 \in J_{i_0}$, where the uniqueness clearly follows from the fact that $\langle J_i \mid i \in I \rangle$ are mutually disjoint. Then simply set $g(j_0) = a_{i_0 j_0} \in A_{j_0}$ so that clearly $g \in \prod_{j \in J} A_j$. If we then set $a'_i = \langle g(j) \mid j \in J_i \rangle = \langle a_{ij} \mid j \in J_i \rangle = b_i$ for all $i \in I$, then $f(g) = \langle a'_i \mid i \in I \rangle = \langle b_i \mid i \in I \rangle = b$. Since b was arbitrary this shows that f is indeed onto.

It may not have been obvious, but the uniqueness of $i_0 \in I$ for any $j_0 \in J$ (such that $j_0 \in J_{i_0}$) when constructing g was critical for this proof. To see why, suppose that for some $j_0 \in J$ there are distinct i_1 and i_2 in I such that $j_0 \in J_{i_1}$ and $j_0 \in J_{i_2}$. Then it could very well be that $a_{i_1 j_0} \neq a_{i_2 j_0}$ (though they would both be in A_{j_0}) and we would have to choose one to be $g(j_0)$. Supposing we choose $g(j_0) = a_{i_1 j_0}$ then we would have $a'_{i_2} = \langle g(j) \mid j \in J_{i_2} \rangle$ so that $a'_{i_2} \neq b_{i_2}$ since $a'_{i_2}(j_0) = g(j_0) = a_{i_1 j_0} \neq a_{i_2 j_0} = b_{i_2}(j_0)$. If we had set $g(j_0) = a_{i_2 j_0}$ instead then, by the same argument, we would have $a'_{i_1} \neq b_{i_1}$. Clearly in either case this would break the proof since we would

have $f(g) = \langle a'_i \mid i \in I \rangle \neq \langle b_i \mid i \in I \rangle = b$. In fact, in this case there would be no $g \in \prod_{j \in J} A_j$ such that $f(g) = b$ since we cannot choose a value for $g(j_0)$ for which $a'_i = b_i$ for all $i \in I$.

Returning from our digression, we have shown that f is a bijection from $\prod_{j \in J} A_j$ to $\prod_{i \in I} B_i$ so that

$$\left| \prod_{j \in J} A_j \right| = \left| \prod_{i \in I} B_i \right|. \quad (7)$$

Therefore we have

$$\begin{aligned} \prod_{i \in I} \left(\prod_{j \in J_i} \kappa_j \right) &= \prod_{i \in I} |B_i| && \text{(by (6))} \\ &= \left| \prod_{i \in I} B_i \right| && \text{(by the definition of cardinal product)} \\ &= \left| \prod_{j \in J} A_j \right| && \text{(by (7))} \\ &= \prod_{j \in J} \kappa_j && \text{(by (5))} \end{aligned}$$

as desired. □

Exercise 9.1.8

If $\kappa_i \leq \lambda_i$ for all $i \in I$, then

$$\prod_{i \in I} \kappa_i \leq \prod_{i \in I} \lambda_i.$$

Solution:

Proof. Suppose sets $\langle A_i \mid i \in I \rangle$ and $\langle B_i \mid i \in I \rangle$ where $|A_i| = \kappa_i$ and $|B_i| = \lambda_i$ for all $i \in I$. Then, by the definition of the cardinal product, we have that

$$\prod_{i \in I} \kappa_i = \left| \prod_{i \in I} A_i \right| \qquad \prod_{i \in I} \lambda_i = \left| \prod_{i \in I} B_i \right|.$$

Now, for any $i \in I$, we also have

$$|A_i| = \kappa_i \leq \lambda_i = |B_i|$$

so that we can choose an injective $f_i : A_i \rightarrow B_i$ (with the help of the Axiom of Choice).

We then construct a function f from $\prod_{i \in I} A_i$ to $\prod_{i \in I} B_i$. So for any $a \in \prod_{i \in I} A_i$ we have that $a = \langle a_i \mid i \in I \rangle$ where each $a_i \in A_i$. Thus $a_i \in A_i = \text{dom}(f_i)$ for each $i \in I$ so that $f_i(a_i) \in B_i$. We then define $f(a) = \langle f_i(a_i) \mid i \in I \rangle$ so that clearly $f(a) \in \prod_{i \in I} B_i$ and hence f is a function into $\prod_{i \in I} B_i$.

We claim that f as defined above is injective. To this end, consider any α and β in $\prod_{i \in I} A_i$ where $\alpha \neq \beta$. It then follows that $\alpha = \langle \alpha_i \mid i \in I \rangle$ and $\beta = \langle \beta_i \mid i \in I \rangle$ where both α_i and β_i

are elements of A_i for each $i \in I$. Since $\alpha \neq \beta$ we must have that there is an $i_0 \in I$ such that $\alpha_{i_0} \neq \beta_{i_0}$. It then follows that $f_{i_0}(\alpha_{i_0}) \neq f_{i_0}(\beta_{i_0})$ since f_{i_0} is injective. Therefore clearly $f(\alpha) = \langle f_i(\alpha_i) \mid i \in I \rangle \neq \langle f_i(\beta_i) \mid i \in I \rangle = f(\beta)$. This shows that f is injective since α and β were arbitrary.

Finally, since f is an injective function from $\prod_{i \in I} A_i$ to $\prod_{i \in I} B_i$, we have that

$$\prod_{i \in I} \kappa_i = \left| \prod_{i \in I} A_i \right| \leq \left| \prod_{i \in I} B_i \right| = \prod_{i \in I} \lambda_i$$

as desired. □

Exercise 9.1.9

Find some cardinals κ_n, λ_n ($n \in \mathbf{N}$) such that $\kappa_n < \lambda_n$ for all n , but $\prod_{n=0}^{\infty} \kappa_n = \prod_{n=0}^{\infty} \lambda_n$.

Solution:

Let $\kappa_n = 2$ and $\lambda_n = \aleph_0$ for all $n \in \mathbf{N}$. We claim that these satisfy the requirements.

Proof. Clearly $\kappa_n = 2 < \aleph_0 = \lambda_n$ for every $n \in \mathbf{N}$. However, by Exercise 9.1.10 and Theorem 5.2.2c, we then have

$$\prod_{n=0}^{\infty} \kappa_n = \prod_{n=0}^{\infty} 2 = \prod_{n < \aleph_0} 2 = 2^{\aleph_0} = \aleph_0^{\aleph_0} = \prod_{n < \aleph_0} \aleph_0 = \prod_{n=0}^{\infty} \aleph_0 = \prod_{n=0}^{\infty} \lambda_n$$

□

Exercise 9.1.10

Prove that $\kappa \cdot \kappa \cdots (\lambda \text{ times}) = \kappa^\lambda$.

Solution:

Proof. First, we clearly have that

$$\kappa \cdot \kappa \cdots (\lambda \text{ times}) = \prod_{\alpha < \lambda} \kappa.$$

So let A and B be sets such that $|A| = \kappa$ and $|B| = \lambda$. Then by definition we have $\prod_{\alpha < \lambda} \kappa = \left| \prod_{\alpha < \lambda} A \right|$ and $\kappa^\lambda = |A^B|$. We also have that there is a bijective $g : \lambda \rightarrow B$ since $|\lambda| = \lambda = |B|$. We construct a bijection f from A^B to $\prod_{\alpha < \lambda} A$. So, for any $h \in A^B$ let $f(h) = h \circ g$. Clearly, since $g : \lambda \rightarrow B$ and $h : B \rightarrow A$, it follows that $f(h) : \lambda \rightarrow A$ and hence clearly $f(h) \in \prod_{\alpha < \lambda} A$ (since $f(h)(\alpha) \in A$ for each $\alpha < \lambda$) so that f is a function from A^B to $\prod_{\alpha < \lambda} A$.

We also claim that f is injective. So consider any h_1 and h_2 in A^B where $h_1 \neq h_2$. It then follows that there must be a $b \in B$ where $h_1(b) \neq h_2(b)$. Then let $\alpha = g^{-1}(b)$, noting that g^{-1} is a bijective function from B to λ since g is bijective. We then have that

$$\begin{aligned} f(h_1)(\alpha) &= (h_1 \circ g)(\alpha) = h_1(g(\alpha)) = h_1(g(g^{-1}(b))) = h_1(b) \\ &\neq h_2(b) = h_2(g(g^{-1}(b))) = h_2(g(\alpha)) = (h_2 \circ g)(\alpha) = f(h_2)(\alpha) \end{aligned}$$

so that $f(h_1) \neq f(h_2)$. Since h_1 and h_2 were arbitrary this shows that f is indeed injective.

We also claim that f is onto. So consider any $h' \in \prod_{\alpha < \lambda} A$ so that h' is a function from λ to A (since $h'(\alpha) \in A$ for each $\alpha < \lambda$). Then let $h = h' \circ g^{-1}$ so that clearly h is a function from B to A since $g^{-1} : B \rightarrow \lambda$ and $h' : \lambda \rightarrow A$. We then have that

$$f(h) = h \circ g = (h' \circ g^{-1}) \circ g = h' \circ (g^{-1} \circ g) = h' \circ i_\lambda = h'$$

where i_λ is the identity function from λ to λ . Since h' was arbitrary, this shows that f is indeed onto.

Thus, since we have shown that f is a bijection, it follows that

$$\kappa \cdot \kappa \cdots (\lambda \text{ times}) = \prod_{\alpha < \lambda} \kappa = \left| \prod_{\alpha < \lambda} A \right| = |A^B| = \kappa^\lambda$$

as desired. □

Exercise 9.1.11

Prove the formula $(\prod_{i \in I} \kappa_i)^\lambda = \prod_{i \in I} (\kappa_i)^\lambda$. [Hint: Generalize the proof of the special case $(\kappa \cdot \lambda)^\mu = \kappa^\mu \cdot \lambda^\mu$, given in Theorem 1.7 of Chapter 5.]

Note that the hint differs from that in the book; see the Errata List.

Solution:

Proof. First suppose that $\langle A_i \mid i \in I \rangle$ are sets where $|A_i| = \kappa_i$ for all $i \in I$. Also suppose that B is a set such that $|B| = \lambda$. It then follows that

$$\left(\prod_{i \in I} \kappa_i \right)^\lambda = \left| \prod_{i \in I} A_i \right|^\lambda = \left| \left(\prod_{i \in I} A_i \right)^B \right|$$

and

$$\prod_{i \in I} \kappa_i^\lambda = \prod_{i \in I} |A_i^B| = \left| \prod_{i \in I} A_i^B \right|.$$

We then aim to construct a bijection F from $\prod_{i \in I} A_i^B$ to $\left(\prod_{i \in I} A_i \right)^B$, which clearly would show the result since we would have by the above that

$$\left(\prod_{i \in I} \kappa_i \right)^\lambda = \left| \left(\prod_{i \in I} A_i \right)^B \right| = \left| \prod_{i \in I} A_i^B \right| = \prod_{i \in I} \kappa_i^\lambda.$$

So suppose that $f \in \prod_{i \in I} A_i^B$ so that $f_i = f(i) \in A_i^B$ for every $i \in I$. Then define a function g such that, for any $b \in B$, $g(b) = \langle f_i(b) \mid i \in I \rangle$. We then have that, for any $i \in I$, $f_i(b) \in A_i$ since f_i is a function from B to A_i . Therefore $g(b) \in \prod_{i \in I} A_i$, and hence g is a function from B to $\prod_{i \in I} A_i$ so that $g \in \left(\prod_{i \in I} A_i \right)^B$. Naturally then we set $F(f) = g$ so that F is indeed a function from $\prod_{i \in I} A_i^B$ to $\left(\prod_{i \in I} A_i \right)^B$.

To show that F is injective consider any f and f' in $\prod_{i \in I} A_i^B$ where $f \neq f'$. It then follows that there is an $i_0 \in I$ such that $f_{i_0} = f(i_0) \neq f'(i_0) = f'_{i_0}$. Then, since f_{i_0} and f'_{i_0} are both in $A_{i_0}^B$, it follows that they are both functions from B to A_{i_0} . Since $f_{i_0} \neq f'_{i_0}$ we have that there is a $b \in B$ such that $f_{i_0}(b) \neq f'_{i_0}(b)$. We then have that $g(b) = \langle f_i(b) \mid i \in I \rangle \neq \langle f'_i(b) \mid i \in I \rangle = g'(b)$ since $f_{i_0}(b) \neq f'_{i_0}(b)$ and $i_0 \in I$. Therefore $F(f) = g \neq g' = F(f')$, which shows that F is injective since f and f' were arbitrary.

To show that F is also onto, consider any $g' \in \left(\prod_{i \in I} A_i\right)^B$ so that g' is a function from B to $\prod_{i \in I} A_i$. Therefore, for any $b \in B$, $g'(b)$ is a function on I such that $g'(b)(i) \in A_i$ for each $i \in I$. Clearly we have $g'(b) = \langle g'(b)(i) \mid i \in I \rangle$. Then, for each $i \in I$, we define a function f_i on B such that $f_i(b) = g'(b)(i)$. Then, since $f_i(b) = g'(b)(i) \in A_i$, clearly each f_i is a function from B to A_i . Therefore $f_i \in A_i^B$. We then set $f = \langle f_i \mid i \in I \rangle$ so that clearly $f \in \prod_{i \in I} A_i^B$. Now, we then set $g = F(f)$ so that, by the definition of F , g is a function on B such $g(b) = \langle f_i(b) \mid i \in I \rangle$ for any $b \in B$. We then have that $g(b) = \langle f_i(b) \mid i \in I \rangle = \langle g'(b)(i) \mid i \in I \rangle = g'(b)$ by the definition of each f_i . Since $b \in B$ was arbitrary, this shows that $F(f) = g = g'$. Thus F is onto since g' was arbitrary.

We have shown that F is a bijection so that the result follows as described above. \square

Exercise 9.1.12

Prove the formula

$$\prod_{i \in I} (\kappa^{\lambda_i}) = \kappa^{\sum_{i \in I} \lambda_i}.$$

[Hint: Generalize the proof of the special case $\kappa^\lambda \cdot \kappa^\mu = \kappa^{\lambda+\mu}$ given in Theorem 1.7(a) of Chapter 5.]

Solution:

Proof. First, suppose that A is a set such that $|A| = \kappa$, and that $\langle B_i \mid i \in I \rangle$ are mutually disjoint sets such that $|B_i| = \lambda_i$ for each $i \in I$. Then, by the definitions of cardinal products and sums, we have

$$\prod_{i \in I} (\kappa^{\lambda_i}) = \prod_{i \in I} |A^{B_i}| = \left| \prod_{i \in I} A^{B_i} \right|$$

and

$$\kappa^{\sum_{i \in I} \lambda_i} = \kappa^{|\cup_{i \in I} B_i|} = \left| A^{\cup_{i \in I} B_i} \right|.$$

We now construct a bijective function F from $\prod_{i \in I} A^{B_i}$ to $A^{\cup_{i \in I} B_i}$, which clearly shows the desired result since we then have

$$\prod_{i \in I} (\kappa^{\lambda_i}) = \left| \prod_{i \in I} A^{B_i} \right| = \left| A^{\cup_{i \in I} B_i} \right| = \kappa^{\sum_{i \in I} \lambda_i}.$$

So, for any $f = \langle f_i \mid i \in I \rangle \in \prod_{i \in I} A^{B_i}$, we have that each f_i is a function from B_i to A . It then follows from the fact that $\langle B_i \mid i \in I \rangle$ are mutually disjoint that $\{f_i\}_{i \in I}$ is a compatible system of functions. We then that $g = \cup_{i \in I} f_i$ is a function from $\cup_{i \in I} \text{dom}(f_i) = \cup_{i \in I} B_i$ to A by Theorem 2.3.12. Hence $g \in A^{\cup_{i \in I} B_i}$, and we of course set $F(f) = g$ so that F is a function from $\prod_{i \in I} A^{B_i}$ to $A^{\cup_{i \in I} B_i}$.

To show that F is injective, consider any f and f' in $\prod_{i \in I} A^{B_i}$ where $f \neq f'$. Let $g = F(f)$ and $g' = F(f')$. It then follows that there is an $i_0 \in I$ such that $f_{i_0} \neq f'_{i_0}$. Then, it has to be that there is a $b \in B_{i_0}$ where $f_{i_0}(b) \neq f'_{i_0}(b)$. Clearly also $b \in \bigcup_{i \in I} B_i$ since $i_0 \in I$ so that $b \in \text{dom}(g)$ and $b \in \text{dom}(g')$ (since by the definition of F we have that g and g' are functions with domain $\bigcup_{i \in I} B_i$). Moreover, we have that $g(b) = f_{i_0}(b)$ and $g'(b) = f'_{i_0}(b)$, which again follows from the fact that $\langle B_i \mid i \in I \rangle$ are mutually disjoint or equivalently that $\{f_i\}_{i \in I}$ is a compatible system. Hence we have $g(b) = f_{i_0}(b) \neq f'_{i_0}(b) = g'(b)$ so that $F(f) = g \neq g' = F(f')$. This shows that F is injective since f and f' were arbitrary.

To see that F is onto, consider any $g' \in A^{\bigcup_{i \in I} B_i}$. For each $i \in I$, define $f_i = g' \upharpoonright B_i$, which is clearly a function from B_i to A so that $f_i \in A^{B_i}$. Then let $f = \langle f_i \mid i \in I \rangle$ so that $F(f) = g = \bigcup_{i \in I} f_i$, and clearly $f \in \prod_{i \in I} A^{B_i} = \text{dom}(F)$. Now consider any $(x, y) \in g = \bigcup_{i \in I} f_i$ so that there is an $i_0 \in I$ where $(x, y) \in f_{i_0} = g' \upharpoonright B_{i_0}$. Therefore $(x, y) \in g'$ since obviously $g' \upharpoonright B_{i_0} \subseteq g'$. Consider next any $(x, y) \in g'$ so that $x \in \text{dom}(g') = \bigcup_{i \in I} B_i$ and $y = g'(x)$. Then there is an $i_0 \in I$ where $x \in B_{i_0}$ so that $(x, y) \in g' \upharpoonright B_{i_0} = f_{i_0}$. Hence clearly $(x, y) \in \bigcup_{i \in I} f_i = g$ since $i_0 \in I$. Therefore $g \subseteq g'$ and $g' \subseteq g$ so that $F(f) = g = g'$, which shows that F is onto since g' was arbitrary.

We have thus shown that F is a bijection so that the result follows. \square

Exercise 9.1.13

Prove that if $1 < \kappa_i \leq \lambda_i$ for all $i \in I$, then $\sum_{i \in I} \kappa_i \leq \prod_{i \in I} \lambda_i$.

Solution:

Lemma 9.1.13.1. *If A is a set and $n < |A|$ for a finite cardinal (i.e. natural number) n , then $n + 1 \leq |A|$.*

Proof. Since $n < |A|$, it follows that there is an injective function $f : n \rightarrow A$ but that f cannot be onto. Thus there is an $a \in A$ such that $a \notin \text{ran}(f)$. Noting that $n + 1 = n \cup \{n\}$, for any $k \in n + 1$, define a mapping g by

$$g(k) = \begin{cases} f(k) & k \in n \\ a & k = n. \end{cases}$$

Clearly in either of these cases we have that $g(k) \in A$ so that g is in fact a function from $n + 1$ into A .

To see that g is injective, consider any k_1 and k_2 in $n + 1$ where $k_1 \neq k_2$. If $k_1 = n$ then $g(k_1) = a$ and it has to be that $k_2 \in n$ since $k_2 \neq k_1 = n$. Hence clearly $g(k_2) = f(k_2) \in \text{ran}(f)$ whereas $g(k_1) = a \notin \text{ran}(f)$ so that $g(k_1) \neq g(k_2)$. If $k_1 \in n$ but $k_2 = n$ then this is analogous the previous case, so assume that also $k_2 \in n$. Here clearly $g(k_1) = f(k_1) \neq f(k_2) = g(k_2)$ since f is injective and $k_1 \neq k_2$. Therefore, in all cases we have that $g(k_1) \neq g(k_2)$, which shows that g is injective since k_1 and k_2 were arbitrary.

The existence of the injection g thus shows that that $n + 1 \leq |A|$ as desired. \square

Main Problem.

Proof. Suppose that $\langle A_i \mid i \in I \rangle$ are mutually disjoint sets where $|A_i| = \kappa_i$ for every $i \in I$, and that $\langle B_i \mid i \in I \rangle$ are sets where $|B_i| = \lambda_i$ for each $i \in I$. Since we have $|A_i| = \kappa_i \leq \lambda_i = |B_i|$, we can

assume that $A_i \subseteq B_i$ (for every $i \in I$). We show the result by constructing an injective function that maps $\bigcup_{i \in I} A_i$ into $\prod_{i \in I} B_i$, since we clearly would then have

$$\sum_{i \in I} \kappa_i = \left| \bigcup_{i \in I} A_i \right| \leq \left| \prod_{i \in I} B_i \right| = \prod_{i \in I} \lambda_i$$

by the definitions of cardinal sum and product, which is the desired result.

First, if $I = \emptyset$ then we have that $\sum_{i \in I} \kappa_i = |\bigcup_{i \in \emptyset} A_i| = |\emptyset| = 0$. The only function with domain $I = \emptyset$ is \emptyset itself so that $\prod_{i \in I} \lambda_i = |\prod_{i \in \emptyset} B_i| = |\{\emptyset\}| = 1$. Hence clearly $\sum_{i \in I} \kappa_i = 0 \leq 1 = \prod_{i \in I} \lambda_i$ so that the hypothesis is true. So assume in what follows that $I \neq \emptyset$ so that there is an $i_0 \in I$.

Now, for every $i \in I$, since $1 < \kappa_i \leq \lambda_i = |B_i|$, it follows from Lemma 9.1.13.1 that $2 \leq |B_i|$. Hence, for each $i \in I$ we can choose two distinct elements α_i and β_i from B_i , though this requires the Axiom of Choice. So, for any $x \in \bigcup_{i \in I} A_i$, there is an $i_x \in I$ such that $x \in A_{i_x}$. Then, for any $i \in I$, set

$$a_i = \begin{cases} x & i = i_x \\ \alpha_i & i \neq i_x \text{ and } i = i_0 \text{ and } x = \alpha_{i_x} \\ \beta_i & i \neq i_x \text{ and } i = i_0 \text{ and } x \neq \alpha_{i_x} \\ \beta_i & i \neq i_x \text{ and } i \neq i_0 \text{ and } x = \alpha_{i_x} \\ \alpha_i & i \neq i_x \text{ and } i \neq i_0 \text{ and } x \neq \alpha_{i_x} \end{cases}$$

and let $f(x) = \langle a_i \mid i \in I \rangle$. Clearly, for any such x and $i \in I$, we have $a_i = x \in A_{i_x} = A_i$ if $i = i_x$ so that $a_i \in B_i$ since $A_i \subseteq B_i$. In the other cases either $a_i = \alpha_i \in B_i$ or $a_i = \beta_i \in B_i$. Therefore $f(x) = \langle a_i \mid i \in I \rangle \in \prod_{i \in I} B_i$ so that f is a function from $\bigcup_{i \in I} A_i$ into $\prod_{i \in I} B_i$.

We also claim that f is injective. To see this, consider x and y in $\bigcup_{i \in I} A_i$ where $x \neq y$, and let i_x and i_y be those elements of I such that $x \in A_{i_x}$ and $y \in A_{i_y}$. Also let $f(x) = \langle a_i \mid i \in I \rangle$ and $f(y) = \langle b_i \mid i \in I \rangle$.

The following involves a lot of messy case work, so we shall number the cases for easy reference:

1. Case: $i_x = i_y$. Then $a_{i_x} = x \neq y = b_{i_y} = b_{i_x}$.
2. Case: $i_x \neq i_y$.
 - (a) Case: $i_x \neq i_0$ and $i_y \neq i_0$.
 - i. Case: $a_{i_x} = x = \alpha_{i_x}$.
 - A. Case: $b_{i_y} = y = \alpha_{i_y}$. Then, since $i_x \neq i_y$ and $i_x \neq i_0$ and $y = \alpha_{i_y}$, we have $b_{i_x} = \beta_{i_x} \neq \alpha_{i_x} = a_{i_x}$.
 - B. Case: $b_{i_y} = y \neq \alpha_{i_y}$. Then, since $i_0 \neq i_x$ and $i_0 = i_0$ and $x = \alpha_{i_x}$, we have $a_{i_0} = \alpha_{i_0}$. Also, since $i_0 \neq i_y$ and $i_0 = i_0$ and $y \neq \alpha_{i_y}$, we have $b_{i_0} = \beta_{i_0} \neq \alpha_{i_0} = a_{i_0}$.
 - ii. Case: $a_{i_x} = x \neq \alpha_{i_x}$.
 - A. Case: $b_{i_y} = y = \alpha_{i_y}$. This is analogous to case 2.a.i.B with $a_{i_x} = x$ and $b_{i_y} = y$ reversed so that again $a_{i_0} \neq b_{i_0}$.
 - B. Case: $b_{i_y} = y \neq \alpha_{i_y}$. Then, since $i_x \neq i_y$ and $i_x \neq i_0$ and $y \neq \alpha_{i_y}$, we have $b_{i_x} = \alpha_{i_x} \neq a_{i_x}$.
 - (b) Case: $i_x \neq i_0$ and $i_y = i_0$.
 - i. Case: $a_{i_x} = x = \alpha_{i_x}$.
 - A. Case: $b_{i_y} = y = \alpha_{i_y}$. Since $i_x \neq i_y$ and $i_x \neq i_0$ and $y = \alpha_{i_y}$, we have $b_{i_x} = \beta_{i_x} \neq \alpha_{i_x} = a_{i_x}$.

- B. Case: $b_{i_y} = y \neq \alpha_{i_y}$. Since $i_y \neq i_x$ and $i_y = i_0$ and $x = \alpha_{i_x}$, we have $a_{i_y} = \alpha_{i_y} \neq b_{i_y}$.
- ii. Case: $a_{i_x} = x \neq \alpha_{i_x}$.
- A. Case: $b_{i_y} = y = \alpha_{i_y}$. Since $i_y \neq i_x$ and $i_y = i_0$ and $x \neq \alpha_{i_x}$, we have $a_{i_y} = \beta_{i_y} \neq \alpha_{i_y} = b_{i_y}$.
- B. Case: $b_{i_y} = y \neq \alpha_{i_y}$. Since $i_x \neq i_y$ and $i_x \neq i_0$ and $y \neq \alpha_{i_y}$, we have $b_{i_x} = \alpha_{i_x} \neq a_{i_x}$.
- (c) Case: $i_x = i_0$ and $i_y \neq i_0$. This is analogous to the previous case 2.b with the roles of i_x and i_y reversed.
- (d) Case: $i_x = i_0$ and $i_y = i_0$. This is impossible since $i_x \neq i_y$.

Thus, in all cases, there is an $i \in I$ such that $a_i \neq b_i$ so that clearly $f(x) = \langle a_i \mid i \in I \rangle \neq \langle b_i \mid i \in I \rangle = f(y)$. This shows that f is injective since x and y were arbitrary. This completes the proof as described above. \square

Exercise 9.1.14

Evaluate the cardinality of $\prod_{0 < \alpha < \omega_1} \alpha$. [Answer: 2^{\aleph_1} .]

Solution:

Lemma 9.1.14.1. *If α and β are ordinals and $\alpha \leq \beta$, then $\aleph_\alpha^{\aleph_\beta} = 2^{\aleph_\beta}$.*

Proof. First we have that

$$2^{\aleph_\beta} \leq \aleph_\alpha^{\aleph_\beta}$$

by property (n) after Lemma 5.6.1 since clearly $2 \leq \aleph_\alpha$. We also have

$$\begin{aligned} \aleph_\alpha^{\aleph_\beta} &\leq (2^{\aleph_\alpha})^{\aleph_\beta} && \text{(again by property (n), and Theorems 5.1.8 and 5.1.9)} \\ &= 2^{\aleph_\alpha \cdot \aleph_\beta} && \text{(by Theorem 5.1.7b)} \\ &= 2^{\aleph_\beta}. && \text{(by Corollary 7.2.2 since } \alpha \leq \beta \text{)} \end{aligned}$$

Hence it follows from the Cantor-Bernstein Theorem that $\aleph_\alpha^{\aleph_\beta} = 2^{\aleph_\beta}$ as desired. \square

Main Problem.

We claim that $\left| \prod_{0 < \alpha < \omega_1} \alpha \right| = 2^{\aleph_1}$.

Proof. First, let $I = \{\alpha \mid 0 < \alpha < \omega_1\}$ so that clearly $\prod_{0 < \alpha < \omega_1} \alpha = \prod_{\alpha \in I} \alpha$. It is trivial to show that the mapping

$$f(\alpha) = \begin{cases} \alpha + 1 & 0 \leq \alpha < \omega_0 \\ \alpha & \omega_0 \leq \alpha < \omega_1 \end{cases}$$

for $\alpha \in \omega_1$ is a bijection from ω_1 to I , and hence $|I| = |\omega_1| = \aleph_1$. Also, since $\alpha < \omega_1$ for any $\alpha \in I$, it follows from Lemma 7.2.1.5 that $|\alpha| \leq \aleph_0$. Therefore, we first have

$$\left| \prod_{\alpha \in I} \alpha \right| = \prod_{\alpha \in I} |\alpha| \quad \text{(by the definition of the cardinal product)}$$

$$\begin{aligned}
&\leq \prod_{\alpha \in I} \aleph_0 && \text{(by Exercise 9.1.8 since } |\alpha| \leq \aleph_0 \text{ for all } \alpha \in I) \\
&= \aleph_0^{|I|} && \text{(by Exercise 9.1.10)} \\
&= \aleph_0^{\aleph_1} && \text{(since we have shown that } |I| = \aleph_1) \\
&= 2^{\aleph_1}. && \text{(by Lemma 9.1.14.1)}
\end{aligned}$$

Now, let $J = \{\alpha \mid 1 < \alpha < \omega_1\}$. For any $a = \langle a_2, a_3, \dots, a_\omega, a_{\omega+1}, \dots \rangle = \langle a_\alpha \mid \alpha \in J \rangle \in \prod_{\alpha \in J} \alpha$, it is easy to show that the function that maps this to $b = \langle 0, a_2, a_3, \dots, a_\omega, a_{\omega+1}, \dots \rangle \in \prod_{\alpha \in I} \alpha$ is bijective so that $|\prod_{\alpha \in J} \alpha| = |\prod_{\alpha \in I} \alpha|$. It should also be clear that $|J| = |I| = \aleph_1$. Lastly, we clearly have $2 \leq |a|$ for all $a \in J$ since $1 < \alpha$. We then have

$$\begin{aligned}
2^{\aleph_1} &= 2^{|J|} && \text{(since } |J| = \aleph_1) \\
&= \prod_{\alpha \in J} 2 && \text{(by Exercise 9.1.10)} \\
&\leq \prod_{\alpha \in J} |\alpha| && \text{(by Exercise 9.1.8 since } 2 \leq |\alpha| \text{ for all } \alpha \in J) \\
&= \left| \prod_{\alpha \in J} \alpha \right| && \text{(by the definition of the cardinal product)} \\
&= \left| \prod_{\alpha \in I} \alpha \right| && \text{(by what was shown above)}
\end{aligned}$$

It therefore follows from the Cantor-Bernstein Theorem that $|\prod_{0 < \alpha < \omega_1} \alpha| = |\prod_{\alpha \in I} \alpha| = 2^{\aleph_1}$ as desired. \square

Exercise 9.1.15

Justify the existence of the function f in the proof of Lemma 9.1.2 in detail by the axioms of set theory.

Solution:

First, for any $i \in I$, we know that $|A_i| = |A'_i|$ so that there exists a bijection from A_i onto A'_i . It was shown in Exercise 2.3.9a that the set $A_i'^{A_i}$ exists so that $B_i = \{h_i \in A_i'^{A_i} \mid h_i \text{ is a bijection}\}$ exists by the Axiom Schema of Comprehension. Clearly then $B_i \neq \emptyset$ for every $i \in I$. Since B_i is uniquely defined for each $i \in I$, it follows from the Axiom Schemas of Replacement and Comprehension that the set $\{B_i \mid i \in I\}$ exists, which is a system of nonempty sets. It then follows from Exercise 2.3.9b that $\prod_{i \in I} B_i$ exists, and by the Axiom of Choice that there is an $F \in \prod_{i \in I} B_i$, i.e. $F = \langle f_i \mid i \in I \rangle$ in the notation of the proof. Now F is a function on I in which $F(i) = f_i \in B_i$ for every $i \in I$. By an application of the Axiom Schema of Comprehension, $\text{ran}(F) = \{f_i \mid i \in I\}$ exists, and we have that $f = \bigcup \text{ran}(F)$, which exists by the Axiom of Union, noting that then $f = \bigcup_{i \in I} f_i$ as in the proof.