Topology
Second Edition
by James Munkres

Solutions Manual

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Chapter 1 Set Theory and Logic

§1 Fundamental Concepts

Exercise 1.1

Check the distributive laws for $\cup$ and $\cap$ and DeMorgan's laws.

Solution:

Suppose that A, B, and C are sets. First we show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof. We show this as a series of logical equivalences:

$$\begin{aligned} x \in A \cap (B \cup C) &\Leftrightarrow x \in A \wedge x \in B \cup C \\ &\Leftrightarrow x \in A \wedge (x \in B \vee x \in C) \\ &\Leftrightarrow (x \in A \wedge x \in B) \vee (x \in A \wedge x \in C) \\ &\Leftrightarrow x \in A \cap B \vee x \in A \cap C \\ &\Leftrightarrow x \in (A \cap B) \cup (A \cap C)\,, \end{aligned}$$

which of course shows the desired result. □

Next we show that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Proof. We show this in the same way:

$$\begin{aligned} x \in A \cup (B \cap C) &\Leftrightarrow x \in A \vee x \in B \cap C \\ &\Leftrightarrow x \in A \vee (x \in B \wedge x \in C) \\ &\Leftrightarrow (x \in A \vee x \in B) \wedge (x \in A \vee x \in C) \\ &\Leftrightarrow x \in A \cup B \wedge x \in A \cup C \\ &\Leftrightarrow x \in (A \cup B) \cap (A \cup C)\,, \end{aligned}$$

which of course shows the desired result. □

Now we show the first DeMorgan's law that $A - (B \cup C) = (A - B) \cap (A - C)$.

Proof. We show this in the same way:

$$\begin{aligned} x \in A - (B \cup C) &\Leftrightarrow x \in A \wedge x \notin B \cup C \\ &\Leftrightarrow x \in A \wedge \neg(x \in B \vee x \in C) \\ &\Leftrightarrow x \in A \wedge (x \notin B \wedge x \notin C) \\ &\Leftrightarrow (x \in A \wedge x \notin B) \wedge (x \in A \wedge x \notin C) \\ &\Leftrightarrow x \in A - B \wedge x \in A - C \\ &\Leftrightarrow x \in (A - B) \cap (A - C)\,, \end{aligned}$$

which is the desired result. □

Lastly we show that $A - (B \cap C) = (A - B) \cup (A - C)$.

Proof. Again we use a sequence of logical equivalences:

$$\begin{aligned} x \in A - (B \cap C) &\Leftrightarrow x \in A \wedge x \notin B \cap C \\ &\Leftrightarrow x \in A \wedge \neg(x \in B \wedge x \in C) \\ &\Leftrightarrow x \in A \wedge (x \notin B \vee x \notin C) \\ &\Leftrightarrow (x \in A \wedge x \notin B) \vee (x \in A \wedge x \notin C) \\ &\Leftrightarrow x \in A - B \vee x \in A - C \\ &\Leftrightarrow x \in (A - B) \cup (A - C)\,, \end{aligned}$$

as desired. ☐

Exercise 1.2

Determine which of the following statements are true for all sets A, B, C, and D. If a double implication fails, determine whether one or the other of the possible implications holds. If an equality fails, determine whether the statement becomes true if the "equals" symbol is replaced by one or the other of the inclusion symbols $\subset$ or $\supset$.

(a) $A \subset B$ and $A \subset C \Leftrightarrow A \subset (B \cup C)$.
(b) $A \subset B$ or $A \subset C \Leftrightarrow A \subset (B \cup C)$.
(c) $A \subset B$ and $A \subset C \Leftrightarrow A \subset (B \cap C)$.
(d) $A \subset B$ or $A \subset C \Leftrightarrow A \subset (B \cap C)$.
(e) $A - (A - B) = B$.
(f) $A - (B - A) = A - B$.
(g) $A \cap (B - C) = (A \cap B) - (A \cap C)$.
(h) $A \cup (B - C) = (A \cup B) - (A \cup C)$.
(i) $(A \cap B) \cup (A - B) = A$.
(j) $A \subset C$ and $B \subset D \Rightarrow (A \times B) \subset (C \times D)$.
(k) The converse of (j).
(l) The converse of (j), assuming that A and B are nonempty.
(m) $(A \times B) \cup (C \times D) = (A \cup C) \times (B \cup D)$.
(n) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.
(o) $A \times (B - C) = (A \times B) - (A \times C)$.
(p) $(A - B) \times (C - D) = (A \times C - B \times C) - A \times D$.
(q) $(A \times B) - (C \times D) = (A - C) \times (B - D)$.

Solution:

(a) We claim that $A \subset B$ and $A \subset C \Rightarrow A \subset (B \cup C)$ but that the converse is not generally true.

Proof. Suppose that $A \subset B$ and $A \subset C$ and consider any $x \in A$. Then clearly also $x \in B$ since $A \subset B$ so that $x \in B \cup C$. Since x was arbitrary, this shows that $A \subset (B \cup C)$ as desired.

To show that the converse is not true, suppose that $A = \{1, 2, 3\}$, $B = \{1, 2\}$, and $C = \{3, 4\}$. Then clearly $A \subset \{1, 2, 3, 4\} = B \cup C$ but it neither true that $A \subset B$ (since $3 \in A$ but $3 \notin B$) nor $A \subset C$ (since $1 \in A$ but $1 \notin C$). ☐

(b) We claim that $A \subset B$ or $A \subset C \Rightarrow A \subset (B \cup C)$ but that the converse is not generally true.

Proof. Suppose that $A \subset B$ or $A \subset C$ and consider any $x \in A$. If $A \subset B$ then clearly $x \in B$ so that $x \in B \cup C$. If $A \subset C$ then clearly $x \in C$ so that again $x \in B \cup C$. Since x was arbitrary, this shows that $A \subset (B \cup C)$ as desired.

The counterexample that disproves the converse of part (a), also serves as a counterexample to the converse here. Again this is because $A \subset B \cup C$ but neither $A \subset B$ nor $A \subset C$, which is to say that $A \not\subset B$ and $A \not\subset C$. Hence it is not true that $A \subset B$ or $A \subset C$. ☐

(c) We claim that this biconditional is true.

Proof. ($\Rightarrow$) Suppose that $A \subset B$ and $A \subset C$ and consider any $x \in A$. Then clearly also $x \in B$ and $x \in C$ since both $A \subset B$ and $A \subset C$. Hence $x \in B \cap C$, which proves that $A \subset B \cap C$ since x was arbitrary.

($\Leftarrow$) Now suppose that $A \subset B \cap C$ and consider any $x \in A$. Then $x \in B \cap C$ as well so that $x \in B$ and $x \in C$. Since x was an arbitrary element of A, this of course shows that both $A \subset B$ and $A \subset C$ as desired. □

(d) We claim that only the converse is true here.

Proof. To show the converse, suppose that $A \subset B \cap C$. It was shown in part (c) that this implies that both $A \subset B$ and $A \subset C$. Thus it is clearly true that $A \subset B$ or $A \subset C$.

As a counterexample to the forward implication, let $A = \{1\}$, $B = \{1, 2\}$, and $C = \{3, 4\}$ so that clearly $A \subset B$ and hence $A \subset B$ or $A \subset C$ is true. However we have that B and C are disjoint so that $B \cap C = \varnothing$, therefore $A \not\subset \varnothing = B \cap C$ since $A \neq \varnothing$. □

(e) We claim that $A - (A - B) \subset B$ but that the other direction is not generally true.

Proof. First consider any $x \in A - (A - B)$ so that $x \in A$ but $x \notin A - B$. Hence it is not true that $x \in A$ and $x \notin B$. So it must be that $x \notin A$ or $x \in B$. However, since we know that $x \in A$, it has to be that $x \in B$. Thus $A - (A - B) \subset B$ since x was arbitrary.

Now let $A = \{1, 2\}$ and $B = \{2, 3\}$. Then we clearly have $A - B = \{1\}$, and thus $A - (A - B) = \{2\}$. So clearly B is not a subset of $A - (A - B)$ since $3 \in B$ but $3 \notin A - (A - B)$. □

(f) Here we claim that $A - (B - A) \supset A - B$ but that the other direction is not generally true.

Proof. First suppose that $x \in A - B$ so that $x \in A$ but $x \notin B$. Then it is certainly true that $x \notin B$ or $x \in A$ so that, by logical equivalence, it is not true that $x \in B$ and $x \notin A$. That is, it is not true that $x \in B - A$, which is to say that $x \notin B - A$. Since also $x \in A$, it follows that $x \in A - (B - A)$, which shows the desired result since x was arbitrary.

To show that the other direction does not hold consider the counterexample $A = \{1, 2\}$ and $B = \{2, 3\}$. Then $B - A = \{3\}$ so that $A - (B - A) = \{1, 2\} = A$. We also have that $A - B = \{1\}$ so that $2 \in A - (B - A)$ but $2 \notin A - B$. This suffices to show that $A - (B - A) \not\subset A - B$. □

(g) We claim that equality holds here, i.e. that $A \cap (B - C) = (A \cap B) - (A \cap C)$.

Proof. ($\subset$) Suppose that $x \in A \cap (B - C)$ so that $x \in A$ and $x \in B - C$. Thus $x \in B$ but $x \notin C$. Since both $x \in A$ and $x \in B$ we have that $x \in A \cap B$. Also since $x \notin C$ it clearly must be that $x \notin A \cap C$. Hence $x \in (A \cap B) - (A \cap C)$, which shows the forward direction since x was arbitrary.

($\supset$) Now suppose that $x \in (A \cap B) - (A \cap C)$. Hence $x \in A \cap B$ but $x \notin A \cap C$. From the former of these we have that $x \in A$ and $x \in B$, and from the latter it follows that either $x \notin A$ or $x \notin C$. Since we know that $x \in A$, it must therefore be that $x \notin C$. Hence $x \in B - C$ since $x \in B$ but $x \notin C$. Since also $x \in A$ we have that $x \in A \cap (B - C)$, which shows the desired result since x was arbitrary. □

(h) Here we claim that $A \cup (B - C) \supset (A \cup B) - (A \cup C)$ but that the forward direction is not generally true.

Proof. First consider any $x \in (A \cup B) - (A \cup C)$ so that $x \in A \cup B$ and $x \notin A \cup C$. From the latter, it follows that $x \notin A$ and $x \notin C$ since otherwise we would have $x \in A \cup C$. From the former, we have that $x \in A$ or $x \in B$ so that it must be that $x \in B$ since $x \notin A$. Therefore we have that $x \in B$ and $x \notin C$ so that $x \in B - C$. From this it obviously follows that $x \in A \cup (B - C)$, which shows that $A \cup (B - C) \supset (A \cup B) - (A \cup C)$ since x was arbitrary.

To show that the forward direction does not always hold, consider the sets $A = \{1, 2\}$, $B = \{2, 3\}$, and $C = \{2\}$. Then we clearly have that $B - C = \{3\}$, and hence $A \cup (B - C) = \{1, 2, 3\}$. On the other hand, we have $A \cup B = \{1, 2, 3\}$ and $A \cup C = \{1, 2\}$ so that $(A \cup B) - (A \cup C) = \{3\}$. Hence, for example, $1 \in A \cup (B - C)$ but $1 \notin (A \cup B) - (A \cup C)$, which suffices to show that $A \cup (B - C) \not\subset (A \cup B) - (A \cup C)$ as desired. □

(i) We claim that equality holds here.

Proof. We show this with a chain of logical equivalences:

$$\begin{aligned} x \in (A \cap B) \cup (A - B) &\Leftrightarrow x \in A \cap B \vee x \in A - B \\ &\Leftrightarrow (x \in A \wedge x \in B) \vee (x \in A \wedge x \notin B) \\ &\Leftrightarrow x \in A \wedge (x \in B \vee x \notin B) \\ &\Leftrightarrow x \in A \wedge \text{True} \\ &\Leftrightarrow x \in A\,, \end{aligned}$$

where we note that "True" denotes the fact that $x \in B \vee x \notin B$ is always true by the excluded middle property of logic. □

(j) We claim that this implication is true.

Proof. Suppose that $A \subset C$ and $B \subset D$. Consider any $(x, y) \in A \times B$ so that $x \in A$ and $y \in B$ by the definition of the cartesian product. Then also clearly $x \in C$ and $y \in D$ since $A \subset C$ and $B \subset D$. Hence $(x, y) \in C \times D$, which shows the result since the ordered pair (x, y) was arbitrary. □

(k) We claim that the converse of (j) is not always true.

Proof. Consider the following sets:

$$\begin{aligned} A &= \varnothing & C &= \{1\} \\ B &= \{1, 2\} & D &= \{2\}\,. \end{aligned}$$

Then we have that $A \times B = \varnothing$ since there are no ordered pairs (x, y) such that $x \in A$ (since $A = \varnothing$). Hence it is vacuously true that $(A \times B) \subset (C \times D)$. However, clearly it is not the case that $B \subset D$, and so, even though $A \subset C$, it is not true that $A \subset C$ and $B \subset D$. □

(l) We claim that the converse of (j) is true with the stipulation that A and B are both nonempty.

Proof. Suppose that $(A \times B) \subset (C \times D)$. First consider any $x \in A$. Then, since $B \neq \varnothing$, there is a $b \in B$. Then $(x, b) \in A \times B$ so that clearly also $(x, b) \in C \times D$. Hence $x \in C$ so that $A \subset C$ since x was arbitrary. An analogous argument shows that $B \subset D$ since A is nonempty. Hence it is true that $A \subset C$ and $B \subset D$ as desired. □

(m) Here we claim that $(A \times B) \cup (C \times D) \subset (A \cup C) \times (B \cup D)$ but that the other direction is not always true.

Proof. First consider any $(x,y) \in (A \times B) \cup (C \times D)$ so that either $(x,y) \in A \times B$ or $(x,y) \in C \times D$. In the first case $x \in A$ and $y \in B$ so that clearly $x \in A \cup C$ and $y \in B \cup D$. Hence $(x,y) \in (A \cup C) \times (B \cup D)$. In the second case we have $x \in C$ and $y \in D$ so that again $x \in A \cup C$ and $y \in B \cup D$ are still both true. Hence of course $(x,y) \in (A \cup C) \times (B \cup D)$ here also. This shows the result in either case since (x,y) was an arbitrary ordered pair.

To show that the other direction does not always hold, consider $A = B = \{1\}$ and $C = D = \{2\}$. Then we clearly have $A \times B = \{(1,1)\}$ and $C \times D = \{(2,2)\}$ so that $(A \times B) \cup (C \times D) = \{(1,1),(2,2)\}$. We also have $A \cup C = B \cup D = \{1,2\}$ so that $(A \cup C) \times (B \cup D) = \{(1,1),(1,2),(2,1),(2,2)\}$. This clearly shows that $(A \times B) \cup (C \times D) \not\supset (A \cup C) \times (B \cup D)$ as desired. □

(n) We claim that the equality holds here.

Proof. We can show this by a series of logical equivalences:

$$\begin{aligned}(x,y) \in (A \times B) \cap (C \times D) &\Leftrightarrow (x,y) \in A \times B \wedge (x,y) \in C \times D \\ &\Leftrightarrow (x \in A \wedge y \in B) \wedge (x \in C \wedge y \in D) \\ &\Leftrightarrow (x \in A \wedge x \in C) \wedge (y \in B \wedge y \in D) \\ &\Leftrightarrow x \in A \cap C \wedge y \in B \cap D \\ &\Leftrightarrow (x,y) \in (A \cap C) \times (B \cap D)\end{aligned}$$

as desired. □

(o) We claim that equivalence holds here as well.

Proof. ($\subset$) First consider any $(x,y) \in A \times (B - C)$ so that $x \in A$ and $y \in B - C$. From the latter of these we have that $y \in B$ but $y \notin C$. We clearly then have that $(x,y) \in A \times B$ since $x \in A$ and $y \in B$. It also has to be that $(x,y) \notin A \times C$ since $y \notin C$ even though it is true that $x\ inA$. Therefore $(x,y) \in (A \times B) - (A \times C)$ as desired.

($\supset$) Now suppose that $(x,y) \in (A \times B) - (A \times C)$ so that $(x,y) \in A \times B$ but $(x,y) \notin (A \times C)$. From the former we have that $x \in A$ and $y \in B$. It then must be that $y \notin C$ since $(x,y) \notin (A \times C)$ but we know that $x \in A$. Then we have $y \in B - C$ since $y \in B$ but $y \notin C$. Since also $x \in A$, it follows that $(x,y) \in A \times (B - C)$ as desired. □

(p) We claim the equivalence hold for this statement.

Proof. ($\subset$) Suppose that $(x,y) \in (A - B) \times (C - D)$ so that $x \in A - B$ and $y \in C - D$. Then we have that $x \in A$, $x \notin B$, $y \in C$, and $y \notin D$. So first, clearly $(x,y) \in A \times C$. Then, since $x \notin B$, we have that $(x,y) \notin B \times C$, and hence $(x,y) \in A \times C - B \times C$. Since $y \notin D$, we also have that $(x,y) \notin A \times D$, and thus $(x,y) \in (A \times C - B \times C) - A \times D$. This clearly shows the desired result since (x,y) was arbitrary.

($\supset$) Now suppose that $(x,y) \in (A \times C - B \times C) - A \times D$ so that $(x,y) \in A \times C - B \times C$ but $(x,y) \notin A \times D$. From the former we have that $(x,y) \in A \times C$ and $(x,y) \notin B \times C$. Thus $x \in A$ and $y \in C$ so that it has to be that $x \notin B$ since $(x,y) \notin B \times C$ but we know that $y \in C$. It also must be that $y \notin D$ since $(x,y) \notin A \times D$ but $x \in A$. Therefore we have that $x \in A$, $x \notin B$, $y \in C$, and $y \notin D$, from which it readily follows that $x \in A - B$ and $y \in C - D$. Thus clearly $(x,y) \in (A - B) \times (C - D)$, which shows the desired result since (x,y) was arbitrary. □

(q) Here we claim that $(A \times B) - (C \times D) \supset (A - C) \times (B - D)$ but that the forward direction is not true in general.

Proof. First consider any $(x, y) \in (A - C) \times (B - D)$ so that $x \in A - C$ and $y \in B - D$. Thus we have $x \in A$, $x \notin C$, $y \in B$, and $y \notin D$. From this clearly $(x, y) \in A \times B$ but $(x, y) \notin C \times D$. Hence $(x, y) \in (A \times B) - (C \times D)$, which clearly shows the desired result since (x, y) was arbitrary.

To show that the forward direction does not hold, consider $A = \{1, 2\}$, $B = \{a, b\}$, $C = \{2, 3\}$, and $D = \{b, c\}$. We then clearly have the following sets:

$$A \times B = \{(1, a), (1, b), (2, a), (2, b)\} \qquad A - C = \{1\}$$
$$C \times D = \{(2, b), (2, c), (3, b), (3, c)\} \qquad B - D = \{a\}$$
$$(A \times B) - (C \times D) = \{(1, a), (1, b), (2, a)\} \qquad (A - C) \times (B - D) = \{(1, a)\}\ .$$

This clearly shows that $(A \times B) - (C \times D)$ is not a subset of $(A - C) \times (B - D)$. □

Exercise 1.3

(a) Write the contrapositive and converse of the following statement: "If $x < 0$, then $x^2 - x > 0$," and determine which (if any) of the three statements are true.

(b) Do the same for the statement "If $x > 0$, then $x^2 - x > 0$."

Solution:

(a) First we claim that the original statement is true.

Proof. Since $x < 0$ we clearly have that $x - 1 < x < 0$ as well. Then, since the product of two negative numbers is positive, we have that $x^2 - x = x(x - 1) > 0$ as desired. □

The contrapositive of this is, "If $x^2 - x \leq 0$, then $x \geq 0$." This is of course also true by virtue of the fact that the contrapositive is logically equivalent to the original implication.

Lastly, the converse of this statement is, "If $x^2 - x > 0$, then $x < 0$." We claim that this is not generally true.

Proof. A simple counterexample of $x = 2$ shows this. We have $x^2 - x = 2^2 - 2 = 4 - 2 = 2 > 0$, but also clearly $x = 2 > 0$ as well so that $x < 0$ is clearly false. □

(b) First we claim that this statement is false.

Proof. As a counterexample, let $x = 1/2$. Then clearly $x > 0$, but we also have $x^2 - x = (1/2)^2 - 1/2 = 1/4 - 1/2 = -1/4 < 0$ so that $x^2 - x > 0$ is obviously not true. □

The contrapositive is then "If $x^2 - x \leq 0$, then $x \leq 0$," which is false since it is logically equivalent to the original statement.

The converse is "If $x^2 - x > 0$, then $x > 0$," which we claim is false.

Proof. As a counterexample, consider $x = -1$ so that $x^2 - x = (-1)^2 - (-1) = 1 + 1 = 2 > 0$. However, we also clearly have $x = -1 < 0$ so that $x > 0$ is not true. □

Exercise 1.4

Let A and B be sets of real numbers. Write the negation of each of the following statements:

(a) For every $a \in A$, it is true that $a^2 \in B$.

(b) For at least one $a \in A$, it is true that $a^2 \in B$.

(c) For every $a \in A$, it is true that $a^2 \notin B$.

(d) For at least one $a \notin A$, it is true that $a^2 \in B$.

Solution:

These are all basic logical negations using existential quantifiers:

(a) There is an $a \in A$ where $a^2 \notin B$.

(b) For every $a \in A$, $a^2 \notin B$.

(c) There is an $a \in A$ where $a^2 \in B$.

(d) For every $a \notin A$, $a^2 \notin B$.

Exercise 1.5

Let $\mathcal{A}$ be a nonempty collection of sets. Determine the truth of each of the following statements and of their converses:

(a) $x \in \bigcup_{A \in \mathcal{A}} A \Rightarrow x \in A$ for at least one $A \in \mathcal{A}$.

(b) $x \in \bigcup_{A \in \mathcal{A}} A \Rightarrow x \in A$ for every $A \in \mathcal{A}$.

(c) $x \in \bigcap_{A \in \mathcal{A}} A \Rightarrow x \in A$ for at least one $A \in \mathcal{A}$.

(d) $x \in \bigcap_{A \in \mathcal{A}} A \Rightarrow x \in A$ for every $A \in \mathcal{A}$.

Solution:

(a) The statement on the right is the definition of the statement on the left so of course the implication and its converse are true.

(b) The implication is generally false.

Proof. As a counterexample, consider $\mathcal{A} = \{\{1\}, \{2\}\}$. Then clearly $\bigcup_{A \in \mathcal{A}} A = \{1, 2\}$ so that $1 \in \bigcup_{A \in \mathcal{A}} A$, but 1 is not in A for every $A \in \mathcal{A}$ since $1 \notin \{2\}$. □

However, the converse *is* true.

Proof. Suppose that $x \in A$ for every $A \in \mathcal{A}$. Since $\mathcal{A}$ is nonempty there is an $A_0 \in \mathcal{A}$. Then $x \in A_0$ since $A_0 \in \mathcal{A}$. Hence by definition $x \in \bigcup_{A \in \mathcal{A}} A$ since $x \in A_0$ and $A_0 \in \mathcal{A}$. □

(c) The implication here is true.

Proof. Suppose that $x \in \bigcap_{A \in \mathcal{A}} A$ so that by definition $x \in A$ for every $A \in \mathcal{A}$. Since $\mathcal{A}$ is nonempty there is an $A_0 \in \mathcal{A}$ so that in particular $x \in A_0$. This shows the desired result since $A_0 \in \mathcal{A}$. □

The converse is not generally true.

Proof. As a counterexample consider $\mathcal{A} = \{\{1, 2\}, \{2, 3\}\}$. Then $1 \in \{1, 2\}$ and $\{1, 2\} \in \mathcal{A}$, but $1 \notin \bigcap_{A \in \mathcal{A}} A$ since clearly $\bigcap_{A \in \mathcal{A}} A = \{2\}$. □

(d) The statement on the right is the definition of the statement on the left so of course the implication and its converse are true.

Exercise 1.6

Write the contrapositive of each of the statements of Exercise 5.

> **Solution:**
>
> Again these involve simple logical negations of both sides of the implications:
>
> (a) $x \notin A$ for every $A \in \mathcal{A} \Rightarrow x \notin \bigcup_{A \in \mathcal{A}} A$.
>
> (b) $x \notin A$ for at least one $A \in \mathcal{A} \Rightarrow x \notin \bigcup_{A \in \mathcal{A}} A$.
>
> (c) $x \notin A$ for every $A \in \mathcal{A} \Rightarrow x \notin \bigcap_{A \in \mathcal{A}} A$.
>
> (d) $x \notin A$ for at least one $A \in \mathcal{A} \Rightarrow x \notin \bigcap_{A \in \mathcal{A}} A$.

Exercise 1.7

Given sets A, B, and C, express each of the following sets in terms of A, B, and C, using the symbols $\cup$, $\cap$ and $-$.

$$
\begin{aligned}
D &= \{x \mid x \in A \text{ and } (x \in B \text{ or } x \in C\} \ , \\
E &= \{x \mid (x \in A \text{ and } x \in B) \text{ or } x \in C\} \ , \\
F &= \{x \in A \text{ and } (x \in B \Rightarrow x \in C)\} \ .
\end{aligned}
$$

> **Solution:**
>
> First, we obviously have
>
> $$
> \begin{aligned}
> D &= A \cap (B \cup C) \\
> E &= (A \cap B) \cup C \, ,
> \end{aligned}
> $$
>
> noting that $D \neq E$ generally though they appear similar. Regarding F we have the following sequence of logical equivalences:
>
> $$
> \begin{aligned}
> x \in F &\Leftrightarrow x \in A \wedge (x \in B \Rightarrow x \in C) \\
> &\Leftrightarrow x \in A \wedge (x \notin B \vee x \in C) \\
> &\Leftrightarrow x \in A \wedge \neg(x \in B \wedge x \notin C) \\
> &\Leftrightarrow x \in A \wedge \neg(x \in B - C) \\
> &\Leftrightarrow x \in A \wedge x \notin B - C \\
> &\Leftrightarrow x \in A - (B - C)
> \end{aligned}
> $$
>
> so that of course $F = A - (B - C)$.

Exercise 1.8

If a set A has two elements, show that $\mathcal{P}(A)$ has four elements. How many elements does $\mathcal{P}(A)$ have if A has one element? Three elements? No elements? Why is $\mathcal{P}(A)$ called the power set of A.

Solution:

We claim that if a finite set has n elements, then its power set has 2^n elements, which is why it is called the power set.

Proof. We show this by induction on the size of the set. For the base case start with the the empty set in which $n = 0$. Clearly the only subset of $\varnothing$ is the trivial subset $\varnothing$ itself so that $\mathcal{P}(\varnothing) = \{\varnothing\}$. This has $1 = 2^0 = 2^n$ element obviously, which shows the base case. Now suppose that the power set of any set with n elements has 2^n elements. Let A be a set with $n+1$ elements, noting that this is nonempty since $n+1 \geq 1$ since $n \geq 0$. Hence there is an $x \in A$. For any subset $B \subset A$, either $x \notin B$ or $x \in B$. In the first case B is a subset of $A - \{x\}$ and in the latter $B = \{x\} \cup C$ for some $C \subset A - \{x\}$. Therefore $\mathcal{P}(A)$ has twice the number of elements of $\mathcal{P}(A - \{x\})$, one half being just the elements of $A - \{x\}$ and the other being those elements with x added in. But $A - \{x\}$ has n elements since A has $n+1$, and hence $\mathcal{P}(A - \{x\})$ has 2^n elements by the induction hypothesis. Thus $\mathcal{P}(A)$ has $2 \cdot 2^n = 2^{n+1}$ elements, which completes the induction. □

Using this, we can answer all of the specific questions. If a set has two elements, than its power set has $2^2 = 4$ elements. If it has one element, then its power set has $2^1 = 2$ elements, namely $\mathcal{P}(\{x\}) = \{\varnothing, \{x\}\}$. If a set has three elements then its power set has $2^3 = 8$ elements. Lastly, if a set has no elements (i.e. it is the empty set), then its power set has $2^0 = 1$ elements. As noted in the proof we have $\mathcal{P}(\varnothing) = \{\varnothing\}$.

Exercise 1.9

Formulate and prove DeMorgan's laws for arbitrary unions and intersections.

Solution:

In following suppose that A is a set and $\mathcal{B}$ is a nonempty collection of sets. For arbitrary unions, we claim that

$$A - \bigcup_{B \in \mathcal{B}} B = \bigcap_{B \in \mathcal{B}} (A - B).$$

Proof. The simplest way to show this is with a series of logically equivalent statements. For any x we have that

$$\begin{aligned} x \in A - \bigcup_{B \in \mathcal{B}} B &\Leftrightarrow x \in A \wedge x \notin \bigcup_{B \in \mathcal{B}} B \\ &\Leftrightarrow x \in A \wedge \neg \exists B \in \mathcal{B}(x \in B) \\ &\Leftrightarrow x \in A \wedge \forall B \in \mathcal{B}(x \notin B) \\ &\Leftrightarrow \forall B \in \mathcal{B}(x \in A \wedge x \notin B) \\ &\Leftrightarrow \forall B \in \mathcal{B}(x \in A - B) \\ &\Leftrightarrow x \in \bigcap_{B \in \mathcal{B}} (A - B), \end{aligned}$$

which of course shows the desired result. □

For intersections, we claim that

$$A - \bigcap_{B \in \mathcal{B}} B = \bigcup_{B \in \mathcal{B}} (A - B).$$

Proof. Similarly, we show this with a series of logically equivalent statements. For any x we have

$$\begin{aligned} x \in A - \bigcap_{B \in \mathcal{B}} B &\Leftrightarrow x \in A \wedge x \notin \bigcap_{B \in \mathcal{B}} B \\ &\Leftrightarrow x \in A \wedge \neg \forall B \in \mathcal{B}(x \in B) \\ &\Leftrightarrow x \in A \wedge \exists B \in \mathcal{B}(x \notin B) \\ &\Leftrightarrow \exists B \in \mathcal{B}(x \in A \wedge x \notin B) \\ &\Leftrightarrow \exists B \in \mathcal{B}(x \in A - B) \\ &\Leftrightarrow x \in \bigcup_{B \in \mathcal{B}} (A - B)\,, \end{aligned}$$

which shows the desired result. □

Exercise 1.10

Let $\mathbb{R}$ denote the set of real numbers. For each of the following subsets of $\mathbb{R} \times \mathbb{R}$, determine whether it is equal to the cartesian product of two subsets of $\mathbb{R}$.

(a) $\{(x, y) \mid x \text{ is an integer}\}$.

(b) $\{(x, y) \mid 0 < y \leq 1\}$.

(c) $\{(x, y) \mid y > x\}$.

(d) $\{(x, y) \mid x \text{ is not an integer and } y \text{ is an integer}\}$.

(e) $\left\{(x, y) \mid x^2 + y^2 < 1\right\}$.

Solution:

(a) This is equal to the set $\mathbb{Z} \times \mathbb{R}$, which is trivial to prove.

(b) It is easy to show that this is equal to $\mathbb{R} \times (0, 1]$, where of course $(a, b]$ denotes the half-open interval $\{x \in \mathbb{R} \mid a < x \leq b\}$.

(c) We claim that this cannot be equal to the cartesian product of subsets of $\mathbb{R}$.

Proof. Let $A = \{(x, y) \mid y > x\}$ and suppose to the contrary that $A = B \times C$ where $B, C \subset \mathbb{R}$. Since $1 > 0$, we have that $(0, 1) \in A$. Then also $0 \in B$ and $1 \in C$ since $A = B \times C$. We also have that $1 \in B$ and $2 \in C$ since $2 > 1$ so that $(1, 2) \in A = B \times C$. Thus $1 \in B$ and $1 \in C$ so that $(1, 1) \in B \times C = A$, but this cannot be since it is not true that $1 > 1$. Hence we have a contradiction so that A cannot be expressed as $B \times C$. □

(d) It is trivial to show that this set is equal to $(\mathbb{R} - \mathbb{Z}) \times \mathbb{Z}$.

(e) We claim that this set cannot be expressed as the cartesian product of subsets of $\mathbb{R}$.

Proof. Let $A = \left\{(x, y) \mid x^2 + y^2 < 1\right\}$ and suppose to the contrary that $A = B \times C$ where $B, C \subset \mathbb{R}$. We then have that $(9/10)^2 + 0^2 = 81/100 + 0 = 81/100 < 1$ so that $(9/10, 0) \in A = B \times C$, and hence $9/10 \in B$ and $0 \in C$. Also $0^2 + (9/10)^2 = (9/10)^2 + 0^2 = 81/100 < 1$ so that $(0, 9/10) \in A = B \times C$, and hence $0 \in B$ and $9/10 \in C$. Hence $(9/10, 9/10) \in B \times C = A$ since $9/10$ is in both B and C. However, we have $(9/10)^2 + (9/10)^2 = 81/100 + 81/100 = 162/100 \geq 1$ so that $(9/10, 9/10)$ cannot be in A, so we have a contradiction. So it must be that A cannot be equal to $B \times C$. □

§2 Functions

Exercise 2.1

Let $f : A \to B$. Let $A_0 \subset A$ and $B_0 \subset B$.

(a) Show that $A_0 \subset f^{-1}(f(A_0))$ and that equality holds if f is injective.

(b) Show that $f(f^{-1}(B_0)) \subset B_0$ and that equality holds if f is surjective.

Solution:

(a)

Proof. Consider any $x \in A_0$ and let $y = f(x)$ so that clearly $y \in f(A_0)$. Then, since $f(x) = y \in f(A_0)$, it follows from the definition of the preimage that $x \in f^{-1}(f(A_0))$. Hence $A_0 \subset f^{-1}(f(A_0))$ as desired since x was arbitrary. Now suppose that f is also injective and consider this time any $x \in f^{-1}(f(A_0))$ so that $y = f(x) \in f(A_0)$ by the definition of a preimage. Then there is an $x' \in A_0$ where $f(x') = y = f(x)$ by the definition of an image. Since f injective though, it must be that $x = x' \in A_0$. This shows that $f^{-1}(f(A_0)) \subset A_0$ since x was arbitrary. The desired equality follows since it was already shown that $A_0 \subset f^{-1}(f(A_0))$ (whether or not f is injective). □

(b)

Proof. First suppose that y is any element of $f(f^{-1}(B_0))$ so that there is an $x \in f^{-1}(B_0)$ where $f(x) = y$. Since $x \in f^{-1}(B_0)$, we then have that $y = f(x) \in B_0$ by the definition of a preimage. Hence $f(f^{-1}(B_0)) \subset B_0$ since y was arbitrary. Now suppose also that f is surjective and suppose that $y \in B_0$ so that also clearly $y \in B$ since $B_0 \subset B$. Since f is surjective, there is an $x \in A$ where $f(x) = y$. We then have that $x \in f^{-1}(B_0)$ since $f(x) = y \in B_0$. Clearly then $y = f(x) \in f(f^{-1}(B_0))$ so that $B_0 \subset f(f^{-1}(B_0))$ since y was arbitrary. This shows equality as desired. □

Exercise 2.2

Let $f : A \to B$ and let $A_i \subset A$ and $B_i \subset B$ for $i = 0$ and $i = 1$. Show that f^{-1} preserves inclusions, unions, intersections, and differences of sets:

(a) $B_0 \subset B_1 \Rightarrow f^{-1}(B_0) \subset f^{-1}(B_1)$.

(b) $f^{-1}(B_0 \cup B_1) = f^{-1}(B_0) \cup f^{-1}(B_1)$.

(c) $f^{-1}(B_0 \cap B_1) = f^{-1}(B_0) \cap f^{-1}(B_1)$.

(d) $f^{-1}(B_0 - B_1) = f^{-1}(B_0) - f^{-1}(B_1)$.

Show that f preserves inclusions and unions only:

(e) $A_0 \subset A_1 \Rightarrow f(A_0) \subset f(A_1)$.

(f) $f(A_0 \cup A_1) = f(A_0) \cup f(A_1)$.

(g) $f(A_0 \cap A_1) \subset f(A_0) \cap f(A_1)$; show that equality holds if f injective.

(h) $f(A_0 - A_1) \supset f(A_0) - f(A_1)$; show that equality holds if f injective.

Solution:

(a)

Proof. Suppose that $B_0 \subset B_1$ and consider any $x \in f^{-1}(B_0)$. Then by the definition of a preimage, we have $f(x) \in B_0$ so that also $f(x) \in B_1$ since $B_0 \subset B_1$. This shows that $x \in f^{-1}(B_1)$ again by the definition of a preimage. Thus $f^{-1}(B_0) \subset f^{-1}(B_1)$ since x was arbitrary as desired. ☐

(b)

Proof. We can show this easily using a string of biconditionals. For any $x \in A$ we have

$$\begin{aligned} x \in f^{-1}(B_0 \cup B_1) &\Leftrightarrow f(x) \in B_0 \cup B_1 \\ &\Leftrightarrow f(x) \in B_0 \vee f(x) \in B_1 \\ &\Leftrightarrow x \in f^{-1}(B_0) \vee x \in f^{-1}(B_1) \\ &\Leftrightarrow x \in f^{-1}(B_0) \cup f^{-1}(B_1)\,, \end{aligned}$$

which shows the desired result. ☐

(c)

Proof. We can show this in a very similar manner to what was done in part (b). We have

$$\begin{aligned} x \in f^{-1}(B_0 \cap B_1) &\Leftrightarrow f(x) \in B_0 \cap B_1 \\ &\Leftrightarrow f(x) \in B_0 \wedge f(x) \in B_1 \\ &\Leftrightarrow x \in f^{-1}(B_0) \wedge x \in f^{-1}(B_1) \\ &\Leftrightarrow x \in f^{-1}(B_0) \cap f^{-1}(B_1)\,, \end{aligned}$$

for any $x \in A$. ☐

(d)

Proof. This is also shown similarly. For $x \in A$ we have

$$\begin{aligned} x \in f^{-1}(B_0 - B_1) &\Leftrightarrow f(x) \in B_0 - B_1 \\ &\Leftrightarrow f(x) \in B_0 \wedge f(x) \notin B_1 \\ &\Leftrightarrow x \in f^{-1}(B_0) \wedge x \notin f^{-1}(B_1) \\ &\Leftrightarrow x \in f^{-1}(B_0) - f^{-1}(B_1)\,. \end{aligned}$$

☐

(e)

Proof. Suppose that $A_0 \subset A_1$ and consider any $y \in f(A_0)$. Then there is an $x \in A_0$ where $y = f(x)$ by the definition of an image set. Then also $x \in A_1$ since $A_0 \subset A_1$, from which it follows that $y = f(x) \in f(A_1)$. Therefore $f(A_0) \subset f(A_1)$ as desired since y was arbitrary. ☐

(f)

Proof. We can show this easily using a string of biconditionals. For any $x \in A$ we have

$$\begin{aligned} y \in f(A_0 \cup A_1) &\Leftrightarrow \exists x(x \in A_0 \cup A_1 \wedge y = f(x)) \\ &\Leftrightarrow \exists x[(x \in A_0 \vee x \in A_1) \wedge y = f(x)] \\ &\Leftrightarrow \exists x[(x \in A_0 \wedge y = f(x)) \vee (x \in A_1 \wedge y = f(x))] \\ &\Leftrightarrow \exists x(x \in A_0 \wedge y = f(x)) \vee \exists x(x \in A_1 \wedge y = f(x)) \end{aligned}$$

$$\Leftrightarrow y \in f(A_0) \vee y \in f(A_1)$$
$$\Leftrightarrow y \in f(A_0) \cup f(A_1)\,,$$

which shows the desired result. □

(g)

Proof. Consider any $y \in f(A_0 \cap A_1)$ so that there is an $x \in A_0 \cap A_1$ where $y = f(x)$. Hence of course $x \in A_0$ and $x \in A_1$. Since also $y = f(x)$, this suffices to show that $y \in f(A_0)$ and $y \in f(A_1)$, and therefore $y \in f(A_0) \cap f(A_1)$ as desired.

Now suppose that f is injective and consider any $y \in f(A_0) \cap f(A_1)$. Then $y \in f(A_0)$ and $y \in f(A_1)$, from which it follows that there is an $x_0 \in A_0$ where $y = f(x_0)$, and an $x_1 \in A_1$ where $y = f(x_1)$. We then have $f(x_0) = y = f(x_1)$ so that $x_0 = x_1$ since f is injective. Hence $x_0 \in A_0$ and $x_0 = x_1 \in A_1$, so of course $x_0 \in A_0 \cap A_1$. Since also $y = f(x_0)$, this shows by definition that $y \in f(A_0 \cap A_1)$. Therefore $f(A_0) \cap f(A_1) \subset f(A_0 \cap A_1)$ since y was arbitrary, which shows the desired equivalence since the other direction was already shown. □

(h)

Proof. Consider any $y \in f(A_0) - f(A_1)$ so that $y \in f(A_0)$ and $y \notin f(A_1)$. Then there is an $x \in A_0$ where $y = f(x)$. We also have that there is no $x' \in A_1$ such that $y = f(x')$. Since we know that $y = f(x)$ it then has to be that $x \notin A_1$. Hence $x \in A_0 - A_1$, so that $y \in f(A_0 - A_1)$ since of course $y = f(x)$. This shows that $f(A_0 - A_1) \supset f(A_0) - f(A_1)$ as desired since y was arbitrary.

Now suppose that f is injective and consider any $y \in f(A_0 - A_1)$. Then there is an $x \in A_0 - A_1$ where $y = f(x)$ by the definition of an image set. Then $x \in A_0$ but $x \notin A_1$. It then follows that $y \in f(A_0)$ since $y = f(x)$ and $x \in A_0$. Consider any $x' \in A_1$. Then it cannot be that $y = f(x')$, because if this were the case then $f(x) = y = f(x')$ so that $x = x'$ since f is injective. But we know that $x' = x \notin A_1$, which would present a contradiction. So it must be that there is no $x' \in A_1$ where $y = f(x')$, which suffices to show that $y \notin f(A_1)$. Therefore $y \in f(A_0) - f(A_1)$ so that $f(A_0 - A_1) \subset f(A_0) - f(A_1)$ since y was arbitrary. This of course shows equivalence as desired. □

Exercise 2.3

Show that (b), (c), (f), and (g) of Exercise 2 hold for arbitrary unions and intersections.

Solution:

In what follows suppose that $f : A \to B$ and that $\mathcal{A}$ and $\mathcal{B}$ are nonempty collections of subsets of A and B, respectively. This is to say that $A' \subset A$ for all $A' \in \mathcal{A}$ and $B' \subset B$ for all $B' \in \mathcal{B}$.

First we show that part in Exercise 2.2 part (b) holds for arbitrary unions, i.e. that

$$f^{-1}\left(\bigcup_{B' \in \mathcal{B}} B'\right) = \bigcup_{B' \in \mathcal{B}} f^{-1}(B')\,.$$

Proof. As before, we again show this with a string of biconditional assertions:

$$x \in f^{-1}\left(\bigcup_{B' \in \mathcal{B}} B'\right) \Leftrightarrow f(x) \in \bigcup_{B' \in \mathcal{B}} B'$$
$$\Leftrightarrow \exists B' \in \mathcal{B}(f(x) \in B')$$

$$\Leftrightarrow \exists B' \in \mathcal{B}(x \in f^{-1}(B'))$$
$$\Leftrightarrow x \in \bigcup_{B' \in \mathcal{B}} f^{-1}(B')$$

as desired. □

Next we show Exercise 2.2 part (c) for arbitrary intersections, that is

$$f^{-1}\left(\bigcap_{B' \in \mathcal{B}} B'\right) = \bigcap_{B' \in \mathcal{B}} f^{-1}(B') \,.$$

Proof. We show this with a string of bijections again:

$$\begin{aligned} x \in f^{-1}\left(\bigcap_{B' \in \mathcal{B}} B'\right) &\Leftrightarrow f(x) \in \bigcap_{B' \in \mathcal{B}} B' \\ &\Leftrightarrow \forall B' \in \mathcal{B}(f(x) \in B') \\ &\Leftrightarrow \forall B' \in \mathcal{B}(x \in f^{-1}(B')) \\ &\Leftrightarrow x \in \bigcap_{B' \in \mathcal{B}} f^{-1}(B') \,, \end{aligned}$$

which shows the desired result. □

Now we show Exercise 2.2 part (f) for arbitrary unions, that is that

$$f\left(\bigcup_{A' \in \mathcal{A}} A'\right) = \bigcup_{A' \in \mathcal{A}} f(A') \,.$$

Proof. Again we utilize a string of biconditionals:

$$\begin{aligned} y \in f\left(\bigcup_{A' \in \mathcal{A}} A'\right) &\Leftrightarrow \exists x \left[x \in \bigcup_{A' \in \mathcal{A}} A' \wedge y = f(x)\right] \\ &\Leftrightarrow \exists x \left[\exists A' \in \mathcal{A}(x \in A') \wedge y = f(x)\right] \\ &\Leftrightarrow \exists x \left[\exists A' \in \mathcal{A}(x \in A' \wedge y = f(x))\right] \\ &\Leftrightarrow \exists x \exists A' \in \mathcal{A}(x \in A' \wedge y = f(x)) \\ &\Leftrightarrow \exists A' \in \mathcal{A} \exists x(x \in A' \wedge y = f(x)) \\ &\Leftrightarrow \exists A' \in \mathcal{A} \left[\exists x(x \in A' \wedge y = f(x))\right] \\ &\Leftrightarrow \exists A' \in \mathcal{A} \left[y \in f(A')\right] \\ &\Leftrightarrow y \in \bigcup_{A' \in \mathcal{A}} f(A') \,, \end{aligned}$$

from which the result follows immediately. □

Lastly, we show Exercise 2.2 part (g) for arbitrary intersections, which is that

$$f\left(\bigcap_{A' \in \mathcal{A}} A'\right) \subset \bigcap_{A' \in \mathcal{A}} f(A') \,,$$

where equality holds if f is injective.

Proof. First suppose that $y \in f\left(\bigcap_{A' \in \mathcal{A}} A'\right)$ so that there is an $x \in \bigcap_{A' \in \mathcal{A}} A'$ where $y = f(x)$. Then $x \in A'$ for every $A' \in \mathcal{A}$. So, for any such $A' \in \mathcal{A}$, we have that $x \in A'$ and $y = f(x)$ so that $y \in f(A')$. Since A' was arbitrary, this shows that $y \in \bigcap_{A' \in \mathcal{A}} f(A')$, which shows the desired result since y was arbitrary.

Now suppose that f is injective and let $y \in \bigcap_{A' \in \mathcal{A}} f(A')$. Then $y \in f(A')$ for every $A' \in \mathcal{A}$. So, for any such $A_0 \in \mathcal{A}$ we have that $y \in f(A_0)$ so that there is a $x_0 \in A_0$ where $y = f(x_0)$. Suppose for the moment that $x_0 \notin \bigcap_{A' \in \mathcal{A}} A'$ so that there is an $A_1 \in \mathcal{A}$ where $x_0 \notin A_1$. However, since $A_1 \in \mathcal{A}$ we have that $y \in f(A_1)$, and hence there is an $x_1 \in A_1$ where $y = f(x_1)$. But then we have $f(x_0) = y = f(x_1)$ so that $x_0 = x_1$ since f is injective, and so we have that both $x_0 \notin A_1$ and $x_0 = x_1 \in A_1$. As this is a contradiction, it has to be that $x_0 \in \bigcap_{A' \in \mathcal{A}} A'$. Since also $y = f(x_0)$, this shows that $y \in f\left(\bigcap_{A' \in \mathcal{A}} A'\right)$. This shows that $f\left(\bigcap_{A' \in \mathcal{A}} A'\right) \supset \bigcap_{A' \in \mathcal{A}} f(A')$ since y was arbitrary, which in turns proves the desired equivalence. □

Exercise 2.4

Let $f : A \to B$ and $g : B \to C$.

(a) If $C_0 \subset C$, show that $(g \circ f)^{-1}(C_0) = f^{-1}(g^{-1}(C_0))$.

(b) If f and g are injective, show that $g \circ f$ is injective.

(c) If $g \circ f$ is injective, what can you say about the injectivity of f and g?

(d) If f and g are surjective, show that $g \circ f$ is surjective.

(e) If $g \circ f$ is surjective, what can you say about the surjectivity of f and g?

(f) Summarize your answers to (b)-(e) in the form of a theorem.

Solution:

(a)

Proof. Suppose that $C_0 \subset C$. We can show this with a string of biconditionals. For any x, we have

$$\begin{aligned} x \in (g \circ f)^{-1}(C_0) &\Leftrightarrow (g \circ f)(x) \in C_0 \\ &\Leftrightarrow g(f(x)) \in C_0 \\ &\Leftrightarrow f(x) \in g^{-1}(C_0) \\ &\Leftrightarrow x \in f^{-1}(g^{-1}(C_0)), \end{aligned}$$

which of course shows the desired result. □

(b)

Proof. Suppose that $x, y \in A$ and $x \neq y$. Then, since f is injective, it has to be that $f(x) \neq f(y)$ by the contrapositive of the definition of an injection. Then again $(g \circ f)(x) = g(f(x)) \neq g(f(y)) = (g \circ f)(y)$ since $f(x) \neq f(y)$ and g is injective. This shows that $g \circ f$ is injective by the contrapositive of the definition. □

(c) Here we claim that if $g \circ f$ is injective, then f must be injective but g may not be.

Proof. Suppose that $g \circ f$ is injective but that f is not. Then there are $x, y \in A$ where $x \neq y$ but $f(x) = f(y)$. Then we have

$$(g \circ f)(x) = g(f(x)) = g(f(y)) = (g \circ f)(y)\,,$$

which contradicts the fact that $g \circ f$ is injective since $x \neq y$. So it must be that f is injective.

To show that g need not be injective, consider the sets

$$A = \{1, 2\} \qquad B = \{1, 2, 3\} \qquad C = \{a, b\}$$

and the function sets

$$f = \{(1, 1), (2, 2)\} \qquad g = \{(1, a), (2, b), (3, b)\}\ .$$

It is easy to see that $f : A \to B$ is injective as is the composition $g \circ f = \{(1, a), (2, b)\}$, but that $g : B \to C$ is not since $g(2) = b = g(3)$. □

(d)

Proof. Suppose that f and g are surjective and consider any $z \in C$. Then there is a $y \in B$ where $z = g(y)$ since g is surjective. Since f is also surjective, there is then an $x \in A$ where $y = f(x)$. Then we have

$$(g \circ f)(x) = g(f(x)) = g(y) = z\,,$$

which shows that $g \circ f$ is surjective as desired since z was arbitrary. □

(e) We claim that if $g \circ f$ is surjective, then g must be surjective, but f may not be.

Proof. Suppose that $g \circ f$ is surjective and consider any $z \in C$ so that there is an $x \in A$ where $(g \circ f)(x) = z$. Then we have that $g(f(x)) = z$ so that $y = f(x)$ is an element of B where $g(y) = z$. This shows that g is surjective since z was arbitrary.

To show that f need not be surjective we can use the same example sets A, B, C and functions f, g used in part (c). It is easy to see there that $g \circ f$ and g are surjective but f is not since there is no element of A that maps to $3 \in B$. □

(f) We can summarize these facts in the following theorem, whose proof is of course found in the previous parts:

Theorem 2.4.1. *Suppose that $f : A \to B$ and $g : B \to C$. We assert the following:*

(1) If f and g are injective then $g \circ f$ is injective.

(2) If $g \circ f$ is injective then f is also injective.

(3) If f and g are surjective then $g \circ f$ is surjective.

(4) If $g \circ f$ is surjective then g is also surjective.

Exercise 2.5

In general, let us denote the ***identity function*** for a set C by i_C. That is, define $i_C : C \to C$ to be the function given by the rule $i_C(x) = x$ for all $x \in C$. Given $f : A \to B$, we say that $g : B \to A$ is a ***left inverse*** for f if $g \circ f = i_A$; and we say that $h : B \to A$ is a ***right inverse*** for f if $f \circ h = i_B$.

(a) Show that if f has a left inverse, f is injective; and if f has a right inverse, f is surjective.

(b) Give an example of a function that has a left inverse but no right inverse.

(c) Give an example of a function that has a right inverse but no left inverse.

(d) Can a function have more than one left inverse? More than one right inverse?

(e) Show that if f has both a left inverse g and a right inverse h, then f is bijective and $g = h = f^{-1}$.

Solution:

In what follows we suppose that $f : A \to B$.

(a)

Proof. First suppose that f has a left inverse $g : B \to A$ so that $g \circ f = i_A$. Consider any $x, y \in A$ where $f(x) = f(y)$. Then we have

$$x = i_A(x) = (g \circ f)(x) = g(f(x)) = g(f(y)) = (g \circ f)(y) = i_A(y) = y\,,$$

which shows that f is injective by definition.

Now suppose that f has a right inverse $h : B \to A$ so that $f \circ h = i_B$. Consider any $y \in B$ so that

$$y = i_B(y) = (f \circ h)(y) = f(h(y))\,.$$

Then $x = h(y)$ is an element of A such that $f(x) = y$, which shows that f must be surjective since y was arbitrary. □

(b) Consider the sets

$$A = \{1, 2\} \qquad\qquad B = \{a, b, c\}$$

and the function $f = \{(1, a), (2, b)\}$. Define the function $g : B \to A$ by $g = \{(a, 1), (b, 2), (c, 2)\}$. It is easy to see that this is a left inverse of f since we have

$$(g \circ f)(1) = g(f(1)) = g(a) = 1 \qquad\qquad (g \circ f)(2) = g(f(2)) = g(b) = 2$$

so that $g \circ f = i_A$.

Also note that clearly f is not surjective since there is no element of A that maps to $c \in B$. This suffices to show that f cannot have a right inverse since, if it did, then it would have to be surjective by part (a).

(c) Now define the sets

$$A = \{1, 2, 3\} \qquad\qquad B = \{a, b\}$$

and the function $f = \{(1, a), (2, b), (3, a)\}$. Define the function $h : B \to A$ by $h = \{(a, 1), (b, 2)\}$. Then we have

$$(f \circ h)(a) = f(h(a)) = f(1) = a \qquad\qquad (f \circ h)(b) = f(h(b)) = f(2) = b$$

so that clearly $f \circ h = i_B$, and hence h is a right inverse of f.

Note, however, that f is not injective since $f(1) = a = f(3)$. This suffices to show that f cannot have a left inverse since, if it did, it would be injective by part (a).

(d) We claim that a function can have more than one right or left inverse.

Proof. To show that a function can have more than one left inverse consider the example constructed in part (b). Recall that this consists of the sets

$$A = \{1, 2\} \qquad\qquad B = \{a, b, c\}$$

and the function $f = \{(1, a), (2, b)\}$. It was shown there that the function $g_1 = \{(a, 1), (b, 2), (c, 2)\}$ is a left inverse. Let $g_2 = \{(a, 1), (b, 2), (c, 1)\}$ so that clearly $g_1 \neq g_2$ since $g_1(c) = 2 \neq 1 = g_2(c)$. It is trivial to show that g_2 is also a left inverse of f, which shows that more than one left inverse exists for this f.

To show that a function can have more than one right inverse, consider the example in part (c), which are the sets

$$A = \{1, 2, 3\} \qquad\qquad B = \{a, b\}$$

and the function $f = \{(1, a), (2, b), (3, a)\}$. It was shown there that the function $h_1 = \{(a, 1), (b, 2)\}$ is a right inverse. Let $h_2 = \{(a, 3), (b, 2)\}$ so that clearly $h_1 \neq h_2$ since $h_1(a) = 1 \neq 3 = h_2(a)$. However, it is trivial to show that h_2 is also a right inverse of f, from which the desired result follows. □

(e) Note that what follows proves Lemma 2.1 in the text, which is not proven there.

Proof. Suppose that f has left inverse g and right inverse h. Then f must be both injective (since it has a left inverse) and surjective (since it has a right inverse) so that it is bijective by definition. Then of course the function $f^{-1} : B \to A$ exists. Consider any $y \in B$ and set $x = f^{-1}(y)$ so that $y = f(x)$. Then we have that

$$g(y) = g(f(x)) = (g \circ f)(x) = i_A(x) = x$$

since g is a left inverse of f. We also have

$$f(h(y)) = (f \circ h)(y) = i_B(y) = y$$

so that

$$h(y) = f^{-1}(f(h(y)) = f^{-1}(y) = x\,.$$

This shows that $x = f^{-1}(y) = g(y) = h(y)$, which in turn shows that $f^{-1} = g = h$ as desired since y was arbitrary. □

Exercise 2.6

Let $f : \mathbb{R} \to \mathbb{R}$ be the function $f(x) = x^3 - x$. By restricting the domain and range of f appropriately, obtain from f a bijective function g. Draw the graphs of g and g^{-1}. (There are several possible choices for g.)

Solution:

Define the subsets of the reals $A = [1, \infty)$ and $B = [0, \infty)$. We claim that the function $g : A \to B$ defined by $g(x) = f(x) = x^3 - x$ for all $x \in A$ is bijective.

Proof. First we will show that B can even be a range for g, i.e. we must show that $g(x) \in B$ for every $x \in A$, as this is not necessarily obvious. So for any $x \in A$ we have that $x \geq 1$, and thus

$x^2 \geq 1$ as well. Then we have $x^2 - 1 \geq 0$ so that the product $x(x^2-1) \geq 0$ since of course $x \geq 1 > 0$. Therefore $g(x) = f(x) = x^3 - x = x(x^2-1) \geq 0$ so that $g(x) \in B$.

Next we show that g is monotonically increasing, from which injectivity follows. Suppose that $x, y \in A$ where $x < y$. Since we have $x, y \geq 1 > 0$, it follows that $x^2 < y^2$, and therefore $x^2 - 1 < y^2 - 1$. Thus we have

$$\begin{gathered} x < y \\ x(x^2-1) \leq y(x^2-1) < y(y^2-1) \\ x^3 - x < y^3 - y \\ g(x) < g(y) \end{gathered}$$

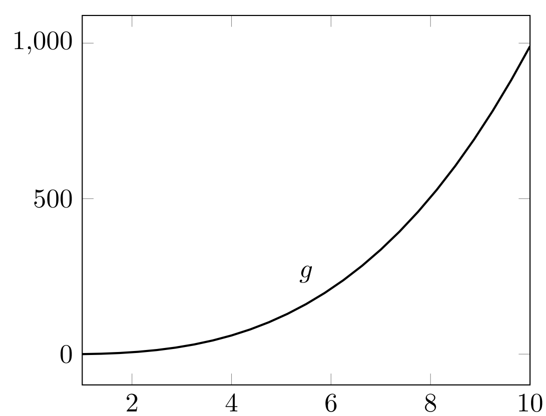
since both $x, y \geq 1 > 0$ and $x^2 - 1, y^2 - 1 \geq 0$. This shows that g is monotonically increasing. It follows that g is injective because, if we consider $x, y \in A$ where $x \neq y$, then it has to be that either $x < y$ or $x > y$. In the former case we have $g(x) < g(y)$ and in the latter $g(x) > g(y)$ so that $g(x) \neq g(y)$ either way.

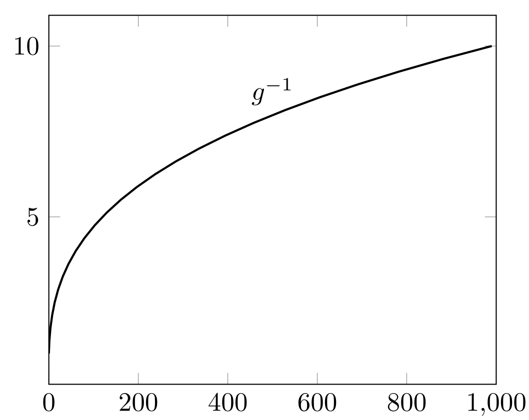
We show that g is surjective in a roundabout way that depends on calculus since g is cubic and so does not yield a simple algebraic inverse function. Consider any $y \in B$ so that of course $y \geq 0$. If $y = 0$ then clearly $g(1) = 0 = y$, so assume that $y > 0$. Let $a_0 = \max\{2, y\}$ so that of course there is a real a such that $a > a_0$ since the reals are unbounded. Then we of course have $a > a_0 \geq 2 \geq 0$ so that $a \in A$. We also have

$$\begin{gathered} a^2 > a_0^2 \geq 2^2 = 4 > 2 \\ a^2 - 1 > 1 \\ a(a^2-1) > a \\ a^3 - a > a \\ g(a) > a > a_0 \geq y\,. \end{gathered}$$

Then we have that $g(1) = 0 < y < g(a)$, and that of course $1 < 2 \leq a_0 < a$. It then follows from the intermediate value theorem that there is an $x \in (1, a)$ such that $f(x) = y$ since clearly g is continuous by elementary calculus. We note that of course $1 \leq x$ so that $x \in A$. This shows that g is surjective since y was arbitrary, which in turn completes the proof that g is a bijection. $\square$

As requested, below are graphs of g and g^{-1} over some subset of their infinite domains and ranges:





One can observe that g (and its inverse for that matter) is monotonically increasing as shown.

§3 Relations

Exercise 3.1

Define two points (x_0, y_0) and (x_1, y_1) of the plane to be equivalent if $y_0 - x_0^2 = y_1 - x_1^2$. Check that this is an equivalence relation and describe the equivalence classes.

Solution:

First we show that this relation, which we shall denote with $\sim$, is an equivalence relation.

Proof. In what follows, suppose that (x_0, y_0), (x_1, y_1), and (x_2, y_2) are all points in the plane.

(Reflexivity) Of course we have $y_0 - x_0^2 = y_0 - x_0^2$, and hence $(x_0, y_0) \sim (x_0, y_0)$.

(Symmetry) Suppose that $(x_0, y_0) \sim (x_1, y_1)$. Then we have $y_0 - x_0^2 = y_1 - x_1^2$ so that of course $y_1 - x_1^2 = y_1 - x_1^2$ since numerical equality is symmetric, and so $(x_1, y_1) \sim (x_0, y_0)$ as well.

(Transitivity) Suppose that $(x_0, y_0) \sim (x_1, y_1)$ and $(x_1, y_1) \sim (x_2, y_2)$. Then $y_0 - x_0^2 = y_1 - x_1^2$ and $y_1 - x_1^2 = y_2 - x_2^2$ so that of course $y_0 - x_0^2 = y_2 - x_2^2$ since numerical equality is transitive. Therefore $(x_0, y_0) \sim (x_2, y_2)$, which shows transitivity.

This suffices to show that $\sim$ is an equivalence relation as we set out to show. □

Each equivalence class formed by this relation is the parabola $y = x^2$ shifted up or down on the y-axis. This is easy to see since two points (x_0, y_0) and (x_1, y_1) are in the same class if $y_0 - x_0^2$ and $y_1 - x_1^2$ have the same value, say c. Then $y_0 - x_0^2 = c$ so that $y_0 = x_0^2 + c$, which is clearly such a parabola, and similarly $y_1 = x_1^2 + c$.

Exercise 3.2

Let C be a relation on a set A. If $A_0 \subset A$, define the ***restriction*** of C to A_0 to be the relation $C \cap (A_0 \times A_0)$. We also note that clearly $C_0 \subset C$ as well. Show that the restriction of an equivalence relation is an equivalence relation.

Solution:

Proof. Define C, A, and A_0 as above and suppose that C is an equivalence relation. Let $C_0 = C \cap (A_0 \times A_0)$ be the restriction of C to A_0, noting that this is in fact a relation on A_0 since clearly $C_0 \subset A_0 \times A_0$. Now we show that C_0 satisfies the three properties of an equivalence relation.

(Reflexivity) Consider any $a \in A_0$ so that of course $(a, a) \in A_0 \times A_0$. Since $A_0 \subset A$ we also have that $a \in A$. Hence aCa since C is an equivalence relation on A and is therefore reflexive. Thus $(a, a) \in C \cap (A_0 \times A_0) = C_0$, which shows that aC_0a so that C_0 is reflexive since a was arbitrary.

(Symmetry) Suppose that $a, b \in A_0$ and that aC_0b. Then of course $(b, a) \in A_0 \times A_0$ and bCa since $C_0 \subset C$. From this it follows that $(b, a) \in C \cap (A_0 \times A_0) = C_0$ so that bC_0a. This of course shows that C_0 is symmetric.

(Transitivity) Now consider $a, b, c \in A_0$ and suppose that both aC_0b and bC_0c. Then we have aCb and bCc since $C_0 \subset C$. Since C is an equivalence relation and therefore transitive, it follows that aCc, and since also clearly $(a, c) \in A_0 \times A_0$, we have $(a, c) \in C \cap (A_0 \times A_0) = C_0$ so that aC_0c. This shows that C_0 is transitive. □

Exercise 3.3

Here is a "proof" that every relation C that is both symmetric and transitive is also reflexive: "Since

C is symmetric, aCb implies bCa. Since C is transitive, aCb and bCa together imply aCa, as desired." Find the flaw in this argument.

Solution:

Suppose that C is a relation on the set A. This argument is perfectly valid for any $a, b \in A$ such that aCb, which is to say that we can conclude that aCa in this case (and by the same argument bCb). However, reflexivity requires aCa to hold for *every* $a \in A$. So if there is no $b \in A$ such that aCb then the above argument cannot be applied and we cannot conclude that aCa. In this case the element a is effectively not involved in the relation at all.

This is perhaps best illustrated with an example: suppose that $A = \{1, 2, 3, 4\}$ and

$$C = \{(1,1), (2,2), (3,3), (1,2), (2,1), (2,3), (3,2), (1,3), (3,1)\} .$$

It is easy to verify that C is both symmetric and transitive on A but it is clearly not reflexive since $(4,4) \notin C$. One can also observe how 4 is not involved in the relation at all and, if it were, it would have to be that $(4,4) \in C$ if C were to remain symmetric and transitive.

Exercise 3.4

Let $f : A \to B$ be a surjective function. Let us define a relation on A by setting $a_0 \sim a_1$ if

$$f(a_0) = f(a_1) .$$

(a) Show that this is an equivalence relation.

(b) Let A^* be the set of equivalence classes. Show that there is a bijective correspondence of A^* with B.

Solution:

(a)

Proof. We show the three properties necessary for $\sim$ to be an equivalence relation:

(Reflexivity) Consider any $a \in A$ so that of course $f(a) = f(a)$ since f is a function. Hence $a \sim a$ so that $\sim$ is reflexive since a was arbitrary.

(Symmetry) Consider $a, b \in A$ and suppose that $a \sim b$. Then by definition $f(a) = f(b)$ so that obviously also $f(b) = f(a)$ since equality is symmetric. So of course $b \sim a$, which shows that $\sim$ is symmetric.

(Transitivity) Consider $a, b, c \in A$ and suppose that $a \sim b$ and $b \sim c$. Then by definition $f(a) = f(b)$ and $f(b) = f(c)$ so that of course $f(a) = f(b) = f(c)$, and hence $a \sim c$. This shows that $\sim$ is transitive. □

(b)

Proof. Define the function $g : A^* \to B$ as follows. For any equivalence class $C \in A^*$, we know that C is nonempty since A^* is a partition of A. Hence there is an $a \in C$, so set $g(C) = f(a)$, noting that clearly $g(C) = f(a) \in B$ so that B can be the range of g.

To show that g is injective, consider two equivalence classes C and D where $g(C) = g(D)$. Then there are elements $c \in C$ and $d \in D$ where $f(c) = g(C) = g(D) = f(d)$. This shows that $c \sim d$ so that they must be in the same equivalence class. Thus $d \in C$ since $c \in C$, but also $d \in D$ so that C and D are not disjoint. Hence it must be that $C = D$ by Lemma 3.1, which shows that g is injective.

To show that g is surjective, consider any $b \in B$. Since f is surjective, there is an $a \in A$ such that $f(a) = b$. Since A^* is a partition, a must belong to an equivalence class $C \in A^*$. Then there is an element $c \in C$ such that $g(C) = f(c)$ by the definition of g. Since a and c are both in the same equivalence class C, we have that $a \sim c$ so that $g(C) = f(c) = f(a) = b$. This shows that g is surjective since $b \in B$ was arbitrary.

Therefore we have shown that g is both injective and surjective, and so is a bijection by definition, as desired. □

Exercise 3.5

Let S and S' be the following subsets of the plane:

$$S = \{(x, y) \mid y = x + 1 \text{ and } 0 < x < 2\} \,,$$
$$S' = \{(x, y) \mid y - x \text{ is an integer}\} \,.$$

(a) Show that S' is an equivalence relation on the real line and $S' \supset S$. Describe the equivalence classes of S'.

(b) Show that given any collection of equivalence relations on a set A, their intersection is an equivalence relation on A.

(c) Describe the equivalence relation T on the real line that is the intersection of all equivalence relations on the real line that contain S. Describe the equivalence classes of T.

Solution:

(a)

Proof. First note that $S' \subset \mathbb{R} \times \mathbb{R}$ and so is a relation on $\mathbb{R}$. We show that S' has the three properties required of an equivalence relation.

(Reflexivity) Consider any $x \in \mathbb{R}$ so that clearly $x - x = 0$ is an integer. Hence $(x, x) \in S'$ by definition. This shows that S' is reflexive since x was arbitrary.

(Symmetry) Suppose that $x, y \in \mathbb{R}$ and $xS'y$. Then $n = y - x$ is an integer so that $x - y = -(y - x) = -n$ is also clearly an integer. Therefore $yS'x$ as well, which shows that S' is symmetric.

(Transitivity) Consider $x, y, z \in \mathbb{R}$ and suppose that both $xS'y$ and $yS'z$. Then $n = y - x$ and $m = z - y$ are both integers. We then have

$$z - x = z - x + y - y = (z - y) + (y - x) = m + n \,,$$

which is clearly an integer since m and n are. Hence $xS'z$ so that S' is transitive.

It is easy to show that $S' \supset S$. Consider any $(x, y) \in S$ so that $0 < x < 2$ and $y = x + 1$. Then $y - x = (x + 1) - x = 1$, which is of course an integer. Hence $(x, y) \in S'$, and thus $S' \supset S$ since (x, y) was arbitrary. □

The equivalence class C containing $x \in \mathbb{R}$ is the countable set $C = \{x + n \mid n \in \mathbb{Z}\}$. While perhaps not immediately obvious, it is almost trivial to show:

$$\begin{aligned} y \in C &\Leftrightarrow \exists n \in \mathbb{Z}(y = x + n) \Leftrightarrow \exists n \in \mathbb{Z}(y - x = n) \\ &\Leftrightarrow xS'y \Leftrightarrow yS'x \end{aligned}$$

$$\Leftrightarrow y \text{ is in the equivalence class determined by } x$$

since S' is symmetric.

(b)

Proof. Let A^* be a collection of equivalence relations on A so that we must show that $C = \bigcap_{D \in A^*} D$ is also an equivalence relation on A. First, suppose that any $(x, y) \in C$ and consider any $D \in A^*$ so that $(x, y) \in D$. Then also $(x, y) \in A \times A$ since D is a relation on A so that $D \subset A \times A$. This shows that $C \subset A \times A$ since (x, y) was arbitrary, and so C is a relation on A. Now we show the three required properties of an equivalence relation:

(Reflexivity) Consider any $x \in A$ so that $(x, x) \in D$ for every $D \in A^*$ since each D is an equivalence relation and so is reflexive. It then follows that $(x, x) \bigcap_{D \in A^*} D = C$, which shows that C is reflexive.

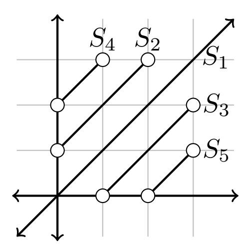
(Symmetric) Suppose that $(x, y) \in C$ and consider any $D \in A^*$ so that also $(x, y) \in D$. Then also $(y, x) \in D$ since D is an equivalence relation and so is symmetric. Since D was arbitrary, this shows that $(y, x) \in \bigcap_{D \in A^*} D = C$ so that C is symmetric.

(Transitivity) Suppose that $(x, y) \in C$ and $(y, z) \in C$. For any $D \in A^*$ we then have that both $(x, y) \in D$ and $(y, z) \in D$. It then follows that $(x, z) \in D$ since D is an equivalence relation and so is transitive. Since D was arbitrary, we have that $(x, z) \in \bigcap_{D \in A^*} D = C$ so that C is transitive as desired. □

(c) First we note that S itself is *not* an equivalence relation on $\mathbb{R}$ since it is not reflexive. In fact $(x, x) \notin S$ for any $x \in \mathbb{R}$ since it is never true that $x = x + 1$. Now define the following subsets of the plane:

$$S_1 = \{(x, y) \mid y = x\} \qquad S_4 = \{(x, y) \mid y = x + 2 \text{ and } 0 < x < 1\}$$
$$S_2 = S = \{(x, y) \mid y = x + 1 \text{ and } 0 < x < 2\} \qquad S_5 = \{(x, y) \mid y = x - 2 \text{ and } 2 < x < 3\} .$$
$$S_3 = \{(x, y) \mid y = x - 1 \text{ and } 1 < x < 3\}$$

We then claim that the intersection we seek is $T = \bigcup_{i \in \{1,\ldots,5\}} S_i$. An illustration of this set in the plane is shown below:



Proof. Let S^* denote the collection of all equivalence relations on $\mathbb{R}$ that contain S so that we must show that $T = \bigcap_{R \in S^*} R$.

($\subset$) Consider any $(x, y) \in T$ and any $R \in S^*$ so that R is an equivalence relation on $\mathbb{R}$ that contains S. We then have the following cases:

Case: $(x, y) \in S_1$. Then of course $y = x$ so that $(x, y) = (x, x) \in R$ since it is an equivalence relation and thus reflexive.

Case: $(x, y) \in S_2 = S$. Then of course $(x, y) \in R$ since R contains S.

Case: $(x, y) \in S_3$. Then we have that $y = x - 1$ with $1 < x < 3$, from which it follows that $x = y + 1$ and $0 < y = x - 1 < 2$. Therefore $(y, x) \in S$ so that $(y, x) \in R$ since R contains S. We then have that $(x, y) \in R$ as well since R is an equivalence relation and therefore symmetric.

Case: $(x, y) \in S_4$. Then $y = x + 2$ with $0 < x < 1$. Let $z = x + 1$ so that $(x, z) \in S = S_4$ since we also have $0 < x < 1 < 2$. We then know that $(x, z) \in R$ since R contains S. We also then have that $y = x + 2 = (x + 1) + 1 = z + 1$ with $0 < 1 < z = x + 1 < 2$ since $0 < x < 1$. Thus $(z, y) \in S = S_4$ so that also $(z, y) \in R$. Since R is an equivalence relation and therefore transitive, we have that $(x, y) \in R$ as well.

Case: $(x, y) \in S_5$. Then we have $y = x - 2$ and $2 < x < 3$. It then follows that $x = y + 2$ and $0 < y = x - 2 < 1$ so that $(y, x) \in S_4$. Then of course $(y, x) \in R$ by the previous case so that also $(x, y) \in R$ since R is an equivalence relation and therefore symmetric.

Thus in every case $(x, y) \in R$ so that also $(x, y) \in \bigcap_{R \in S^*} R$ since R was arbitrary. It then follows that $T \subset \bigcap_{R \in S^*} R$ since (x, y) was arbitrary.

($\supset$) All we really need to show is that T is an equivalence relation on $\mathbb{R}$ that contains S. From this it follows that $T \in S^*$ so that of course $T \supset \bigcap_{R \in S^*} R$. First we note that clearly $T \subset \mathbb{R}^2$ so that it is a relation on $\mathbb{R}$. Also clearly it contains S since $S = S_2 \subset T$. Now we show that it has the three properties that an equivalence relation require.

(Reflexivity) For any $x \in \mathbb{R}$ clearly $(x, x) \in S_1 \subset T$ so that T is reflexive.

(Symmetry) Suppose that xTy so that $(x, y) \in T$. We then have the following cases:

Case: $(x, y) \in S_1$. Then of course $y = x$ so $(y, x) = (x, x) = (x, y) \in S_1 \subset T$.

Case: $(x, y) \in S_2 = S$. Then $y = x + 1$ and $0 < x < 2$ so that $x = y - 1$ and $1 < y = x + 1 < 3$. Hence $(y, x) \in S_3 \subset T$.

Case: $(x, y) \in S_3$. Then we have that $y = x - 1$ with $1 < x < 3$, from which it follows that $x = y + 1$ and $0 < y = x - 1 < 2$. Therefore $(y, x) \in S = S_2 \subset T$.

Case: $(x, y) \in S_4$. Then $y = x + 2$ with $0 < x < 1$ so that $x = y - 2$ and $2 < y = x + 2 < 3$. Hence $(y, x) \in S_5 \subset T$.

Case: $(x, y) \in S_5$. Then we have $y = x - 2$ and $2 < x < 3$ so that $x = y + 2$ and $0 < y = x - 2 < 1$. Hence $(y, x) \in S_4 \subset T$.

So in all cases $(y, x) \in T$, which shows that T is symmetric.

(Transitivity) Now suppose that xTy and yTz. If $x = y$ then of course we have $(x, z) = (y, z) \in T$. Similarly if $y = z$ then $(x, z) = (x, y) \in T$. So assume that $x \neq y$ and $y \neq z$ so that it can neither be that $(x, y) \in S_1$ nor $(y, z) \in S_1$. Thus there are four sets (i.e. S_i where $i \in \{2, 3, 4, 5\}$) that (x, y) and (y, z) can be in, which results in sixteen different possibilities, though not all are possible:

Case: $(x, y) \in S_2$. Then $y = x + 1$ and $0 < x < 2$ so that $1 < y = x + 1 < 3$.

> Case: $(y, z) \in S_2$. Then also $z = y + 1$ and $0 < y < 2$ so that $z = y + 1 = (x + 1) + 1 = x + 2$ and $y = x + 1 < 2$ means that $x < 1$. Hence $z = x + 2$ and $0 < x < 1$ so that $(x, z) \in S_4 \subset T$.
>
> Case: $(y, z) \in S_3$. Then $z = y - 1$ and $1 < y < 3$ so that $z = y - 1 = (x + 1) - 1 = x$, and hence $(x, z) = (x, x) \in S_1 \subset T$.
>
> Case: $(y, z) \in S_4$. This case is not possible because $1 < y < 3$ so that it cannot be that $0 < y < 1$ and hence (y, z) cannot be in S_4.
>
> Case: $(y, z) \in S_5$. Then $z = y - 2$ and $2 < y < 3$ so that $z = y - 2 = (x + 1) - 2 = x - 1$ and $2 < y = x + 1 < 3$ means that $1 < x < 2 < 3$. Hence $z = x - 1$ and $1 < x < 3$ so that $(x, z) \in S_3 \subset T$.

Case: $(x, y) \in S_3$. Then $y = x - 1$ and $1 < x < 3$ so that $0 < y = x - 1 < 2$.

> Case: $(y, z) \in S_2$. Then $z = y + 1$ and $0 < y < 2$ so that $z = y + 1 = (x - 1) + 1 = x$ and hence $(x, z) = (x, x) \in S_1 \subset T$.
>
> Case: $(y, z) \in S_3$. Then $z = y - 1$ and $1 < y < 3$ so that $z = y - 1 = (x - 1) - 1 = x - 2$ and $1 < y = x - 1$ and hence $2 < x$. Therefore $z = x - 2$ and $2 < x < 3$ so that $(x, z) \in S_5 \subset T$.

Case: $(y,z) \in S_4$. Then $z = y+2$ and $0 < y < 1$ so that $z = y+2 = (x-1)+2 = x+1$ and $y = x-1 < 1$ and hence $x < 2$. Thus $z = x+1$ and $0 < 1 < x < 2$ so that $(x,z) \in S_2 \subset T$.

Case: $(y,z) \in S_5$. This is case is not possible because $y < 2$ so that it cannot be that $2 < y < 3$, and hence (y,z) cannot be in S_5.

Case: $(x,y) \in S_4$. Then $y = x+2$ and $0 < x < 1$ so that $2 < y = x+2 < 3$.

Case: $(y,z) \in S_2$. This case is also impossible because $2 < y$ so that it cannot be that $0 < y < 2$, and hence (y,z) cannot be in S_2.

Case: $(y,z) \in S_3$. Then $z = y-1$ and $1 < y < 3$ so that $z = y-1 = (x+2)-1 = x+1$ and $y = x+2 < 3$ so that $x < 1 < 2$. Thus $z = x+1$ and $0 < x < 2$ so that $(x,z) \in S_2 \subset T$.

Case: $(y,z) \in S_4$. This case is not possible because again $2 < y$ so that it cannot be that $0 < y < 1$, and hence (y,z) cannot be in S_4.

Case: $(y,z) \in S_5$. Then $z = y-2$ and $0 < y < 1$ so that $z = y-2 = (x+2)-2 = x$ and hence $(x,z) = (x,x) \in S_1 \subset T$.

Case: $(x,y) \in S_5$. Then $y = x-2$ and $2 < x < 3$ so that $0 < y = x-2 < 1$.

Case: $(y,z) \in S_2$. Then $z = y+1$ and $0 < y < 2$ so that $z = y+1 = (x-2)+1 = x-1$ and $0 < y = x-2$ so that $1 < 2 < x$. Therefore $z = x-1$ and $1 < x < 3$ so that $(x,z) \in S_3 \subset T$.

Case: $(y,z) \in S_3$. This case is not possible because $y < 1$ so that it cannot be that $1 < y < 3$, and hence (y,z) cannot be in S_3.

Case: $(y,z) \in S_4$. Then $z = y+2$ and $0 < y < 1$ so that $z = y+2 = (x-2)+2$ and hence $(z,x) = (x,x) \in S_1 \subset T$.

Case: $(y,z) \in S_5$. This case is also not possible because again $y < 1$ so that it cannot be that $2 < y < 3$, and hence (y,z) cannot be in S_5.

Thus in every case that is actually possible we have that $(x,z) \in T$, which shows that T is transitive.

Therefore T is an equivalence relation that contains S so that $T \supset \bigcap_{R \in S^*} R$ as discussed above, which of course completes the proof that $T = \bigcap_{R \in S^*} R$. □

As far as the equivalence classes formed by T are concerned, refer to the illustration above. Consider the equivalence class C contains $x \in \mathbb{R}$. If $x \leq 0$ or $x \geq 3$ then $C = \{x\}$ because there is no other y for which yTx except $y = x$. So suppose that $0 < x < 3$. If x is an integer so that $x = 1$ or $x = 2$, then $C = \{1,2\}$. If x is not an integer, then C always has three elements. We have that

$$C = \begin{cases} \{x, x+1, x+2\} & 0 < x < 1 \\ \{x-1, x, x+1\} & 1 < x < 2 \\ \{x-2, x-1, x\} & 2 < x < 3 \end{cases}.$$

These facts can be deduced by examining where the vertical line intersecting the x-axis at x intersects the graph of T.

Exercise 3.6

Define a relation on the plane by setting

$$(x_0, y_0) < (x_1, y_1)$$

if either $y_0 - x_0^2 < y_1 - x_1^2$ or $y_0 - x_0^2 = y_1 - x_1^2$ and $x_0 < x_1$. Show that this is an order relation on the plane, and describe it geometrically.

Solution:

First we show that $<$ is an order relation on the plane.

Proof. As clearly $<$ is a relation on the plane, we need only show that it has the three required properties of an order relation:

(Comparability) Consider distinct (x_0, y_0) and (x_1, y_1) in the plane so that either $x_0 \neq x_1$ or $y_0 \neq y_1$. Obviously if $y_0 - x_0^2 < y_1 - x_1^2$ (or $y_1 - x_1^2 < y_0 - x_0^2$) then of course $(x_0, y_0) < (x_1, y_1)$ (or $(x_1, y_1) < (x_0, y_0)$) so we are done. So assume that $y_0 - x_0^2 = y_1 - x_1^2$. It it were the case that $x_0 = x_1$ it would have to be that $y_0 \neq y_1$, but we would have

$$y_0 - x_0^2 = y_1 - x_1^2 = y_1 - x_0^2$$
$$y_0 = y_1 ,$$

which is a contradiction. So it must be that $x_0 \neq x_1$. So either $x_0 < x_1$ and so $(x_0, y_0) < (x_0, y_0)$ or $x_1 < x_0$ and so $(x_1, y_1) < (x_0, y_0)$. This shows that $<$ is comparable in the plane.

(Nonreflexivity) Consider (x, y) in the plane so that obviously $y - x^2 = y - x^2$. As we also have that $x = x$, it is not the case that $x < x$ so that it is not true that $(x, y) < (x, y)$.

(Transitivity) Suppose that $(x_0, y_0) < (x_1, y_1)$ and $(x_1, y_1) < (x_2, y_2)$. We then have the following:

Case: $y_0 - x_0^2 < y_1 - x_1^2$.

Case: $y_1 - x_1^2 < y_2 - x_2^2$. Then of course $y_0 - x_0^2 < y_1 - x_1^2 < y_2 - x_2^2$ so that $(x_0, y_0) < (x_2, y_2)$.

Case: $y_1 - x_1^2 = y_2 - x_2^2$ and $x_1 < x_2$. Then we have $y_0 - x_0^2 < y_1 - x_1^2 = y_2 - x_2^2$ so that again $(x_0, y_0) < (x_2, y_2)$.

Case: $y_0 - x_0^2 = y_1 - x_1^2$ and $x_0 < x_1$.

Case: $y_1 - x_1^2 < y_2 - x_2^2$. Then $y_0 - x_0^2 = y_1 - x_1^2 < y_2 - x_2^2$ so that $(x_0, y_0) < (x_2, y_2)$.

Case: $y_1 - x_1^2 = y_2 - x_2^2$ and $x_1 < x_2$. Then $y_0 - x_0^2 = y_1 - x_1^2 = y_2 - x_2^2$ and $x_0 < x_1 < x_2$ so that $(x_0, y_0) < (x_2, y_2)$.

Thus in all cases $(x_0, y_0) < (x_2, y_2)$, which shows that $<$ is transitive in the plane. □

Geometrically, we refer back to Exercise 3.1 and consider a parabola $y = x^2$ shifted up or down on the y-axis be some amount c. Then $y = x^2 + c$ so that $y - x^2 = c$, and hence every (x, y) point on the parabola has the same value for $y - x^2$, namely c. Therefore if two distinct points (x_0, y_0) and (x_1, y_1) lie on the different parabolas then $y_0 - x_0^2$ and $y_1 - x_1^2$ will have different values, say c and d, respectively. Then clearly if (x_1, y_1) is on a higher parabola on the y-axis then $c < d$ so that $y_0 - x_0^2 = c < d = y_1 - x_1^2$ so that $(x_0, y_0) < (x_1, y_1)$ in our order. If the points lie on the same parabola then $y_0 - x_0^2 = y_1 - x_1^2$ and whichever points is further to the right will be larger in our order since then, for example, $x_0 < x_1$ so that $(x_0, y_0) < (x_1, y_1)$.

Exercise 3.7

Show that the restriction of an order relation is an order relation.

Solution:

Proof. Suppose that A is a set with order relation $<$. Also let A_0 be a subset of A so that $\prec = < \cap (A_0 \times A_0)$ is the restriction of $<$ to A_0. Clearly $\prec \subset A_0 \times A_0$ so that it s a relation on A_0. So we must show that $\prec$ satisfies the three properties of an order relation:

(Comparability) Consider any $x, y \in A_0$ so that also $x, y \in A$ since $A_0 \subset A$. Then x and y are comparable in $<$ since it is an order. So, without loss of generality, we can assume that $x < y$ and so $(x, y) \in <$. Since clearly also $(x, y) \in A_0 \times A_0$, it follows that $(x, y) \in < \cap (A_0 \times A_0) = \prec$ and hence $x \prec y$. This shows that x and y are comparable in $\prec$.

(Nonreflexivity) Suppose that $x \in A_0$ so that also $x \in A$ since $A_0 \subset A$. Then it cannot be true that $x < x$ since A is an order and so is nonreflexive. Thus $(x, x) \notin <$ so that also $(x, x) \notin < \cap (A_0 \times A_0) = \prec$. Hence it is not true that $x \prec x$ so that $\prec$ is also nonreflexive since x was arbitrary.

(Transitivity) Suppose that $x \prec y$ and $y \prec z$. Then of course $(x, y) \in A_0 \times A_0$ and $(x, y) \in <$, and similarly $(y, z) \in A_0 \times A_0$ and $(y, z) \in <$. Hence $x < y$ and $y < z$ so that $x < z$ since $<$ is an order and therefore transitive. Thus $(x, z) \in <$ so that $(x, z) \in < \cap (A_0 \times A_0) = \prec$ since clearly $(x, z) \in A_0 \times A_0$. So then $x \prec z$, which shows that $\prec$ is transitive. □

Exercise 3.8

Check that the relation defined in Example 7 is an order relation.

Solution:

Recall that Example 7 is the relation on C on the real line such that xCy if $x^2 < y^2$ or $x^2 = y^2$ and $x < y$.

Proof. We show that this satisfies the three properties of an order:

(Comparability) Suppose that x and y are distinct real numbers. If $x^2 < y^2$ (or $y^2 < x^2$) then of course xCy (or yCx) so we are done. So assume that $x^2 = y^2$. Since we know that $x \neq y$, it has to be that $y = -x$ so that still $x^2 = y^2$. This also implies that $x, y \neq 0$ since otherwise we would have $0 = y = -x = 0 = x$. If $x > 0$ then we have $y = -x < 0 < x$. If $x < 0$ then we have $x < 0 < -x = y$. Hence either way $x^2 = y^2$ but $x < y$ (or $y < x$) so that xC (or yCx). This shows that x and y are comparable in C.

(Nonreflexivity) Suppose that $x \in \mathbb{R}$ so that of course $x^2 = x^2$. However clearly it is not the case that $x < x$ so that it cannot be that xCx in this relation.

(Transitivity) Suppose that xCy and yCz. We then have the following cases:

Case: $x^2 < y^2$.

Case: $y^2 < z^2$. Then clearly $x^2 < y^2 < z^2$ so that xCz.

Case: $y^2 = z^2$ and $y < z$. Then $x^2 < y^2 = z^2$ so that again xCz.

Case: $x^2 = y^2$ and $x < y$.

Case: $y^2 < z^2$. Then we have $x^2 = y^2 < z^2$ so that xCz.

Case: $y^2 = z^2$ and $y < z$. Then $x^2 = y^= z^2$ and $x < y < z$ so that again xCz.

Hence in all cases xCz so that C is also transitive. □

We note that this order relation differs from the normal order on $\mathbb{R}$. For example if $x = -2$ and $y = 1$ then clearly $x < y$ in the normal order. However, we have that $y^2 = 1^2 = 1 < 4 = (-2)^2 = x^2$ so that yCx.

Exercise 3.9

Check that the dictionary order is an order relation.

Solution:

Suppose that A and B are two sets with order relations $<_A$ and $<_B$. Recall that the dictionary order $<$ on $A \times B$ is defined by

$$a_1 \times b_1 < a_2 \times b_2$$

if $a_1 <_A a_2$ or if $a_1 = a_2$ and $b_1 <_B b_2$.

Proof. Clearly $<$ is a relation on $A \times B$ so we just need to show the three properties of an order relation:

(Comparability) Consider distinct points $a_1 \times b_1$ and $a_2 \times b_2$ in $A \times B$ so that $a_1 \neq a_2$ or $b_1 \neq b_1$. If $a_1 \neq a_2$ then they are comparable in $<_A$ (since it is an order relation) so that, without loss of generality, we can assume that $a_1 <_A a_2$. Then of course $a_1 \times b_1 < a_2 \times b_2$ by definition. So assume that $a_1 = a_2$ so that it must be that $b_1 \neq b_2$. Then b_1 and b_2 are comparable in $<_B$ since it is an order relation. So without loss of generality we can assume that $b_1 <_B b_2$. Then of course $a_1 \times b_1 < a_2 \times b_2$ since also $a_1 = a_2$. Thus either way $a_1 \times b_1$ and $a_2 \times b_2$ are comparable in $<$.

(Nonreflexivity) Suppose that $a \times b$ is any element of $A \times B$. Since $<_B$ is an order relation, it is nonreflexive so that it is not true that $b <_B b$. Since of course $a = a$, it follows that it cannot be that $a \times b < a \times b$ since it would have to be that $b <_B b$. Hence $<$ is nonreflexive since $a \times b$ was arbitrary.

(Transitivity) Suppose that $a_1 \times b_1 < a_2 \times b_2$ and $a_2 \times b_2 < a_3 \times b_3$. We then have the following cases:

Case: $a_1 <_A a_2$.

- Case: $a_2 <_A a_3$. Then $a_1 <_A a_2$ and $a_2 <_A a_3$ so that $a_1 <_A a_3$ since $<_A$ is transitive. Thus by definition $a_1 \times b_1 < a_3 \times b_3$.
- Case: $a_2 = a_3$ and $b_2 <_B b_3$. Then $a_1 <_A a_2 = a_3$ so that $a_1 \times b_1 < a_3 \times b_3$ by definition.

Case: $a_1 = a_2$ and $b_1 <_B b_2$.

- Case: $a_2 <_A a_3$. Then $a_1 = a_2 <_A a_3$ so that by definition $a_1 \times b_1 < a_3 \times b_3$.
- Case: $a_2 = a_3$ and $b_2 <_B b_3$. Then $a_1 = a_2 = a_3$ and $b_1 <_B b_2$ and $b_2 <_B b_3$ so that $b_1 <_B b_3$ since $<_B$ is transitive. Therefore again $a_1 \times b_1 < a_3 \times b_3$ by definition.

Thus in all cases $a_1 \times b_1 < a_3 \times b_3$, which shows that $<$ is transitive. □

Exercise 3.10

(a) Show that the map $f : (-1, 1) \to \mathbb{R}$ of Example 9 is order preserving.

(b) Show that the equation $g(y) = 2y/[1 + (1 + 4y^2)^{1/2}]$ defines a function $g : \mathbb{R} \to (-1, 1)$ that is both a left and right inverse for f.

Solution:

(a) Recall that f from Example 3.9 is defined by

$$f(x) = \frac{x}{1 - x^2}$$

for $x \in (-1, 1)$. We show that f preserves order.

Proof. Suppose that $x, y \in (-1, 1)$ and that $x < y$. Now, since $-1 < x < 1$, we have $0 \leq |x| < 1$ so that

$$x^2 = |x|^2 = |x|\,|x| < |x| \cdot 1 = |x| < 1.$$

Hence $0 < 1 - x^2$ so that also $0 < 1/(1-x^2)$ as well. By the same argument $0 < 1 - y^2$ so that $0 < 1/(1-y^2)$. We then have the following cases:

Case: $x \geq 0$. Then also $y > 0$ since $y > x \geq 0$. Thus

$$\begin{aligned} x^2 = x \cdot x \leq x \cdot y < y \cdot y &= y^2 && \text{(since } x < y\text{, } x \geq 0\text{, and } y > 0\text{)} \\ -x^2 &> -y^2 \\ 1 - x^2 &> 1 - y^2 \\ \frac{1}{1-y^2} &> \frac{1}{1-x^2}\,. \end{aligned}$$

From this it follows that

$$f(x) = \frac{x}{1-x^2} < \frac{y}{1-x^2} < \frac{y}{1-y^2} = f(y)$$

since $x < y$ and $1/(1-x^2)$ and $1/(1-y^2)$ are both positive.

Case: $x < 0$. Then $f(x) = x/(1-x^2) < 0$ since $x < 0$ and $1/(1-x^2) > 0$.

> Case: $y \leq 0$. Then we have that $-x > -y \geq 0$ so that $f(-y) < f(-x)$ by the previous case. But we have $f(-x) = -x/(1-(-x)^2) = -x/(1-x^2) = -f(x)$, and similarly $f(-y) = -f(y)$. Hence $-f(y) = f(-y) < f(-x) = -f(x)$ so that $f(y) > f(x)$.
>
> Case: $y > 0$. Then $f(y) = y/(1-y^2) > 0$ since $y > 0$ and $1/(1-y^2) > 0$. Hence $f(x) < 0 < f(y)$.

Therefore in all cases we have $f(x) < f(y)$, which shows that f preserves order. □

(b)

Proof. First we need to show that g is even a function from $\mathbb{R}$ to $(-1, 1)$. We note that

$$\begin{aligned} y^2 &\geq 0 \\ 4y^2 &\geq 0 \\ 1 + 4y^2 &\geq 1 > 0 \\ \sqrt{1+4y^2} &> 0 \\ 1 + \sqrt{1+4y^2} &> 1 > 0 \end{aligned}$$

so that $g(y)$ is well-defined for all $y \in \mathbb{R}$ since the denominator of $g(y)$ is always nonzero. Now, if $y = 0$ then clearly $|g(y)| = |g(0)| = |0| = 0 < 1$. So in what follows assume that $y \neq 0$ so that $|y| > 0$ and $y^2 > 0$. Then we have

$$\begin{aligned} 1 &> 0 \\ 1 + 4y^2 &> 4y^2 > 0 \\ \sqrt{1+4y^2} &> \sqrt{4y^2} = 2\,|y| \\ 1 + \sqrt{1+4y^2} &> 1 + 2\,|y| > 2\,|y| > 0 \\ \frac{1}{2\,|y|} &> \frac{1}{1+\sqrt{1+4y^2}} \end{aligned}$$

$$1 = \frac{2|y|}{2|y|} > \frac{2|y|}{1+\sqrt{1+4y^2}} = \left|\frac{2y}{1+\sqrt{1+4y^2}}\right| = |g(y)|$$

so that $-1 < g(y) < 1$ and hence $g(y) \in (-1,1)$. Thus g is in fact a well-defined function from $\mathbb{R}$ into $(-1,1)$.

To show that g is a left inverse of f consider any $x \in (-1,1)$. First we have

$$\begin{aligned}
\sqrt{1+4f(x)^2} &= \sqrt{1+\frac{4x^2}{(1-x^2)^2}} = \sqrt{\frac{(1-x^2)^2+4x^2}{(1-x^2)^2}} \\
&= \sqrt{\frac{1-2x^2+x^4+4x^2}{(1-x^2)^2}} = \sqrt{\frac{1+2x^2+x^4}{(1-x^2)^2}} \\
&= \sqrt{\frac{(1+x^2)^2}{(1-x^2)^2}} = \frac{1+x^2}{1-x^2}
\end{aligned}$$

so that

$$\begin{aligned}
(g\circ f)(x) = g(f(x)) &= \frac{2f(x)}{1+\sqrt{1+4f(x)^2}} = \frac{2x/(1-x^2)}{1+\frac{1+x^2}{1-x^2}} \\
&= \frac{2x}{(1-x^2)\left(1+\frac{1+x^2}{1-x^2}\right)} = \frac{2x}{1-x^2+1+x^2} \\
&= \frac{2x}{2} = x\,,
\end{aligned}$$

which shows that $g \circ f = i_{(-1,1)}$ since x was arbitrary so that g is a left inverse of f.

Now consider any $y \in \mathbb{R}$. Then we first have

$$\begin{aligned}
g(y)^2 = \left(\frac{2y}{1+\sqrt{1+4y^2}}\right)^2 &= \frac{4y^2}{(1+\sqrt{1+4y^2})^2} \\
&= \frac{4y^2}{1+2\sqrt{1+4y^2}+1+4y^2} = \frac{4y^2}{2+4y^2+2\sqrt{1+4y^2}} \\
&= \frac{2y^2}{1+2y^2+\sqrt{1+4y^2}}
\end{aligned}$$

so that

$$\begin{aligned}
(f\circ g)(y) = f(g(y)) &= \frac{g(y)}{1-g(y)^2} = \frac{2y/(1+\sqrt{1+4y^2})}{1-\frac{2y^2}{1+2y^2+\sqrt{1+4y^2}}} \\
&= \frac{2y}{(1+\sqrt{1+4y^2})\left(\frac{1+2y^2+\sqrt{1+4y^2}-2y^2}{1+2y^2+\sqrt{1+4y^2}}\right)} \\
&= \frac{2y}{(1+\sqrt{1+4y^2})\left(\frac{1+\sqrt{1+4y^2}}{1+2y^2+\sqrt{1+4y^2}}\right)} \\
&= \frac{2y}{\frac{1+2\sqrt{1+4y^2}+1+4y^2}{1+2y^2+\sqrt{1+4y^2}}} = \frac{2y}{\frac{2+4y^2+2\sqrt{1+4y^2}}{1+2y^2+\sqrt{1+4y^2}}}
\end{aligned}$$

$$= \frac{2y}{2\left(\frac{1+2y^2+\sqrt{1+4y^2}}{1+2y^2+\sqrt{1+4y^2}}\right)} = \frac{2y}{2} = y\,,$$

which shows that $f \circ g = i_{\mathbb{R}}$ since y was arbitrary so that g is also a right inverse of f. □

Note that what was shown implies that f is bijective and that g is its inverse, by Exercise 2.5 part (e).

Exercise 3.11

Show that an element in an ordered set has at most one immediate successor and at most one immediate predecessor. Show that a subset of an ordered set has at most one smallest element and at most one largest element.

Solution:

Lemma 3.11.1. *Suppose that A is a set with order $<$ and that a and b are two elements of A. Then $a < b$ if and only if it is not true that $b \leq a$.*

Proof. ($\Rightarrow$) Suppose that $a < b$. If it were the case that $a = b$ then we would have $a < b = a$, which would violate the nonreflexivity property of the order. If it were the case that $b < a$ then we would have $a < b$ and $b < a$ so that $a < a$ by the transitive property of the order. This again violates nonreflexivity. Hence neither $a = b$ nor $b < a$ so that it is not true that $b \leq a$.

($\Leftarrow$) Now suppose that it is *not* true that $b \leq a$. Then neither $b = a$ nor $b < a$. Since $b \neq a$, it must be that either $a < b$ or $b < a$ by the comparability property. However, we know that cannot be that $b < a$, so it must be that $a < b$. □

Main Problem.

Proof. In what follows Suppose that A is a set with order $<$.

First let a be an element of A and suppose that b_1 and b_2 are both immediate successors of a, which of course means that $a < b_1, b_2$. Then by definition the open intervals (a, b_1) and (a, b_2) are both empty. Now, suppose that b_1 and b_2 are distinct so that they must be comparable since $<$ is an order. Without loss of generality we can assume that $b_1 < b_2$, but then we have $a < b_1 < b_2$ so that $b_1 \in (a, b_2)$. This is a contradiction since we know that (a, b_2) is empty, so it has to be that $b_1 = b_2$. This of course shows that the immediate successor is unique. An analogous argument shows that the immediate predecessor, if it exists, is also unique.

Now suppose that A_0 is a subset of A with smallest elements a_1 and a_2. If a_1 and a_2 were to be distinct then they must be comparable so that we can assume $a_1 < a_2$. However, then it is not true that $a_2 \leq a_1$ by Lemma 3.11.1, but this means that a_2 cannot be a smallest element of A_0 since $a_1 \in A_0$. As this is a contradiction, it must be that $a_1 = a_2$, which shows that the smallest element is unique if there is one. An analogous argument shows any largest element is also unique. □

Exercise 3.12

Let $\mathbb{Z}_+$ denote the set of positive integers. Consider the following order relations on $\mathbb{Z}_+ \times \mathbb{Z}_+$.

(i) The dictionary order.

(ii) $(x_0, y_0) < (x_1, y_1)$ if either $x_0 - y_0 < x_1 - y_1$ or $x_0 - y_0 = x_1 - y_1$ and $y_0 < y_1$.

(iii) $(x_0, y_0) < (x_1, y_1)$ if either $x_0 + y_0 < x_1 + y_1$ or $x_0 + y_0 = x_1 + y_1$ and $y_0 < y_1$.

In these order relations, which elements have immediate predecessors? Does the set have a smallest element? Show that the three order types are different.

Solution:

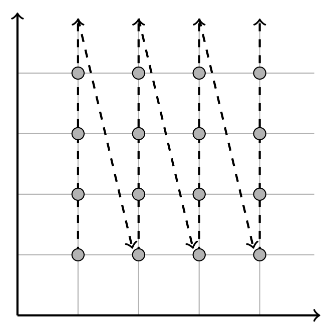
Lemma 3.12.1. *If A and B are ordered sets and A has a smallest element but B does not, then A and B do not have the same order type.*

Proof. This may seem fairly obvious but we show it formally anyway. Suppose that $<_A$ and $<_B$ are the orders on A and B, respectively. Let a be the smallest element of A and suppose to the contrary that they *do* have the same order type. Then there is a bijection $f : A \to B$ that preserves order, noting that f^{-1} is also then a bijection. Now, since B has no smallest element, there must be a $b \in B$ where $b <_B f(a)$. Setting $a' = f^{-1}(b)$ so that $f(a') = b$ we have that $f(a') = b <_B f(a)$, and hence it has to be that $a' <_A a$ since otherwise we'd have $a \leq_A a'$ so that $f(a) \leq_B f(a')$. However, $a' < a$ means that it is not true that $a \leq a'$ by Lemma 3.11.1, which contradicts the fact that a is the smallest element of A. Therefore it must be that no such f exists so that A and B have different order types.

□

Main Problem.

The figure below illustrates the dictionary order of part (i):



We claim that every point has an immediate predecessor except points in the subset

$$A = \{(x, y) \mid y = 1\} .$$

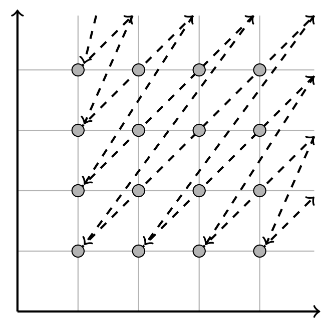
We also claim that $(1, 1)$ is the smallest element in $\mathbb{Z}_+ \times \mathbb{Z}_+$ with this order.

Proof. First consider any point $(x_1, y_1) \in \mathbb{Z}_+ \times \mathbb{Z}_+$ where $(x_1, y_1) \notin A$ so that $y_0 \neq 1$. It then follows that $y_1 > 1$ since 1 is the smallest positive integer. Then set $x_0 = x_1$ and $y_0 = y_1 - 1$ so that clearly $(x_0, y_0) \in \mathbb{Z}_+ \times \mathbb{Z}_+$ since $y_0 > 0$ because $y_1 > 1$. Clearly also $x_0 = x_1$ and $y_0 < y_1$ so that $(x_0, y_0) < (x_1, y_1)$ in the dictionary order. We claim that (x_0, y_0) is the immediate predecessor of (x_1, y_1). So suppose to the contrary that there is an $(x_2, y_2) \in \mathbb{Z}_+ \times \mathbb{Z}_+$ such that $(x_0, y_0) < (x_2, y_2) < (x_1, y_1)$. It cannot be that $x_0 < x_2$ since then we would have $x_1 = x_0 < x_2$ so that $(x_1, y_1) < (x_2, y_2)$, which by Lemma 3.11.1 contradicts the fact that $(x_2, y_2) < (x_1, y_1)$. So it has to be that $x_0 = x_2$ and $y_0 < y_2$. Since then $x_2 = x_0 = x_1$, it must also be that $y_2 < y_1$ since $(x_2, y_2) < (x_1, y_1)$. But then we have $y_0 < y_2 < y_1 = y_0 + 1$, which is not possible since $y_1 = y_0 + 1$ is the immediate successor of y_0 in $\mathbb{Z}_+$ so that there can be no integers between them. So it must be that no (x_2, y_2) exists so that (x_0, y_0) is the immediate predecessor of (x_1, y_1). Thus every element not in A has an immediate predecessor.

Now consider any $(x_1, y_1) \in A$ so that $y_1 = 1$, and consider any point $(x_0, y_0) < (x_1, y_1)$. It cannot be that $x_0 = x_1$ since then we would have $y_0 < y_1 = 1$, which is not possible since 1 is the smallest positive integer. So it must be that $x_0 < x_1$. Then let $x_2 = x_0$ and $y_2 = y_0 + 1$ so that clearly $(x_2, y_2) \in \mathbb{Z}_+ \times \mathbb{Z}_+$ since $y+1$ is always still a positive integer if y is. Then we have that $x_0 = x_2$ and $y_0 < y_0+1 = y_2$, and hence $(x_0, y_0) < (x_2, y_2)$. We also have $x_2 = x_0 < x_1$ so that $(x_2, y_2) < (x_1, y_1)$. Therefore $(x_0, y_0) < (x_2, y_2) < (x_1, y_1)$, which shows that (x_0, y_0) is not the immediate predecessor of (x_1, y_1). This shows that (x_1, y_1) has no immediate predecessor since $(x_0, y_0) < (x_1, y_1)$ was arbitrary. Since $(x_1, y_1) \in A$ was arbitrary, this completes the proof that every element in $\mathbb{Z}_+ \times \mathbb{Z}_+$ has an immediate predecessor except those in A.

It is easy to prove that $(1, 1)$ is the smallest element in the dictionary order. Consider any $(x_0, y_0) \in \mathbb{Z}_+ \times \mathbb{Z}_+$ and suppose that $(x_0, y_0) < (1, 1)$. It cannot be that $x_0 < 1$ since 1 is the smallest positive integer. So then $x_0 = 1$ and $y_0 < 1$, but this is also not possible, again since 1 is the smallest positive integer. Thus it cannot be that $(x_0, y_0) < (1, 1)$, so it must be that $(1, 1) \leq (x_0, y_0)$ by Lemma 3.11.1. This shows that $(1, 1)$ is the smallest element since (x_0, y_0) was arbitrary. □

Below is shown an illustration for the order in part (ii):



We claim that every element has an immediate predecessor except those in the subset

$$A = \{(x, y) \mid x = 1 \text{ or } y = 1\} .$$

We also claim that the set $\mathbb{Z}_+ \times \mathbb{Z}_+$ has no smallest element in this order.

Proof. First consider any $(x_1, y_1) \notin A$ so that $x_1 \neq 1$ and $y_1 \neq 1$. Then it has to be that $x_1, y_1 > 1$. So set $(x_0, y_0) = (x_1 - 1, y_1 - 1)$ so that clearly still $(x_0, y_0) \in \mathbb{Z}_+ \times \mathbb{Z}_+$. Then $x_0 - y_0 = (x_1 - 1) - (y_1 - 1) = x_1 - 1 - y_1 + 1 = x_1 - y_1$. We also have $y_0 = y_1 - 1 < y_1$ so that $(x_0, y_0) < (x_1, y_1)$. Now suppose that there is an $(x_2, y_2) \in \mathbb{Z}_+ \times \mathbb{Z}_+$ where $(x_0, y_0) < (x_2, y_2) < (x_1, y_1)$. It cannot be that $x_0 - y_0 < x_2 - y_2$ since then we would have $x_1 - y_1 = x_0 - y_0 < x_2 - y_2$ so that $(x_1, y_1) < (x_2, y_2)$, which we know cannot be the case since $(x_2, y_2) < (x_1, y_1)$. So it must be that $x_2 - y_2 = x_0 - y_0 = x_1 - y_1$ and $y_0 < y_2$, but then we must have $y_0 < y_2 < y_1 = y_0 + 1$, noting that $y_2 < y_1$ because $x_2 - y_2 = x_1 - y_1$ and $(x_2, y_2) < (x_1, y_1)$. However, this is not possible since of course $y_0 + 1$ is the immediate successor of y_0 in $\mathbb{Z}_+$. So then it has to be that no such (x_2, y_2) exists so that (x_0, y_0) is the immediate predecessor of (x_1, y_1). This shows that every point not in A has an immediate predecessor since (x_1, y_1) was arbitrary.

Now suppose that $(x_1, y_1) \in A$ so that $x_1 = 1$ or $y_1 = 1$, and consider any $(x_0, y_0) < (x_1, y_1)$.

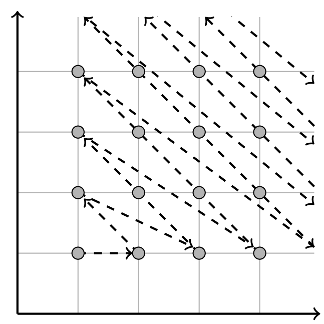
Case: $x_1 = 1$. Suppose that $x_0 - y_0 = x_1 - y_1$ and $y_0 < y_1$. Then we would have $-y_0 > -y_1$ and $x_0 - y_0 = x_1 - y_1 = 0 - y_1 = -y_1 < -y_0$ so that $x_0 < 0$ (by adding y_0 to both sides), which is not possible.

Case: $y_1 = 1$. It clearly cannot be that case that $x_0 - y_0 = x_1 - y_1$ and $y_0 < y_1$ since then $y_0 < y_1 = 1$, which is not possible.

So in either case it must be that $x_0 - y_0 < x_1 - y_1$ since $(x_0, y_0) < (x_1, y_1)$. So set the point $(x_2, y_2) = (x_0 + 1, y_0 + 1)$, which is clearly still an element of $\mathbb{Z}_+ \times \mathbb{Z}_+$. We then have $x_2 - y_2 = (x_0 + 1) - (y_0 + 1) = x_0 + 1 - y_0 - 1 = x_0 - y_0 < x_1 - y_1$ so that $(x_2, y_2) < (x_1, y_1)$. We also have $x_0 - y_0 = x_2 - y_2$ and $y_0 < y_0 + 1 = y_2$ so that also $(x_0, y_0) < (x_2, y_2)$. Hence $(x_0, y_0) < (x_2, y_2) < (x_1, y_1)$ so that (x_0, y_0) is not the immediate predecessor of (x_1, y_1). Since $(x_0, y_0) < (x_1, y_1)$ was arbitrary, this shows that (x_1, y_1) has no immediate predecessor at all. Since $(x_1, y_1) \in A$ was arbitrary, this shows that no element of A has an immediate predecessor.

To show that $\mathbb{Z}_+$ in this order has no smallest element, consider absolutely any $(x_1, y_1) \in \mathbb{Z}_+ \times \mathbb{Z}_+$. Let $(x_0, y_0) = (x_1, y_1 + 1)$ so that of course $(x_0, y_0) \in \mathbb{Z}_+ \times \mathbb{Z}_+$. We then have that $x_0 - y_0 = x_1 - (y_1 + 1) = (x_1 - y_1) - 1 < x_1 - y_1$ so that $(x_0, y_0) < (x_1, y_1)$. Then of course is it not true that $(x_1, y_1) \leq (x_0, y_0)$ by Lemma 3.11.1 so that (x_1, y_1) cannot be the smallest element. Then, since (x_1, y_1) was arbitrary, this shows that $\mathbb{Z}_+ \times \mathbb{Z}_+$ has no smallest element in this order. □

An illustration of the order of part (iii) is shown below:



We claim that every element has an immediate predecessor except for $(1, 1)$, which is the smallest element.

Proof. First we show that $(x_0, y_0) = (1, 1)$ is the smallest element, from which it follows that it cannot have an immediate predecessor since it has no predecessors at all. Consider any (x_1, y_1) in $\mathbb{Z}_+ \times \mathbb{Z}_+$. If $(x_1, y_1) = (1, 1)$ then of course $(x_0, y_0) = (1, 1) \leq (x_1, y_1)$ is true, so assume that $(x_1, y_1) \neq (1, 1)$ so that either $x_1 \neq 1$ or $y_1 \neq 1$. If $x_1 \neq 1$ then it has to be that $x_1 > 1$ so that $x_0 + y_0 = 1 + 1 \leq 1 + y_1 < x_1 + y_1$, and hence $(x_0, y_0) < (x_1, y_1)$. If $y_1 \neq 1$ then $y_1 > 1$ so that again $x_0 + y_0 = 1 + 1 \leq x_1 + 1 < x_1 + y_1$, and hence $(x_0, y_0) < (x_1, y_1)$. Thus in every case $(x_0, y_0) \leq (x_1, y_1)$, which shows that $(x_0, y_0) = (1, 1)$ is the smallest element since (x_1, y_1) was arbitrary.

Now we show that every other element of $\mathbb{Z}_+$ has an immediate predecessor in this order. So consider any $(x_1, y_1) \in \mathbb{Z}_+ \times \mathbb{Z}_+$ where $(x_1, y_1) \neq (1, 1)$. Hence either $x_1 \neq 1$ or $y_1 \neq 1$.

Case: $y_1 = 1$. Then it has to be that $x_1 \neq 1$ so that $x_1 > 1$. We claim that $(x_0, y_0) = (1, x_1 - 1)$ is the immediate predecessor of (x_1, y_1). First we note that clearly $(x_0, y_0) \in \mathbb{Z}_+ \times \mathbb{Z}_+$ since $x_1 > 1$. We also have that $x_0 + y_0 = 1 + x_1 - 1 = x_1 < x_1 + y_1$ since $0 < 1 = y_1$, and so $(x_0, y_0) < (x_1, y_1)$. Now suppose that there is a point $(x_2, y_2) \in \mathbb{Z}_+ \times \mathbb{Z}_+$ where $(x_0, y_0) < (x_2, y_2) < (x_1, y_1)$. It cannot be that $x_1 + y_1 = x_2 + y_2$ and $y_1 < y_2$ because then we would have $y_2 < y_1 = 1$, which is not possible since $y_2 \in \mathbb{Z}_+$. So it must be that $x_2 + y_2 < x_1 + y_1 = x_1 + 1$ since $(x_2, y_2) < (x_1, y_1)$. Now, since also $(x_0, y_0) < (x_2, y_2)$, it must be that $x_0 + y_0 \leq x_2 + y_2$, but then we have $x_1 = 1 + (x_1 - 1) = x_0 + y_0 \leq x_2 + y_2 < x_1 + 1$, which is impossible. So it has to be that no such (x_2, y_2) exists so that (x_0, y_0) is the immediate predecessor of (x_1, y_1).

Case: $y_1 \neq 1$. Then it has to be that $y_1 > 1$ so that the point $(x_0, y_0) = (x_1 + 1, y_1 - 1)$ is still an element of $\mathbb{Z}_+ \times \mathbb{Z}_+$. We show that (x_0, y_0) is the immediate predecessor of (x_1, y_1). First we have that $x_0 + y_0 = (x_1 + 1) + (y_1 - 1) = x_1 + y_1$ and $y_0 = y_1 - 1 < y_1$ so that $(x_0, y_0) < (x_1, y_1)$. Now suppose that there a point $(x_2, y_2) \in \mathbb{Z}_+ \times \mathbb{Z}_+$ where $(x_0, y_0) < (x_2, y_2) < (x_1, y_1)$. Then it has to be that $x_0+y_0 \leq x_2+y_2 \leq x_1+y_1$ so that we have $x_1+y_1 = (x_1+1)+(y_1-1) = x_0+y_0 \leq x_2+y_2 \leq x_1+y_1$ so that $x_0 + y_0 = x_2 + y_2 = x_1 + y_1$. But then we must have $y_1 - 1 = y_0 < y_2 < y_1$, which is not possible since $y_1 - 1$ is the immediate predecessor of y_1 in $\mathbb{Z}_+$. So no such (x_2, y_2) can exists, and hence (x_0, y_0) is the immediate predecessor of (x_1, y_1).

This in all cases (x_1, y_1) has an immediate predecessor, which shows the desired result since $(x_1, y_1) \neq (1, 1)$ was arbitrary. □

Now we show that all three orders have different order types.

Proof. It follows immediately from Lemma 3.12.1 that order (i) and order (ii) do not have the same order type since (i) has a smallest element while (ii) does not. Similarly order (iii) and order (ii) cannot have the same order type for the same reason. So all that remains to be shown is that orders (i) and (iii) have different order types.

So denote order (i) with $<$ and order (iii) with $\prec$ and suppose to the contrary that they do have the same order type. Then there is a bijection $f : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{Z}_+ \times \mathbb{Z}_+$ that preserves order, supposing that the domain has the dictionary order $<$ and the range has the order $\prec$. Then of course f^{-1} is also a bijection that preserves order. It was shown above that $\mathbb{Z}_+ \times \mathbb{Z}_+$ with $<$ has countably many elements with no immediate predecessor, whereas $\mathbb{Z}_+ \times \mathbb{Z}_+$ with $\prec$ has only a single such element, namely the smallest element $(1, 1)$.

Thus we can choose an element (x_1, y_1) of $\mathbb{Z}_+ \times \mathbb{Z}_+$ that has no immediate predecessor in $<$ but also such that $f(x_1, y_1) \neq (1, 1)$ so that $f(x_1, y_1)$ does have an immediate predecessor in $\prec$. So let (u_0, v_0) be the immediate predecessor of $f(x_1, y_1)$ in $\prec$ and set $(x_0, y_0) = f^{-1}(u_0, v_0)$ so that $(u_0, v_0) = f(x_0, y_0)$. Then of course $(x_0, y_0) < (x_1, y_1)$ since $f(x_0, y_0) = (u_0, v_0) \prec f(x_1, y_1)$ and f preserves order. But since (x_1, y_1) has no immediate predecessor in $<$, there is a point (x_2, y_2) such that $(x_0, y_0) < (x_2, y_2) < (x_1, y_1)$. We then have that $(u_0, v_0) = f(x_0, y_0) \prec f(x_2, y_2) \prec f(x_1, y_1)$ since f preserves order, which is a contradiction since (u_0, v_0) is the immediate predecessor of $f(x_1, y_1)$. So it must be that no such order-preserving f exists and hence the two orders do not have the same order type. □

Exercise 3.13

Prove the following:

Theorem. *If an ordered set A has the least upper bound property, then it has the greatest lower bound property.*

Solution:

Proof. Suppose that A_0 is any nonempty subset of A that is bounded below so that b is a lower bound of A_0. Let B_0 be the set of lower bounds of A_0 so that B_0 is nonempty since $b \in B_0$. Since A_0 is nonempty there is an $a \in A_0$. Now, for any $x \in B_0$ we have that x is a lower bound of A_0 so that $x \leq a$, which shows that a is an upper bound of B_0. Hence B_0 is a nonempty subset of A that is bounded above, and so has a least upper bound c since A has the least upper bound property. We claim that c is also the greatest lower bound of A_0.

Consider any $x \in A_0$ and any $y \in B_0$. Then y is a lower bound of A_0 so that $y \leq x$ since $x \in A_0$. Since $y \in B_0$ was arbitrary, this shows that x is an upper bound of B_0. Thus we have $c \leq x$ since c is the least upper bound of B_0. Since $x \in A_0$ was arbitrary, this shows that c is a lower bound of

A_0. If y is any other lower bound then $y \in B_0$ so that $y \le c$ since c is an upper bound of B_0. Since y was an arbitrary lower bound, this shows that in fact c is the greatest lower bound of A_0. Hence A also has the greatest lower bound property since the nonempty subset A_0 was arbitrary. □

Exercise 3.14

If C is a relation on a set A, define a new relation D on A by letting $(b, a) \in D$ if $(a, b) \in C$.

(a) Show that C is symmetric if and only if $C = D$.
(b) Show that if C is an order relation, D is also an order relation.
(c) Prove the converse of the theorem in Exercise 13.

Solution:

(a)

Proof. ($\Rightarrow$) Suppose that C is symmetric. Then we simply have

$$\begin{aligned} (a,b) \in C &\Leftrightarrow (b,a) \in C && \text{(since } C \text{ is symmetric)} \\ &\Leftrightarrow (a,b) \in D\,, && \text{(by definition)} \end{aligned}$$

which shows that $C = D$.

($\Leftarrow$) Now suppose that $C = D$ and consider any $(a, b) \in C$. Then $(b, a) \in D$ by definition. Hence also $(b, a) \in C$ since $C = D$, which shows that C is symmetric. □

(b)

Proof. Suppose that C is an order relation. Since clearly D is a relation on A, we need only show that it has the three required properties:

(Comparability) Consider any distinct $a, b \in A$ so that aCb or bCa since C has comparability. Hence either bDa or aDb, respectively, by definition so that a and b are comparable in D as well.

(Nonreflexivity) Consider any $a \in A$. Then $(a, a) \notin C$ since it is nonreflexive, thus also $(a, a) \notin D$ since, if it were, it would also be that $(a, a) \in C$ by definition. Hence D is also nonreflexive since a was arbitrary.

(Transitivity) Suppose that aDb and bDc. Then by definition we have bCa and cCb. That is, cCb and bCa so that cCa since C is transitive. Therefore aDc by definition, which shows that D is also transitive. □

(c) The converse follows from an argument directly analogous to the proof of Exercise 3.13, which we give here for completeness.

Proof. Suppose that A has the greatest lower bound property and that A_0 is any nonempty subset of A that is bounded above so that b is an upper lower bound of A_0. Let B_0 be the set of upper bounds of A_0 so that B_0 is nonempty since $b \in B_0$. Since A_0 is nonempty there is an $a \in A_0$. Now, for any $x \in B_0$ we have that x is an upper bound of A_0 so that $a \le x$, which shows that a is a lower bound of B_0. Hence B_0 is a nonempty subset of A that is bounded below, and so has a greatest lower bound c since A has the greatest lower bound property. We claim that c is also the least upper bound of A_0.

Consider any $x \in A_0$ and any $y \in B_0$. Then y is an upper bound of A_0 so that $x \le y$ since $x \in A_0$. Since $y \in B_0$ was arbitrary, this shows that x is a lower bound of B_0. Thus we have $x \le c$ since c is the greatest lower bound of B_0. Since $x \in A_0$ was arbitrary, this shows that c is an upper bound of A_0. If y is any other upper bound then $y \in B_0$ so that $c \leq y$ since c is a lower bound of B_0. Since y was an arbitrary upper bound, this shows that in fact c is the lest upper bound of A_0. Hence A also has the lest upper bound property since the nonempty subset A_0 was arbitrary. □

Exercise 3.15

Assume that the real line has the least upper bound property.

(a) Show that the sets

$$[0,1] = \{x \mid 0 \leq x \leq 1\} \ ,$$
$$[0,1) = \{x \mid 0 \leq x < 1\}$$

have the least upper bound property.

(b) Does $[0,1] \times [0,1]$ in the dictionary order have the least upper bound property? What about $[0,1] \times [0,1)$? What about $[0,1) \times [0,1]$?

Solution:

(a) We show this for both sets simultaneously as their proofs are nearly identical. We note the minor differences in parentheses.

Proof. Let $A = [0,1]$ (or $A = [0,1)$) and consider any nonempty subset A_0 of A that is bounded above in A. So suppose that b is an upper bound of A_0 in A so that $0 \leq b \leq 1$ (or $0 \leq b < 1$). Now, of course A_0 has a least upper bound c in $\mathbb{R}$ since it is also a nonempty subset of $\mathbb{R}$ that is bounded above. Obviously $c \leq b$ since it is the *least* upper bound. For any $x \in A_0$ we of course have that $x \in A$ so that $0 \leq x \leq 1$ (or $0 \leq x < 1$), and clearly $x \leq c$ since it is an upper bound of A_0. Thus we have $0 \leq x \leq c \leq b \leq 1$ (or $0 \leq x \leq c \leq b < 1$) so that $c \in A$. Hence A_0 has a least upper bound in A as desired. □

(b) First we claim that both $[0,1] \times [0,1]$ and $[0,1) \times [0,1]$ have the least upper bound property. We show this for both sets simultaneously as their proofs are identical.

Proof. Let $X = [0,1] \times [0,1]$ (or $X = [0,1) \times [0,1]$). Suppose that A is a nonempty subset of X that is bounded above, and that $x_1 \times y_1$ is an upper bound of A in X. Then of course we have $x_1 \in [0,1]$ (or $x_1 \in [0,1)$). Define $A_x = \{x \mid x \times y \in A\}$ so that clearly $A_x \subset [0,1]$ (or $A_x \subset [0,1)$). It then follows that $x \leq x_1$ for any $x \in A_x$ since $x_1 \times y_1$ is an upper bound of A in the dictionary order. Also clearly A_x is nonempty since A is. Thus A_x is a nonempty subset of $[0,1]$ (or $[0,1)$) that is bounded above (by x_1) so that it has a least upper bound a by what was shown in part (a).

Now, if $a \notin A_x$ then set $b = 0$. Otherwise define $A_y = \{y \mid a \times y \in A\}$. Then, since $a \in A_x$, there is a $y \in [0,1]$ where $a \times y \in A$, which shows that $y \in A_y$ and hence $A_y \neq \varnothing$. We also clearly have that $A_y \subset [0,1]$ so that A_y is bounded above by 1. Then A_y has a least upper bound b, again by what was shown in part (a). In either case we assert that $a \times b$ is the least upper bound of A in X in the dictionary order.

First, it is obvious that $a \times b \in X$ based on how a and b were defined. Consider any $x \times y \in A$ so that $x \in A_x$, and hence $x \leq a$ since a is the least upper bound of A_x. If $x < a$ then of course $x \times y \leq a \times b$, so assume that $x = a$. Then $a = x \in A_x$ so that b was defined as the least upper bound of A_y. Then we have that $y \in A_y$ since $x \times y = a \times y \in A$, and thus $y \leq b$ since b is the least upper bound of A_y. This shows that $x \times y \leq a \times b$ so that we have shown that $a \times b$ is an upper bound of A since $x \times y$ was arbitrary.

To show that it is the *least* upper bound consider any $x_0 \times y_0 \in X$ where $x_0 \times y_0 < a \times b$. If $x_0 < a$ then there must be an $x \in A_x$ where $x_0 < x$ since x_0 cannot be an upper bound of A_x since $x_0 < a$ and a is the least upper bound of A_x. Then, since $x \in A_x$, there is a $y \in [0,1]$ where $x \times y \in A$. Hence $x_0 \times y_0 < x \times y$ since $x_0 < x$ so that $x_0 \times y_0$ is not an upper bound of A. On the other hand if $x_0 = a$ we must have $y_0 < b$ so that it cannot be that $b = 0$, and hence it must be that $a \in A_x$. Then there must be a $y \in A_y$ where $y_0 < y$ since y_0 cannot be an upper bound of A_y since $y_0 < b$ and b is the least upper bound of A_y. Hence $a \times y \in A$, $x_0 = a$, and $y_0 < y$ so that $x_0 \times y_0 < a \times y \in A$, which shows that $x_0 \times y_0$ is not an upper bound of A. Thus in either case $x_0 \times y_0$ is not an upper bound of A, which shows that $a \times b$ is the least upper bound since $x_0 \times y_0 < a \times b$ was arbitrary.

This of course completes the proof that A has a least upper bound, which shows that X has the least upper bound property since A was an arbitrary nonempty subset. □

We also claim that $[0,1] \times [0,1)$ does *not* have the least upper bound property.

Proof. Let $X = [0,1] \times [0,1)$. Consider the set $A = \{0\} \times [0,1)$, which is clearly a nonempty subset of X. This subset also obviously has an upper bound in X in the dictionary order, for example the point 1×0. So let $x_1 \times y_1$ be any upper bound of A in X and suppose for the moment that $x_1 = 0$. Then $y_1 \in [0,1)$ so that $0 \leq y_1 < 1$, but then there is a $y_0 \in \mathbb{R}$ such that $0 \leq y_1 < y_0 < 1$. Then $y_0 \in [0,1)$ so that $x_1 \times y_0 = 0 \times y_0 \in A$, but also $x_1 \times y_1 < x_1 \times y_0$ so that $x_1 \times y_1$ cannot be an upper bound of A. So it must be that in fact $x_1 \neq 0$ and hence $x_1 > 0$. Now let $x_0 = x_1/2 > 0$ and $y_0 = 0$ so that clearly $x \times y < x_0 \times y_0 < x_1 \times y_1$ for any $x \times y \in A$ since $x = 0 < x_1/2 = x_0 < x_1$. This shows that $x_0 \times y_0$ is still an upper bound of A but that $x_0 \times y_0 < x_1 \times y_1$. Since $x_1 \times y_1$ was an arbitrary upper bound of A, this proves that A can have no *least* upper bound! □

§4 The Integers and the Real Numbers

Exercise 4.1

Prove the following "laws of algebra" for $\mathbb{R}$, using only axioms (1)-(5):

(a) If $x + y = x$, then $y = 0$.

(b) $0 \cdot x = 0$. [Hint: Compute $(x + 0) \cdot x$.]

(c) $-0 = 0$.

(d) $-(-x) = x$.

(e) $x(-y) = -(xy) = (-x)y$.

(f) $(-1)x = -x$.

(g) $x(y - z) = xy - xz$.

(h) $-(x + y) = -x - y$; $-(x - y) = -x + y$.

(i) If $x \neq 0$ and $x \cdot y = x$, then $y = 1$.

(j) $x/x = 1$ if $x \neq 0$.

(k) $x/1 = x$.

(l) $x \neq 0$ and $y \neq 0 \Rightarrow xy \neq 0$.

(m) $(1/y)(1/z) = 1/(yz)$ if $y, z \neq 0$.

(n) $(x/y)(w/z) = (xw)/(yz)$ if $y, z \neq 0$.

(o) $(x/y) + (w/z) = (xz + wy)/(yz)$ if $y, z \neq 0$.

(p) $x \neq 0 \Rightarrow 1/x \neq 0$.

(q) $1/(w/z) = z/w$ if $w, z \neq 0$.

(r) $(x/y)/(w/z) = (xz)/(yw)$ if $y, w, z \neq 0$.

(s) $(ax)/y = a(x/y)$ if $y \neq 0$.

(t) $(-x)/y = x/(-y) = -(x/y)$ if $y \neq 0$.

Solution:

Lemma 4.1.1. *$x + y = x + z$ if and only if $y = z$.*

Proof. (⇐) Clearly if $y = z$ then $x + y = x + z$ since the + operation is a function.
(⇒) If $x + y = x + z$ then we have

$$\begin{aligned}
y &= y + 0 && \text{(by (3))} \\
&= 0 + y && \text{(by (2))} \\
&= (x + (-x)) + y && \text{(by (4))} \\
&= (-x + x) + y && \text{(by (2))} \\
&= -x + (x + y) && \text{(by (1))} \\
&= -x + (x + z) && \text{(by what was just shown for } (\Leftarrow)) \\
&= (-x + x) + z && \text{(by (1))} \\
&= (x + (-x)) + z && \text{(by (2))} \\
&= 0 + z && \text{(by (4))} \\
&= z + 0 && \text{(by (2))} \\
&= z && \text{(by (3))}
\end{aligned}$$

as desired. □

Lemma 4.1.2. *If $x \neq 0$ then $x \cdot y = x \cdot z$ if and only if $y = z$.*

Proof. (⇐) Clearly if $y = z$ then $x \cdot y = x \cdot z$ since the $\cdot$ operation is a function.
(⇒) If $x \cdot y = x \cdot z$ then we have

$$\begin{aligned}
y &= y \cdot 1 && \text{(by (3))} \\
&= 1 \cdot y && \text{(by (2))} \\
&= \left(x \cdot \frac{1}{x}\right) \cdot y && \text{(by (4), noting that } x \neq 0) \\
&= \left(\frac{1}{x} \cdot x\right) \cdot y && \text{(by (2))} \\
&= \frac{1}{x} \cdot (x \cdot y) && \text{(by (1))} \\
&= \frac{1}{x} \cdot (x \cdot z) && \text{(by what was just shown for } (\Leftarrow)) \\
&= \left(\frac{1}{x} \cdot x\right) \cdot z && \text{(by (1))} \\
&= \left(x \cdot \frac{1}{x}\right) \cdot z && \text{(by (2))} \\
&= 1 \cdot z && \text{(by (4))} \\
&= z \cdot 1 && \text{(by (2))} \\
&= z && \text{(by (3))}
\end{aligned}$$

as desired. □

Lemma 4.1.3. $1/(yz) = 1/(zy)$ *if* $y, z \neq 0$.

Proof. We have $(zy) \cdot 1/(yz) = (yz) \cdot 1/(yz) = 1$ by (2) followed by (4) so that $1/(yz)$ is a reciprocal of zy. Since this reciprocal is unique, however, it must be that $1/(yz) = 1/(zy)$ as desired. □

Main Problem.

(a) If $x + y = x$, then $y = 0$.

Proof. Clearly by (3) we have $x + 0 = x = x + y$ so that it has to be that $y = 0$ by Lemma 4.1.1. □

(b) $0 \cdot x = 0$. [Hint: Compute $(x + 0) \cdot x$.]

Proof. We have

$$\begin{aligned} x \cdot x + 0 \cdot x &= x \cdot x + x \cdot 0 && \text{(since } 0 \cdot x = x \cdot 0 \text{ by (2))} \\ &= x \cdot (x + 0) && \text{(by (5))} \\ &= x \cdot x\,. && \text{(since } x + 0 = x \text{ by (3))} \end{aligned}$$

Thus it must be that $0 \cdot x = 0$ by part (a). □

(c) $-0 = 0$.

Proof. By (4) we have $0 + (-0) = 0$ so that it has to be that $-0 = 0$ by part (a). □

(d) $-(-x) = x$.

Proof. We have

$$\begin{aligned} -(-x) &= -(-x) + 0 && \text{(by (3))} \\ &= -(-x) + (x + (-x)) && \text{(by (4))} \\ &= -(-x) + ((-x) + x) && \text{(by (2))} \\ &= (-(-x) + (-x)) + x && \text{(by (1))} \\ &= ((-x) + (-(-x))) + x && \text{(by (2))} \\ &= 0 + x && \text{(by (4))} \\ &= x + 0 && \text{(by (2))} \\ &= x && \text{(by (3))} \end{aligned}$$

as desired. □

(e) $x(-y) = -(xy) = (-x)y$.

Proof. First we have

$$\begin{aligned} x(-y) &= x(-y) + 0 && \text{(by (3))} \\ &= x(-y) + (xy + (-(xy)) && \text{(by (4))} \\ &= (x(-y) + xy) + (-(xy)) && \text{(by (1))} \\ &= x(-y + y) + (-(xy)) && \text{(by (5))} \\ &= x(y + (-y)) + (-(xy)) && \text{(by (2))} \\ &= x \cdot 0 + (-(xy)) && \text{(by (4))} \\ &= 0 \cdot x + (-(xy)) && \text{(by (2))} \\ &= 0 + (-(xy)) && \text{(by part(b))} \\ &= -(xy) + 0 && \text{(by (2))} \\ &= -(xy)\,. && \text{(by (3))} \end{aligned}$$

We also have

$$\begin{aligned}(-x)y &= y(-x) && \text{(by (2))}\\ &= -(yx) && \text{(by what was just shown)}\\ &= -(xy) && \text{(by (2))}\end{aligned}$$

so that the result follows since equality is transitive. ☐

(f) $(-1)x = -x$.

Proof. We have

$$\begin{aligned}(-1)x &= -(1\cdot x) && \text{(by part(e))}\\ &= -(x\cdot 1) && \text{(by (2))}\\ &= -x && \text{(since } x\cdot 1 = x \text{ by (3))}\end{aligned}$$

as desired. ☐

(g) $x(y-z) = xy - xz$.

Proof. We have

$$\begin{aligned}x(y-z) &= x(y+(-z)) && \text{(by the definition of subtraction)}\\ &= xy + x(-z) && \text{(by (5))}\\ &= xy + (-(xz)) && \text{(by part(e))}\\ &= xy - xz && \text{(by the definition of subtraction)}\end{aligned}$$

as desired. ☐

(h) $-(x+y) = -x-y$; $-(x-y) = -x+y$.

Proof. We have

$$\begin{aligned}-(x+y) &= (-1)(x+y) && \text{(by part (f))}\\ &= (-1)x + (-1)y && \text{(by (5))}\\ &= -x + (-y) && \text{(by part (f) twice)}\\ &= -x - y && \text{(by the definition of subtraction)}\end{aligned}$$

and

$$\begin{aligned}-(x-y) &= -(x+(-y)) && \text{(by the definition of subtraction)}\\ &= -x-(-y)) && \text{(by what was just shown)}\\ &= -x+(-(-y)) && \text{(by the definition of subtraction)}\\ &= -x+y && \text{(by part (d))}\end{aligned}$$

as desired. ☐

(i) If $x \neq 0$ and $x \cdot y = x$, then $y = 1$.

Proof. By (3) we have $x \cdot 1 = x = x \cdot y$ so that it has to be that $y = 1$ by Lemma 4.1.2, noting that this applies since $x \neq 0$. ☐

(j) $x/x = 1$ if $x \neq 0$.

Proof. By the definition of division we have $x/x = x \cdot (1/x) = 1$ by (4) since $x \neq 0$ and $1/x$ is defined as the reciprocal (i.e. the multiplicative inverse) of x. □

(k) $x/1 = x$.

Proof. First, we have by (4) that $1 \cdot (1/1) = 1$, where $1/1$ is the reciprocal of 1. We also have that $1 \cdot (1/1) = (1/1) \cdot 1 = 1/1$ by (2) and (3). Therefore $1/1 = 1 \cdot (1/1) = 1$ so that 1 is its own reciprocal. Then, by the definition of division, we have $x/1 = x \cdot (1/1) = x \cdot 1 = x$ by (3). □

(l) $x \neq 0$ and $y \neq 0 \Rightarrow xy \neq 0$.

Proof. Suppose that $x \neq 0$ and $y \neq 0$. Also suppose to the contrary that $xy = 0$. Since $y \neq 0$ it follows from (4) that $1/y$ exists. So, we have $(xy) \cdot (1/y) = 0 \cdot (1/y) = 0$ by part (b). We also have

$$\begin{aligned}
(xy) \cdot \frac{1}{y} &= x\left(y \cdot \frac{1}{y}\right) && \text{(by (1))} \\
&= x \cdot 1 && \text{(by (4))} \\
&= x && \text{(by (3))}
\end{aligned}$$

so that $x = (xy) \cdot (1/y) = 0$, which is a contradiction since we supposed that $x \neq 0$. Hence it must be that $xy \neq 0$ as desired. □

(m) $(1/y)(1/z) = 1/(yz)$ if $y, z \neq 0$.

Proof. We have

$$\begin{aligned}
(yz)\left(\frac{1}{y} \cdot \frac{1}{z}\right) &= (yz)\left(\frac{1}{z} \cdot \frac{1}{y}\right) && \text{(by (2))} \\
&= \left((yz) \cdot \frac{1}{z}\right)\frac{1}{y} && \text{(by (1))} \\
&= \left(y\left(z \cdot \frac{1}{z}\right)\right)\frac{1}{y} && \text{(by (1))} \\
&= (y \cdot 1)\,\frac{1}{y} && \text{(by (4))} \\
&= y \cdot \frac{1}{y} && \text{(by (3))} \\
&= 1 && \text{(by (4))}
\end{aligned}$$

so that $(1/y)(1/z)$ is a multiplicative inverse of yz. Since this inverse is *unique* by (4), however, it has to be that $(1/y)(1/z) = 1/(yz)$ as desired. □

(n) $(x/y)(w/z) = (xw)/(yz)$ if $y, z \neq 0$.

Proof. We have

$$\begin{aligned}
\frac{x}{y} \cdot \frac{w}{z} &= \left(x \cdot \frac{1}{y}\right)\left(w \cdot \frac{1}{z}\right) && \text{(by the definition of division)} \\
&= \left(x \cdot \frac{1}{y}\right)\left(\frac{1}{z} \cdot w\right) && \text{(by (2))}
\end{aligned}$$

$$
\begin{aligned}
&= \left(\left(x \cdot \frac{1}{y}\right) \frac{1}{z}\right) w && \text{(by (1))} \\
&= \left(x \left(\frac{1}{y} \cdot \frac{1}{z}\right)\right) w && \text{(by (1))} \\
&= \left(x \cdot \frac{1}{yz}\right) w && \text{(by part (m) since } y, z \neq 0\text{)} \\
&= \left(\frac{1}{yz} \cdot x\right) w && \text{(by (2))} \\
&= \frac{1}{yz}(xw) && \text{(by (1))} \\
&= (xw)\frac{1}{yz} && \text{(by (2))} \\
&= \frac{xw}{yz} && \text{(by the definition of division)}
\end{aligned}
$$

as desired. □

(o) $(x/y) + (w/z) = (xz + wy)/(yz)$ if $y, z \neq 0$.

Proof. We have

$$
\begin{aligned}
\frac{x}{y} + \frac{w}{z} &= \frac{x}{y} \cdot 1 + \frac{w}{z} \cdot 1 && \text{(by (3))} \\
&= \frac{x}{y} \cdot \frac{z}{z} + \frac{w}{z} \cdot \frac{y}{y} && \text{(by part (j))} \\
&= \frac{xz}{yz} + \frac{wy}{zy} && \text{(by part(n))} \\
&= (xz)\frac{1}{yz} + (wy)\frac{1}{zy} && \text{(by the definition of division)} \\
&= (xz)\frac{1}{yz} + (wy)\frac{1}{yz} && \text{(by Lemma 4.1.3)} \\
&= \frac{1}{yz}(xz) + \frac{1}{yz}(wy) && \text{(by (2))} \\
&= \frac{1}{yz}(xz + wy) && \text{(by (5))} \\
&= (xz + wy)\frac{1}{yz} && \text{(by (2))} \\
&= \frac{xz + wy}{yz} && \text{(by the definition of division)}
\end{aligned}
$$

as desired. □

(p) $x \neq 0 \Rightarrow 1/x \neq 0$.

Proof. Suppose that $x \neq 0$ but $1/x = 0$. Then we first have that $x \cdot (1/x) = x \cdot 0 = 0 \cdot x = 0$ by (2) and part (b). However, we also have $x \cdot (1/x) = 1$ by (4). Hence we have $0 = x \cdot (1/x) = 1$, which is a contradiction since we know that 0 and 1 are distinct by (3). So, if we accept that $x \neq 0$, then it must be that $1/x \neq 0$ also. □

(q) $1/(w/z) = z/w$ if $w, z \neq 0$.

Proof. We have

$$\begin{aligned}
\frac{w}{z} \cdot \frac{z}{w} &= \frac{wz}{zw} && \text{(by part (n) since } w, z \neq 0\text{)} \\
&= (wz)\frac{1}{zw} && \text{(by the definition of division)} \\
&= (wz)\frac{1}{wz} && \text{(by Lemma 4.1.3 since } w, z \neq 0\text{)} \\
&= 1 && \text{(by (4))}
\end{aligned}$$

so that by definition z/w is the reciprocal of w/z. Since this is unique by (4) we then have $z/w = 1/(w/z)$ as desired. □

(r) $(x/y)/(w/z) = (xz)/(yw)$ if $y, w, z \neq 0$.

Proof. We have

$$\begin{aligned}
\frac{x/y}{w/z} &= \frac{x}{y} \cdot \frac{1}{w/z} && \text{(by the definition of division)} \\
&= \frac{x}{y} \cdot \frac{z}{w} && \text{(by part (q) since } w, z \neq 0\text{)} \\
&= \frac{xz}{yw} && \text{(by part (n) since } y, w \neq 0\text{)}
\end{aligned}$$

as desired. □

(s) $(ax)/y = a(x/y)$ if $y \neq 0$.

Proof. We have

$$\begin{aligned}
\frac{ax}{y} &= (ax) \cdot \frac{1}{y} && \text{(by the definition of division)} \\
&= a\left(x \cdot \frac{1}{y}\right) && \text{(by (1))} \\
&= a \cdot \frac{x}{y} && \text{(by the definition of division)}
\end{aligned}$$

as desired. □

(t) $(-x)/y = x/(-y) = -(x/y)$ if $y \neq 0$.

Proof. We have

$$\begin{aligned}
\frac{-x}{y} &= (-x) \cdot \frac{1}{y} && \text{(by the definition of division)} \\
&= ((-1)x) \cdot \frac{1}{y} && \text{(by part (f))} \\
&= (-1)\left(x \cdot \frac{1}{y}\right) && \text{(by (1))} \\
&= (-1)\frac{x}{y} && \text{(by the definition of division)} \\
&= -\left(\frac{x}{y}\right). && \text{(by part (f))}
\end{aligned}$$

Now, we have $(-1)(-1) = -(-1) = 1$ by parts (f) and (d) so that -1 is its own reciprocal, since the reciprocal is unique, i.e. $1/(-1) = -1$. We also have

$$\begin{aligned}
\frac{-x}{y} &= (-x) \cdot \frac{1}{y} && \text{(by the definition of division)} \\
&= ((-1)x) \cdot \frac{1}{y} && \text{(by part (f))} \\
&= (x(-1)) \cdot \frac{1}{y} && \text{(by (2))} \\
&= x\left((-1)\frac{1}{y}\right) && \text{(by (1))} \\
&= x\left(\frac{1}{-1} \cdot \frac{1}{y}\right) && \text{(by what was just shown above)} \\
&= x\frac{1}{(-1)y} && \text{(part (m) since } y \neq 0\text{)} \\
&= x\frac{1}{-y} && \text{(by part (f))}
\end{aligned}$$

so that $-(x/y) = (-x)/y = x/(-y)$ as desired. □

Exercise 4.2

Prove the following "laws of inequalities" for $\mathbb{R}$, using axioms (1)-(6) along with the results of Exercise 1:

(a) $x > y$ and $w > z \Rightarrow x + w > y + z$.
(b) $x > 0$ and $y > 0 \Rightarrow x + y > 0$ and $x \cdot y > 0$.
(c) $x > 0 \Leftrightarrow -x < 0$.
(d) $x > y \Leftrightarrow -x < -y$.
(e) $x > y$ and $z < 0 \Rightarrow xz < yz$.
(f) $x \neq 0 \Rightarrow x^2 > 0$, where $x^2 = x \cdot x$.
(g) $-1 < 0 < 1$.
(h) $xy > 0 \Leftrightarrow x$ and y are both positive or both negative.
(i) $x > 0 \Rightarrow 1/x > 0$.
(j) $x > y > 0 \Rightarrow 1/x < 1/y$.
(k) $x < y \Rightarrow x < (x + y)/2 < y$.

Solution:

Lemma 4.2.1. $x + x = 2x$ *for any real* x.

Proof. We simply have

$$\begin{aligned}
x + x &= x \cdot 1 + x \cdot 1 && \text{(by (3))} \\
&= x(1 + 1) && \text{(by (5))} \\
&= x \cdot 2 && \text{(since 2 is defined as } 1 + 1\text{)} \\
&= 2x && \text{(by (2))}
\end{aligned}$$

as desired. □

Main Problem.

(a) $x > y$ and $w > z \Rightarrow x + w > y + z$.

Proof. We have

$$\begin{aligned} x + w &> y + w && \text{(by (6) since } x > y\text{)} \\ &= w + y && \text{(by (2))} \\ &> z + y && \text{(by (6) since } w > z\text{)} \\ &= y + z && \text{(by (2))} \end{aligned}$$

as desired. □

(b) $x > 0$ and $y > 0 \Rightarrow x + y > 0$ and $x \cdot y > 0$.

Proof. First we have

$$\begin{aligned} x + y &> 0 + y && \text{(by (6) since } x > 0\text{)} \\ &= y + 0 && \text{(by (2))} \\ &= y && \text{(by (3))} \\ &> 0\,. \end{aligned}$$

Also

$$\begin{aligned} x \cdot y &> 0 \cdot y && \text{(by (6) since } x > 0 \text{ and } y > 0\text{)} \\ &= 0 && \text{(by Exercise 4.1 part (b))} \end{aligned}$$

as desired. □

(c) $x > 0 \Leftrightarrow -x < 0$.

Proof. ($\Rightarrow$) Suppose that $x > 0$. Then we have

$$\begin{aligned} -x &= -x + 0 && \text{(by (3))} \\ &= 0 + (-x) && \text{(by (2))} \\ &< x + (-x) && \text{(by (6) since } 0 < x\text{)} \\ &= 0\,. && \text{(by (4))} \end{aligned}$$

($\Leftarrow$) Suppose now that $-x < 0$. Then we have

$$\begin{aligned} x &= x + 0 && \text{(by (3))} \\ &= 0 + x && \text{(by (2))} \\ &> -x + x && \text{(by (6) since } 0 > -x\text{)} \\ &= x + (-x) && \text{(by (2))} \\ &= 0 && \text{(by (4))} \end{aligned}$$

as desired. □

(d) $x > y \Leftrightarrow -x < -y$.

Proof. ($\Rightarrow$) Suppose that $x > y$. Then we have

$$\begin{aligned} -y &= -y + 0 && \text{(by (3))} \\ &= -y + (x + (-x)) && \text{(by (4))} \end{aligned}$$

$$
\begin{aligned}
&= (x + (-x)) + (-y) && \text{(by (2))}\\
&= x + (-x + (-y)) && \text{(by (1))}\\
&> y + (-x + (-y)) && \text{(by (6) since } x > y)\\
&= y + (-y + (-x)) && \text{(by (2))}\\
&= (y + (-y)) + (-x) && \text{(by (1))}\\
&= 0 + (-x) && \text{(by (4))}\\
&= -x + 0 && \text{(by (2))}\\
&= -x\,. && \text{(by (3))}
\end{aligned}
$$

($\Leftarrow$) Now suppose that $-x < -y$. Then we have

$$
\begin{aligned}
x &= x + 0 && \text{(by (3))}\\
&= x + (y + (-y)) && \text{(by (4))}\\
&= (y + (-y)) + x && \text{(by (2))}\\
&= (-y + y) + x && \text{(by (2))}\\
&= -y + (y + x) && \text{(by (1))}\\
&> -x + (y + x) && \text{(by (6) since } -y > -x)\\
&= -x + (x + y) && \text{(by (2))}\\
&= (-x + x) + y && \text{(by (1))}\\
&= (x + (-x)) + y && \text{(by (2))}\\
&= 0 + y && \text{(by (4))}\\
&= y + 0 && \text{(by (2))}\\
&= y && \text{(by (3))}
\end{aligned}
$$

as desired. □

(e) $x > y$ and $z < 0 \Rightarrow xz < yz$.

Proof. First, by Exercise 4.1 part (d), we have $-(-z) = z < 0$ so that $-z > 0$ by part (c). Then, since $x > y$, it follows from (6) that

$$
\begin{aligned}
x(-z) &> y(-z)\\
-(xz) &> -(yz) && \text{(by Exercise 4.1 part (e) applied to both sides)}\\
xz &< yz && \text{(by part (d))}
\end{aligned}
$$

as desired. □

(f) $x \neq 0 \Rightarrow x^2 > 0$, where $x^2 = x \cdot x$.

Proof. Since $x \neq 0$ we either have that $x > 0$ or $x < 0$ since the $<$ relation is an order (in particular a linear order since this is part of the definition of order in this text). If $x > 0$ then we have $x^2 = x \cdot x > 0 \cdot x = 0$ by (6) (since $x > 0$) and Exercise 4.1 part (b). If $x < 0$ then we have $0 = 0 \cdot x < x \cdot x = x^2$ by part (e) (since $0 > x$) and Exercise 4.1 part (b). Together these show the desired result. □

(g) $-1 < 0 < 1$.

Proof. By (4) we know that $1 \neq 0$ so that $1^2 > 0$ by part (f). However, we have $1^2 = 1 \cdot 1 = 1$ by (3). Hence $1 = 1^2 > 0$. It then follows from part (c) that $-1 < 0$ so that we have $-1 < 0 < 1$ as desired. □

(h) $xy > 0 \Leftrightarrow x$ and y are both positive or both negative.

Proof. ($\Rightarrow$) Suppose that $xy > 0$. It cannot be that $x = 0$, for then we would have $0 = 0 \cdot y = xy > 0$ by Exercise 4.1 part (b), which is impossible by the definition of an order. Hence we have $x \neq 0$, and an analogous argument shows that $y \neq 0$ as well. We then have the following:

Case: $x > 0$. Suppose that $y < 0$. Then, by part (e) and Exercise 4.1 part (b), we have $xy < 0 \cdot y = 0$ since $x > 0$ and $y < 0$, which contradicts our initial supposition. Thus, since we know that $y \neq 0$, it has to be that $y > 0$ as well.

Case: $x < 0$. Suppose that $y > 0$. Then, by (6) and Exercise 4.1 part (b), we have $0 = 0 \cdot y > xy$ since $0 > x$ and $y > 0$, which again contradicts the initial supposition. So it must be that $y < 0$ also since $y \neq 0$.

Therefore in every case either both x and y are positive or they are both negative. Since $x \neq 0$, these cases are exhaustive so that this shows the result.

($\Leftarrow$) Suppose that either $x > 0, y > 0$ or $x < 0, y < 0$. In the case where both $x > 0$ and $y > 0$ we clearly have $xy > 0 \cdot y = 0$ by (6) and Exercise 4.1 part (b). In the other case in which $x < 0$ and $y < 0$ we have $0 = 0 \cdot y < xy$ by part (e) and Exercise 4.1 part (b) since $0 > x$ and $y < 0$. Hence $xy > 0$ in both cases. □

(i) $x > 0 \Rightarrow 1/x > 0$.

Proof. First, it cannot be that $1/x = 0$ because then we would have $1 = x(1/x) = x \cdot 0 = 0 \cdot x = 0$ by (4), (2), and Exercise 4.1 part (b). This is clearly a contradiction since we know that $1 \neq 0$ by (3). Hence $1/x \neq 0$. Now suppose that $1/x < 0$ so that $1 = x(1/x) < 0 \cdot (1/x) = 0$ by part (e) since $x > 0$ and $1/x < 0$, and we have also used Exercise 4.1 part (b). This is also a contradiction since it was proved in part (g) that $1 > 0$. Hence the only remaining possibility is that $1/x > 0$ as desired. □

(j) $x > y > 0 \Rightarrow 1/x < 1/y$.

Proof. First, since the order is transitive, we have $x, y > 0$. It then follows from part (i) that $1/x, 1/y > 0$. Then $(1/x)(1/y) > 0$ by part (h). We then have

$$\begin{aligned}
\frac{1}{x} &= \frac{1}{x} \cdot 1 && \text{(by (3))} \\
&= \frac{1}{x}\left(y \cdot \frac{1}{y}\right) && \text{(by (4))} \\
&= \left(\frac{1}{x} \cdot y\right)\frac{1}{y} && \text{(by (1))} \\
&= \left(y \cdot \frac{1}{x}\right)\frac{1}{y} && \text{(by (2))} \\
&= y\left(\frac{1}{x} \cdot \frac{1}{y}\right) && \text{(by (1))} \\
&< x\left(\frac{1}{x} \cdot \frac{1}{y}\right) && \text{(by (6) since } y < x \text{ and } (1/x)(1/y) > 0) \\
&= \left(x \cdot \frac{1}{x}\right)\frac{1}{y} && \text{(by (1))}
\end{aligned}$$

$$\begin{aligned}
&= 1 \cdot \frac{1}{y} && \text{(by (4))} \\
&= \frac{1}{y} \cdot 1 && \text{(by (2))} \\
&= \frac{1}{y} && \text{(by (3))}
\end{aligned}$$

as desired. □

(k) $x < y \Rightarrow x < (x + y)/2 < y$.

Proof. First, we know by part (g) that $1 > 0$ so that

$$\begin{aligned}
2 &= 1 + 1 && \text{(by the definition of 2)} \\
&> 0 + 1 && \text{(by (6) since } 1 > 0\text{)} \\
&= 1 + 0 && \text{(by (2))} \\
&= 1 && \text{(by (3))} \\
&> 0\,. && \text{(by part (g))}
\end{aligned}$$

To summarize, $0 < 1 < 2$. It then follows from part (i) that $1/2 > 0$. We then have

$$\begin{aligned}
x &< y \\
x + x &< x + y && \text{(by (6))} \\
2x &< x + y && \text{(by Lemma 4.2.1)} \\
(2x)\frac{1}{2} &< (x + y)\frac{1}{2} && \text{(by (6) since } 1/2 > 0) \\
(x \cdot 2)\frac{1}{2} &< \frac{x + y}{2} && \text{(by (2) and the definition of division)} \\
x\left(2 \cdot \frac{1}{2}\right) &< \frac{x + y}{2} && \text{(by (1))} \\
x \cdot 1 &< \frac{x + y}{2} && \text{(by (4))} \\
x &< \frac{x + y}{2}\,. && \text{(by (3))}
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
x &< y \\
x + y &< y + y && \text{(by (6))} \\
x + y &< 2y && \text{(by Lemma 4.2.1)} \\
(x + y)\frac{1}{2} &< (2y)\frac{1}{2} && \text{(by (6) since } 1/2 > 0) \\
\frac{x + y}{2} &< (y \cdot 2)\frac{1}{2} && \text{(by the definition of division and (2))} \\
\frac{x + y}{2} &< y\left(2 \cdot \frac{1}{2}\right) && \text{(by (1))} \\
\frac{x + y}{2} &< y \cdot 1 && \text{(by (4))} \\
\frac{x + y}{2} &< y\,. && \text{(by (3))}
\end{aligned}$$

This shows that $x < (x + y)/2 < y$ as desired. □

Exercise 4.3

(a) Show that if $\mathcal{A}$ is a collection of inductive sets, then the intersection of the elements of $\mathcal{A}$ is an inductive set.

(b) Prove the basic properties (1) and (2) of $\mathbb{Z}_+$.

Solution:

(a) We must show that $\bigcap_{A \in \mathcal{A}} A$ is inductive.

Proof. First, consider any $A \in \mathcal{A}$. Then, since A is inductive, $1 \in A$. Since A was arbitrary, this shows that $1 \in \bigcap_{A \in \mathcal{A}} A$. Now suppose that $x \in \bigcap_{A \in \mathcal{A}} A$ and again consider arbitrary $A \in \mathcal{A}$. Then $x \in A$ so that $x + 1 \in A$ also since A is inductive. Since A was arbitrary, this shows that $x + 1 \in \bigcap_{A \in \mathcal{A}} A$. Hence by definition $\bigcap_{A \in \mathcal{A}} A$ is inductive. □

(b)

Proof. Let $\mathcal{A}$ be the collection of all inductive sets of $\mathbb{R}$ so that by definition $\mathbb{Z}_+ = \bigcap_{A \in \mathcal{A}} A$. It then follows immediately from part (a) that $\mathbb{Z}_+$ is inductive since $\mathcal{A}$ is a collection of inductive sets. This shows property (1).

Now suppose that A is an inductive set of positive integers. That is, A is inductive and $A \subset \mathbb{Z}_+$. Consider any $x \in \mathbb{Z}_+ = \bigcap_{B \in \mathcal{A}} B$, where again $\mathcal{A}$ is the the collection of all inductive subsets of $\mathbb{R}$. Clearly we have that $A \subset \mathbb{Z}_+ \subset \mathbb{R}$ so that $A \in \mathcal{A}$ since A is an inductive subset of $\mathbb{R}$. Hence $x \in A$ (since $x \in \bigcap_{B \in \mathcal{A}} B$ and $A \in \mathcal{A}$) so that $\mathbb{Z}_+ \subset A$ since x was arbitrary. This shows that $A = \mathbb{Z}_+$ as desired since also $A \subset \mathbb{Z}_+$. This shows property (2). □

Exercise 4.4

(a) Prove by induction that given $n \in \mathbb{Z}_+$, every nonempty subset of $\{1, \ldots, n\}$ has a largest element.

(b) Explain why you cannot conclude from (a) that every nonempty subset of $\mathbb{Z}_+$ has a largest element.

Solution:

(a)

Proof. Let A be the set of integers such that the hypothesis is true. Clearly the result is then shown if we can prove that $A = \mathbb{Z}_+$. So first, clearly $1 \in A$ since the set $\{1\}$ has only a single nonempty subset, i.e. $\{1\}$ itself, in which 1 is clearly the largest element. Now suppose that $n \in A$ so that every nonempty subset of $S_{n+1} = \{1, \ldots, n\}$ has a largest element. Consider any nonempty subset B of $S_{n+2} = \{1, \ldots, n + 1\}$, noting that $S_{n+2} = S_{n+1} \cup \{n + 1\}$.

Case: $n + 1 \in B$. Then, for any other $k \in B$, $k \in S_{n+2}$ so that either $k = n + 1$ or $k \in S_{n+1}$ so that $k < n + 1$ by the definition of S_{n+1}. Thus in either case $k \leq n + 1$ so that $n + 1$ is the largest element of B since k was arbitrary.

Case: $n+1 \notin B$. Then clearly $B \subset S_{n+1}$ so that B has a largest element by the induction hypothesis since B is nonempty.

Hence in either case B has a largest element so that $n + 1 \in A$ since B was an arbitrary nonempty subset of $S_{n+2} = \{1, \ldots, n + 1\}$. This shows that A is an inductive set of positive integers so that $A = \mathbb{Z}_+$ as desired by the Principle of Induction. □

(b) There could be nonempty subsets of $\mathbb{Z}_+$ that are *not* subsets of $S_{n+1} = \{1, \ldots, n\}$ for any $n \in \mathbb{Z}_+$, in which cases the hypothesis of part (a) is not satisfied so that the conclusion does not necessarily apply. In fact, $\mathbb{Z}_+$ itself is an example of such a set where both the hypothesis and the conclusion are false.

Exercise 4.5

Prove the following properties of $\mathbb{Z}$ and $\mathbb{Z}_+$:

(a) $a, b \in \mathbb{Z}_+ \Rightarrow a + b \in \mathbb{Z}_+$. [Hint: Show that given $a \in \mathbb{Z}_+$, the set $X = \{x \mid x \in \mathbb{R} \text{ and } a + x \in \mathbb{Z}_+\}$ is inductive.]

(b) $a, b \in \mathbb{Z}_+ \Rightarrow a \cdot b \in \mathbb{Z}_+$.

(c) Show that $a \in \mathbb{Z}_+ \Rightarrow a - 1 \in \mathbb{Z}_+ \cup \{0\}$. [Hint: Let $X = \{x \mid x \in \mathbb{R} \text{ and } x - 1 \in \mathbb{Z}_+ \cup \{0\}\}$; show that X is inductive.]

(d) $c, d \in \mathbb{Z} \Rightarrow c + d \in \mathbb{Z}$ and $c - d \in \mathbb{Z}$. [Hint: Prove it first for $d = 1$.]

(e) $c, d \in \mathbb{Z} \Rightarrow c \cdot d \in \mathbb{Z}$.

Solution:

Lemma 4.5.1. *If $x \in \mathbb{Z}$ then $-x \in \mathbb{Z}$.*

Proof. Let $\mathbb{Z}_- = \{-x \mid x \in \mathbb{Z}_+\}$ so that by definition $\mathbb{Z} = \mathbb{Z}_+ \cup \{0\} \cup \mathbb{Z}_-$. Suppose that $x \in \mathbb{Z}$ so that $x \in \mathbb{Z}_+ \cup \{0\} \cup \mathbb{Z}_-$.

Case: $x \in \mathbb{Z}_+$. Then $-x \in \mathbb{Z}_-$ by definition.

Case: $x = 0$. Then by Exercise 4.1 part (c) we have $-x = -0 = 0 \in \{0\}$.

Case: $x \in \mathbb{Z}_-$. Then by definition there is a $y \in \mathbb{Z}_+$ such that $x = -y$. Then $-x = -(-y) = y \in \mathbb{Z}_+$ by Exercise 4.1 part (d).

Hence in all cases either $-x \in \mathbb{Z}_+$, $-x \in \{0\}$, or $-x \in \mathbb{Z}_-$ so that $-x \in \mathbb{Z}_+ \cup \{0\} \cup \mathbb{Z}_- = \mathbb{Z}$ as desired. □

Main Problem.

(a)

Proof. Consider any $a \in \mathbb{Z}_+$ and define $X_a = \{x \in \mathbb{R} \mid a + x \in \mathbb{Z}_+\}$. We show that X_a is inductive. First, since $a \in \mathbb{Z}_+$ we have that $a + 1 \in \mathbb{Z}_+$ since $\mathbb{Z}_+$ is inductive. Hence $1 \in X_a$ by definition. Now suppose that $x \in X_a$ so that $a + x \in \mathbb{Z}_+$. Then we have $a + (x + 1) = (a + x) + 1 \in \mathbb{Z}_+$ since $a + x \in \mathbb{Z}_+$ and $\mathbb{Z}_+$ is inductive. This shows by definition that $x + 1 \in X_a$ and therefore that X_a is inductive. It follows that $\mathbb{Z}_+ \subset X_a$ since $\mathbb{Z}_+$ is defined as the intersection of all inductive subsets of reals, of which X_a is one.

Therefore, for any $a, b \in \mathbb{Z}_+$, we have that $b \in X_a$ since $\mathbb{Z}_+ \subset X_a$. Thus by definition $a + b \in \mathbb{Z}_+$ as desired □

(b)

Proof. Consider any $a \in \mathbb{Z}_+$ and define $X_a = \{x \in \mathbb{R} \mid a \cdot x \in \mathbb{Z}_+\}$. We show that X_a is inductive. To this end, we first have that $a \cdot 1 = a \in \mathbb{Z}_+$ so that $1 \in X_a$ by definition. Now suppose that $x \in X_a$ so that $ax \in \mathbb{Z}_+$. Then we have $a \cdot (x + 1) = a \cdot x + a \cdot 1 = ax + a \in \mathbb{Z}_+$ by part (a) since we know both ax and a are in $\mathbb{Z}_+$. Hence $x+1 \in X_a$ by definition. This shows that X_a is inductive so that again $\mathbb{Z}_+ \subset X_a$.

Hence for any $a, b \in \mathbb{Z}_+$ we have that $b \in X_a$ since $\mathbb{Z}_+ \subset X_a$. It then follows by definition that $a \cdot b \in \mathbb{Z}_+$ as desired. □

(c)

Proof. Let $X = \{x \in \mathbb{R} \mid x - 1 \in \mathbb{Z}_+ \cup \{0\}\}$, which we show is inductive. First, we have $1 - 1 = 1 + (-1) = 0$ so that clearly $1 \in \mathbb{Z}_+ \cup \{0\}$ and hence $1 \in X$. Now suppose that $x \in X$ so that $x - 1 \in \mathbb{Z}_+ \cup \{0\}$.

Case: $x - 1 \in \{0\}$. Then it must be that $x - 1 = 0$, which clearly implies $x = 1 \in \mathbb{Z}_+$ since $\mathbb{Z}_+$ is inductive. Then $(x+1) - 1 = x + (1-1) = x + 0 = x \in \mathbb{Z}_+$ so that $(x+1) - 1 \in \mathbb{Z}_+ \cup \{0\}$ and therefore $x + 1 \in X$.

Case: $x - 1 \in \mathbb{Z}_+$. Then $(x+1) - 1 = x + (1-1) = x + ((-1)+1) = (x-1) + 1 \in \mathbb{Z}_+$ since $x - 1 \in \mathbb{Z}_+$ and $\mathbb{Z}_+$ is inductive. Thus clearly $(x+1) - 1 \in \mathbb{Z}_+ \cup \{0\}$ so that $x + 1 \in X$ by definition.

Hence in both cases $x + 1 \in X$, which shows that X is inductive, and so $\mathbb{Z}_+ \subset X$. Therefore, for any $a \in \mathbb{Z}_+$, we have that also $x \in X$ since $\mathbb{Z}_+ \subset X$. Then, by the definition of X, it follows that $a - 1 \in \mathbb{Z}_+ \cup \{0\}$ as desired. □

(d)

Proof. First we show that the set $X_c = \{x \in \mathbb{R} \mid c + x \in \mathbb{Z} \text{ and } c - x \in \mathbb{Z}\}$ is inductive for any $c \in \mathbb{Z}$. So consider any c and b in $\mathbb{Z}$ so that $c, b \in \mathbb{Z}_+ \cup \{0\} \cup \mathbb{Z}_-$.

Case: $b \in \mathbb{Z}_+$. Then $b + 1 \in \mathbb{Z}_+$ since $\mathbb{Z}_+$ is inductive and $b - 1 \in \mathbb{Z}_+ \cup \{0\}$ by part (c).

Case: $b = 0$. Then $b + 1 = 0 + 1 = 1 \in \mathbb{Z}_+$ since it is inductive, and $b - 1 = 0 - 1 = -1 \in \mathbb{Z}_-$ since $1 \in \mathbb{Z}_+$.

Case: $b \in \mathbb{Z}_-$. Then $b = -a$ for $a \in \mathbb{Z}_+$, and we then have that $a + 1 \in \mathbb{Z}_+$ since $\mathbb{Z}_+$ is inductive. Hence $b - 1 = -a - 1 = -(a+1) \in \mathbb{Z}_-$. We also have that $a - 1 \in \mathbb{Z}_+ \cup \{0\}$ by part (c), from which it is trivial to show that $-(a-1) \in \mathbb{Z}_- \cup \{0\}$. Therefore $b + 1 = -a + 1 = -(a-1) \in \mathbb{Z}_- \cup \{0\}$.

Thus in all cases we have that $b + 1$ and $b - 1$ are in $\mathbb{Z}_+$ or $\{0\}$ or $\mathbb{Z}_-$ so that they are both in $\mathbb{Z}$, and so $1 \in X_b$. Note that this is the case for any $b \in \mathbb{Z}$ so that it is clearly true for c, i.e. $1 \in X_c$. Now suppose that $x \in X_c$ so that $c + x$ and $c - x$ are both in $\mathbb{Z}$. It then follows that $1 \in X_{c+x}$ and $1 \in X_{c-x}$ so that $c + (x+1) = (c+x) + 1 \in \mathbb{Z}$ and $c - (x+1) = (c-x) - 1 \in \mathbb{Z}$. This then shows that $x + 1 \in X_c$. Hence X_c is inductive for any $c \in \mathbb{Z}$ so that $\mathbb{Z}_+ \subset X_c$.

Now consider $c, d \in \mathbb{Z}$.

Case: $d \in \mathbb{Z}_+$. Then clearly $d \in X_c$ since $\mathbb{Z}_+ \subset X_c$. Hence by definition $c + d$ and $c - d$ are both in $\mathbb{Z}$.

Case: $d = 0$. Then $c + d = c + 0 = c \in \mathbb{Z}$ and $c - d = c - 0 = c \in \mathbb{Z}$.

Case: $d \in \mathbb{Z}_-$. Then by definition $d = -a$ for $a \in \mathbb{Z}_+$ so that $a \in X_c$ since $\mathbb{Z}_+ \subset X_c$. Then $c + a$ and $c - a$ are both in $\mathbb{Z}$ by the definition of X_c. Hence $c + d = c + (-a) = c - a \in \mathbb{Z}$ and $c - d = c - (-a) = c + a \in \mathbb{Z}$.

Therefore we have shown that $c + d$ and $c - d$ are both integers in all cases, which is the desired result. □

(e)

Proof. For any $c \in \mathbb{Z}$, define $X_c = \{x \in \mathbb{R} \mid c \cdot x \in \mathbb{Z}\}$. We first show that X_c is inductive for any such $c \in \mathbb{Z}$. We have $c \cdot 1 = c \in \mathbb{Z}$ so that $1 \in X_c$. Now suppose that $x \in X_c$ so that $c \cdot x \in \mathbb{Z}$. Then $c \cdot (x+1) = c \cdot x + c \cdot 1 = c \cdot x + c \in \mathbb{Z}$ by part (d) since both $c \cdot x$ and c are integers. This shows that X_c is inductive so that $\mathbb{Z}_+ \subset X_c$.

Now consider any $c, d \in \mathbb{Z}$.

Case: $d \in \mathbb{Z}_+$. Then $d \in X_c$ since $\mathbb{Z}_+ \subset X_c$. Thus $c \cdot d \in \mathbb{Z}$.

Case: $d = 0$. The $c \cdot d = c \cdot 0 = 0 \in \mathbb{Z}$.

Case: $d \in \mathbb{Z}_-$. Then there is an $a \in \mathbb{Z}_+$ such that $d = -a$. Hence $a \in X_c$ since $\mathbb{Z}_+ \subset X_c$, from which it follows that $c \cdot a \in \mathbb{Z}$. We then have $c \cdot d = c \cdot (-a) = -(c \cdot a) \in \mathbb{Z}$ as well by Lemma 4.5.1.

Thus in all cases $c \cdot d \in \mathbb{Z}$ as desired. □

Exercise 4.6

Let $a \in \mathbb{R}$. Define inductively

$$a^1 = a\,,$$
$$a^{n+1} = a^n \cdot a$$

for $n \in \mathbb{Z}_+$. (See §7 for a discussion of the process of inductive definition.) Show that for $n, m \in \mathbb{Z}_+$ and $a, b \in \mathbb{R}$,

$$a^n a^m = a^{n+m}$$
$$(a^n)^m = a^{nm}$$
$$a^m b^m = (ab)^m\,.$$

These are called the ***laws of exponents***. [Hint: For fixed n, prove the formulas by induction on m.]

Solution:

The following lemma is the familiar proof by induction, which is more straightforward than having to frame everything in terms of inductive sets. Henceforth we use this whenever induction is required.

Lemma 4.6.1. *(Proof by Induction) Suppose that $P(x)$ is a statement with parameter x. Suppose also that $P(1)$ is true and that $P(x)$ implies $P(x+1)$. Then $P(n)$ is true for all $n \in \mathbb{Z}_+$.*

Proof. Define the set $X = \{x \in \mathbb{R} \mid P(x)\}$. We show that X is inductive. Clearly since $P(1)$ is true we have $1 \in X$. Now suppose that $x \in X$ so that $P(x)$ is true. Then $P(x+1)$ is also true so that $x + 1 \in X$. This shows that X is inductive so that $\mathbb{Z}_+ \subset X$. So, for any positive integer n we have that $n \in X$ since $\mathbb{Z}_+ \subset X$. Therefore $P(n)$ is true. Since n was arbitrary, this shows the desired result. □

Main Problem.

In what follows, suppose that $a, b \in \mathbb{R}$.

First we show that $a^n a^m = a^{n+m}$ for all $n, m \in \mathbb{Z}_+$.

Proof. Fix $n \in \mathbb{Z}_+$. We show the result by induction on m. First, we clearly have $a^n a^1 = a^n \cdot a = a^{n+1}$ by the inductive definition. Now suppose that $a^n a^m = a^{n+m}$. Then

$$a^n a^{m+1} = a^n \cdot (a^m \cdot a) \qquad \text{(by the inductive definition)}$$

$$
\begin{aligned}
&= (a^n a^m) \cdot a && \text{(since multiplication is associative)} \\
&= a^{n+m} \cdot a && \text{(by the induction hypothesis)} \\
&= a^{(n+m)+1} && \text{(by the inductive definition)} \\
&= a^{n+(m+1)}\,, && \text{(since addition is associative)}
\end{aligned}
$$

which completes the induction step. Therefore the result holds for all $m \in \mathbb{Z}_+$ by induction. □

Next we show that $(a^n)^m = a^{nm}$ for all $n, m \in \mathbb{Z}_+$.

Proof. We again fix $n \in \mathbb{Z}_+$ and use induction on m. First, we have $(a^n)^1 = a^n = a^{n \cdot 1}$ by the inductive definition. Supposing now that $(a^n)^m = a^{n \cdot m}$, we have

$$
\begin{aligned}
(a^n)^{m+1} &= (a^n)^m \cdot a^n && \text{(by the inductive definition)} \\
&= a^{n \cdot m} a^n && \text{(by the induction hypothesis)} \\
&= a^{n \cdot m + n} && \text{(by what was shown above)} \\
&= a^{n \cdot m + n \cdot 1} \\
&= a^{n \cdot (m+1)}\,. && \text{(by the distributive property)}
\end{aligned}
$$

This completes the induction so that the result holds for all $m \in \mathbb{Z}_+$. □

Lastly, we show that $a^m b^m = (ab)^m$ for all $m \in \mathbb{Z}_+$.

Proof. We show this by induction on m. First, we have $a^1 b^1 = ab = (ab)^1$ by the inductive definition. Now suppose that $a^m b^m = (ab)^m$ so that

$$
\begin{aligned}
a^{m+1} b^{m+1} &= (a^m \cdot a)(b^m \cdot b) && \text{(by the inductive definition)} \\
&= (a \cdot a^m)(b^m \cdot b) && \text{(since multiplication is commutative)} \\
&= ((a \cdot a^m) b^m) \cdot b && \text{(since multiplication is associative)} \\
&= (a \cdot (a^m b^m)) \cdot b && \text{(since multiplication is associative)} \\
&= (a(ab)^m) \cdot b && \text{(by the induction hypothesis)} \\
&= ((ab)^m a) \cdot b && \text{(since multiplication is commutative)} \\
&= (ab)^m (ab) && \text{(since multiplication is associative)} \\
&= (ab)^{m+1}\,. && \text{(by the inductive definition)}
\end{aligned}
$$

This completes the induction. □

Exercise 4.7

Let $a \in \mathbb{R}$ and $a \neq 0$. Define $a^0 = 1$, and for $n \in \mathbb{Z}_+$, $a^{-n} = 1/a^n$. Show that the laws of exponents hold for $a, b \neq 0$ and $n, m \in \mathbb{Z}$.

Solution:

Lemma 4.7.1. *For any $n \in \mathbb{Z}$, $1^n = 1$.*

Proof. We show this for $n \in \mathbb{Z}_+$ by simple induction on n. First, clearly $1^1 = 1$ by the inductive definition of exponentiation. Next, if $1^n = 1$, then we have $1^{n+1} = 1^n \cdot 1 = 1^n = 1$ by the inductive definition of exponentiation and the inductive hypothesis. This completes the induction so that the result holds for all $n \in \mathbb{Z}_+$.

Clearly if $n = 0$ then, by the definition of 0 as an exponent, $1^n = 1^0 = 1$.

Lastly, if $n \in \mathbb{Z}_-$ then there is a $k \in \mathbb{Z}_+$ where $n = -k$. Then we have

$$\begin{aligned}
1^n &= 1^{-k} \\
&= \frac{1}{1^k} && \text{(by the definition of negative exponentiation)} \\
&= \frac{1}{1} && \text{(by what was just shown by induction since } k \in \mathbb{Z}_+\text{)} \\
&= 1\,. && \text{(since 1 is its own reciprocal)}
\end{aligned}$$

Thus the result has been shown for all the resulting cases when $n \in \mathbb{Z}$. □

Lemma 4.7.2. *$1/a^n = (1/a)^n$ for any real $a \neq 0$ and $n \in \mathbb{Z}_+$.*

Proof. We have

$$\begin{aligned}
\left(\frac{1}{a}\right)^n a^n &= \left(\frac{1}{a} \cdot a\right)^n && \text{(by Exercise 4.6 since } n \in \mathbb{Z}_+\text{)} \\
&= 1^n && \text{(by the definition of the reciprocal)} \\
&= 1\,. && \text{(by Lemma 4.7.1)}
\end{aligned}$$

Thus $(1/a)^n$ must be the unique reciprocal of a^n, that is $(1/a)^n = 1/a^n$ as desired. □

Lemma 4.7.3. *$a^n a^{-n} = 1$ for any real $a \neq 0$ and $n \in \mathbb{Z}_+$.*

Proof. We have

$$\begin{aligned}
a^n a^{-n} &= a^n \left(\frac{1}{a^n}\right) && \text{(by the definition of negative exponentiation)} \\
&= a^n \left(\frac{1}{a}\right)^n && \text{(by Lemma 4.7.2)} \\
&= \left(a \cdot \frac{1}{a}\right)^n && \text{(by Exercise 4.6 since } n \in \mathbb{Z}_+\text{)} \\
&= 1^n && \text{(by the definition of the reciprocal)} \\
&= 1 && \text{(by Lemma 4.7.1)}
\end{aligned}$$

as desired. □

Main Problem.

First we show that $a^n a^m = a^{n+m}$ for all real $a \neq 0$ and $n, m \in \mathbb{Z}$.

Proof. Consider any real $a \neq 0$ and $n, m \in \mathbb{Z}$. We number the following cases for reference:

1. Case: $n \in \mathbb{Z}_+$.
   (a) Case: $m \in \mathbb{Z}_+$. Then the result immediately applies by Exercise 4.6.
   (b) Case: $m = 0$. Then we have $a^n a^m = a^n a^0 = a^n \cdot 1 = a^n = a^{n+0} = a^{n+m}$.
   (c) Case: $m \in \mathbb{Z}_-$. Then $m = -k$ for some $k \in \mathbb{Z}_+$.

i. Case: $n > k$. Then $n - k > 0$ so that $n - k \in \mathbb{Z}_+$ since $n - k \in \mathbb{Z}$ by Exercise 4.5 part (d). We then have

$$\begin{aligned}
a^n a^m &= a^n a^{-k} \\
&= a^{n-k+k} a^{-k} && \text{(since } n = n + 0 = n - k + k\text{)} \\
&= (a^{n-k} a^k) a^{-k} && \text{(by Exercise 4.6 since } k, n-k \in \mathbb{Z}_+\text{)} \\
&= a^{n-k}(a^k a^{-k}) && \text{(since multiplication is associative)} \\
&= a^{n-k} \cdot 1 && \text{(by Lemma 4.7.3 since } k \in \mathbb{Z}_+\text{)} \\
&= a^{n-k} \\
&= a^{n+m}\,.
\end{aligned}$$

ii. Case: $n = k$. Then clearly $n + m = n - k = k - k = 0$, so that we have $a^n a^m = a^k a^{-k} = 1 = a^0 = a^{n+m}$ by Lemma 4.7.3 and the definition of 0 as an exponent.

iii. Case: $n < k$. Then $n - k < 0$ so that $n - k \in \mathbb{Z}_-$ since $n - k \in \mathbb{Z}$ by Exercise 4.5 part (d). Also, clearly $-n \in \mathbb{Z}_-$ since $n \in \mathbb{Z}_+$. Then we have

$$\begin{aligned}
a^n a^m &= a^n a^{-k} \\
&= a^n a^{-k+n-n} && \text{(since } -k = -k + 0 = -k + n - n\text{)} \\
&= a^n a^{n-k-n} && \text{(since addition is commutative)} \\
&= a^n (a^{n-k} a^{-n}) && \text{(by case 3c below since } n-k, -n \in \mathbb{Z}_-\text{)} \\
&= a^n (a^{-n} a^{n-k}) && \text{(since multiplication is commutative)} \\
&= (a^n a^{-n}) a^{n-k} && \text{(since multiplication is associative)} \\
&= 1 \cdot a^{n-k} && \text{(by Lemma 4.7.3)} \\
&= a^{n-k} \\
&= a^{n+m}\,.
\end{aligned}$$

2. Case: $n = 0$.

(a) Case: $m \in \mathbb{Z}_+$. Since $a^n a^m = a^m a^n$ and $a^{n+m} = a^{m+n}$, this the same as case 1b above.

(b) Case: $m = 0$. Then we have $a^n a^m = a^0 a^0 = 1 \cdot 1 = 1 = a^0 = a^{0+0} = a^{n+m}$.

(c) Case: $m \in \mathbb{Z}_-$. Then there is a $k \in \mathbb{Z}_+$ such that $m = -k$, and $a^n a^m = a^0 a^{-k} = 1 \cdot (1/a^k) = 1/a^k = a^{-k} = a^m = a^{0+m} = a^{n+m}$.

3. Case: $n \in \mathbb{Z}_-$.

(a) Case: $m \in \mathbb{Z}_+$. This is the same as case 1c above.

(b) Case: $m = 0$. This is the same as case 2c above.

(c) Case: $m \in \mathbb{Z}_-$. Here we have that $n = -k$ and $m = -l$ for some $k, l \in \mathbb{Z}_+$. Hence we have

$$\begin{aligned}
a^n a^m &= a^{-k} a^{-l} \\
&= \left(\frac{1}{a^k}\right)\left(\frac{1}{a^l}\right) && \text{(by the definition of negative exponents)} \\
&= \left(\frac{1}{a}\right)^k \left(\frac{1}{a}\right)^l && \text{(by Lemma 4.7.2)} \\
&= \left(\frac{1}{a}\right)^{k+l} && \text{(by Exercise 4.6 since } k, l \in \mathbb{Z}_+\text{)}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{a^{k+l}} && \text{(by Lemma 4.7.2)} \\
&= a^{-(k+l)} && \text{(by the definition of negative exponents)} \\
&= a^{-k-l} \\
&= a^{n+m}.
\end{aligned}
$$

Thus in all cases we have shown the result. □

Next we show that $(a^n)^m = a^{nm}$ for all real $a \neq 0$ and $n, m \in \mathbb{Z}$.

Proof. Consider any real $a \neq 0$ and $n, m \in \mathbb{Z}$. We again number the cases for reference:

1. Case: $n \in \mathbb{Z}_+$.
   (a) Case: $m \in \mathbb{Z}_+$. Then the result immediately applies by Exercise 4.6.
   (b) Case: $m = 0$. Then we have $(a^n)^m = (a^n)^0 = 1 = a^0 = a^{n \cdot 0} = a^{nm}$ by the definition of a 0 exponent.
   (c) Case: $m \in \mathbb{Z}_-$. Then there is a $k \in \mathbb{Z}_+$ such that $m = -k$. Then we have

$$
\begin{aligned}
(a^n)^m &= (a^n)^{-k} \\
&= \frac{1}{(a^n)^k} && \text{(by the definition of negative exponents)} \\
&= \frac{1}{a^{nk}} && \text{(by Exercise 4.6 since } n, k \in \mathbb{Z}_+) \\
&= a^{-(nk)} && \text{(by the definition of negative exponents)} \\
&= a^{n(-k)} \\
&= a^{nm}.
\end{aligned}
$$

2. Case: $n = 0$. Then we have $(a^n)^m = (a^0)^m = 1^m = 1 = a^0 = a^{0 \cdot m} = a^{nm}$ by the definition of 0 as an exponent and Lemma 4.7.1.
3. Case: $n \in \mathbb{Z}_-$. Then $n = -k$ for some $k \in \mathbb{Z}_+$.
   (a) Case: $m \in \mathbb{Z}_+$. Then we have

$$
\begin{aligned}
(a^n)^m &= (a^{-k})^m \\
&= \left(\frac{1}{a^k}\right)^m && \text{(by the definition of negative exponents)} \\
&= \left[\left(\frac{1}{a}\right)^k\right]^m && \text{(by Lemma 4.7.2)} \\
&= \left(\frac{1}{a}\right)^{km} && \text{(by Exercise 4.6 since } k, m \in \mathbb{Z}_+) \\
&= \frac{1}{a^{km}} && \text{(by Lemma 4.7.2)} \\
&= a^{-(km)} && \text{(by the definition of negative exponents)} \\
&= a^{(-k)m} \\
&= a^{nm}.
\end{aligned}
$$

(b) Case: $m = 0$. The same argument as in case 1b above applies here as it does not depend on n being positive.

(c) Case: $m \in \mathbb{Z}_-$. Then $m = -l$ for some $l \in \mathbb{Z}_+$, and we have

$$\begin{aligned}
(a^n)^m &= (a^{-k})^{-l} \\
&= \left(\frac{1}{a^k}\right)^{-l} && \text{(by the definition of negative exponents)} \\
&= \frac{1}{(1/a^k)^l} && \text{(by the definition of negative exponents)} \\
&= \frac{1}{[(1/a)^k]^l} && \text{(by Lemma 4.7.2)} \\
&= \frac{1}{(1/a)^{kl}} && \text{(by Exercise 4.6 since } k, l \in \mathbb{Z}_+) \\
&= \left(\frac{1}{1/a}\right)^{kl} && \text{(by Lemma 4.7.2)} \\
&= a^{kl} \\
&= a^{(-k)(-l)} \\
&= a^{nm} .
\end{aligned}$$

Thus in all cases we have shown the result. □

Lastly, we show that $a^m b^m = (ab)^m$ for all real $a, b \neq 0$ and $m \in \mathbb{Z}$.

Proof. We have the following cases:

Case: $m \in \mathbb{Z}_+$. The result then follows immediately from Exercise 4.6.

Case: $m = 0$. Then we have $a^m b^m = a^0 b^0 = 1 \cdot 1 = 1 = (ab)^0 = (ab)^m$ by the definition of a 0 exponent.

Case: $m \in \mathbb{Z}_-$. Then there is a $k \in \mathbb{Z}_+$ such that $m = -k$. Then we have

$$\begin{aligned}
a^m b^m &= a^{-k} b^{-k} \\
&= \frac{1}{a^k} \cdot \frac{1}{b^k} && \text{(by the definition of negative exponents)} \\
&= \left(\frac{1}{a}\right)^k \left(\frac{1}{b}\right)^k && \text{(by Lemma 4.7.2)} \\
&= \left(\frac{1}{a} \cdot \frac{1}{b}\right)^k && \text{(by Exercise 4.6 since } k \in \mathbb{Z}_+) \\
&= \left(\frac{1}{ab}\right)^k && \text{(by Exercise 4.1 part (m))} \\
&= \frac{1}{(ab)^k} && \text{(by Lemma 4.7.2)} \\
&= (ab)^{-k} && \text{(by the definition of negative exponents)} \\
&= (ab)^m .
\end{aligned}$$

Therefore in all cases the result has been shown. □

Exercise 4.8

(a) Show that $\mathbb{R}$ has the greatest lower bound property.

(b) Show that $\inf\{1/n \mid n \in \mathbb{Z}_+\} = 0$.

(c) Show that given a with $0 < a < 1$, $\inf\{a^n \mid n \in \mathbb{Z}_+\} = 0$. [Hint: Let $h = (1-a)/a$, and show that $(1+h)^n \geq 1 + nh$.]

Solution:

(a)

Proof. Suppose that A is an arbitrary nonempty set of real number that is bounded below by a. Now let $B = \{-x \mid x \in A\}$ and $b = -a$. First, we claim that b is an upper bound of B. So consider any $y \in B$ so that $y = -x$ for some $x \in A$. Then $a \leq x$ since a a lower bound of A. It then follows from Exercise 4.2 part (d) that $y = -x \leq -a = b$. Since $y \in B$ was arbitrary, this shows that b is an upper bound of B.

Since B is clearly nonempty (since A is), we have that B has a least upper bound $d = \sup B$ since the reals have the least upper bound property. We claim that $c = -d$ is the greatest lower bound of A. So first consider any $x \in A$ so that $y = -x \in B$. Then we have $y \leq d$ since $d = \sup B$. Hence $c = -d \leq -y = x$ again by Exercise 4.2 part (d). Since $x \in A$ was arbitrary, this shows that c is in fact a lower bound of A.

Now suppose that x is any lower bound of A. Then, by the same argument as above for $b = -a$, we have that $y = -x$ is an upper bound of B. It then follows that $d \leq y$ since d is the *least* upper bound of B. Then, again by Exercise 4.2 part (d), we have $x = -(-x) = -y \leq -d = c$, which shows that c is in fact the greatest lower bound since x was arbitrary. This completes the proof. □

(b)

Proof. First, let $A = \{1/n \mid n \in \mathbb{Z}_+\}$ so that we must show that $\inf A = 0$. For any $x \in A$ we have that $x = 1/n$ for some $n \in \mathbb{Z}_+$. Then $n > 0$ so that $x = 1/n > 0$ also by Exercise 4.2 part (i). Hence $0 \leq x$ is true, which shows that 0 is a lower bound of A since x was arbitrary.

Now consider any $x > 0$ so that also $1/x > 0$ by Exercise 4.2 part (i). Then, by the Archimedean ordering property there is an $n \in \mathbb{Z}_+$ such that $n > 1/x > 0$ (since otherwise $1/x$ would be an upper bound of $\mathbb{Z}_+$). It then follows from Exercise 4.2 part (j) that $1/n < 1/(1/x) = x$. Since clearly $1/n \in A$ we have that x is *not* a lower bound of A. Since $x > 0$ was arbitrary, this shows that 0 is the greatest lower bound of A since, by the contrapositive, x being a lower bound of A implies that $x \leq 0$. □

(c)

Proof. Consider any real a where $0 < a < 1$. First we show that the set $\{1/a^n \mid n \in \mathbb{Z}_+\}$ has no upper bound. To this end define $h = (1-a)/a = 1/a - 1$ so that $1 + h = 1 + (1/a - 1) = 1/a$. Clearly we have

$$\begin{aligned}
a &< 1 \\
-a &> -1 \\
1 - a &> 1 - 1 = 0 \\
\frac{1-a}{a} &> \frac{0}{a} = 0 && \text{(since } a > 0\text{)} \\
h &> 0
\end{aligned}$$

so that $1 + h > 1 > 0$ and

$$\begin{aligned}
h &> 0 \\
h^2 &> h \cdot 0 = 0 && \text{(since } h > 0) \\
nh^2 &> n \cdot 0 = 0
\end{aligned}$$

for any $n \in \mathbb{Z}_+$ since $n > 0$.

We show by induction that $(1+h)^n \geq 1 + nh$ for all $n \in \mathbb{Z}_+$. For $n = 1$ we clearly have $(1+h)^n = (1+h)^1 = 1 + h \geq 1 + h = 1 + 1 \cdot h = 1 + nh$. Now, supposing that $(1+h)^n \geq 1 + nh$,we have

$$\begin{aligned}
(1+h)^{n+1} &= (1+h)^n(1+h) \\
&\geq (1+nh)(1+h) && \text{(since } 1 + h > 0) \\
&= 1 + nh + h + nh^2 \\
&\geq 1 + nh + h && \text{(since } nh^2 > 0) \\
&= 1 + (n+1)h\,,
\end{aligned}$$

which completes the induction. So consider any real x. Then, since we know that $\mathbb{Z}_+$ has no upper bound, there is an $n \in \mathbb{Z}_+$ where $n > x/h$ (noting that $h > 0$) so that

$$\begin{aligned}
n &> x/h \\
nh &> (x/h)h = x && \text{(since } h > 0) \\
1 + nh &> 1 + x > x\,.
\end{aligned}$$

Then we have $1/a^n = (1/a)^n = (1+h)^n \geq 1 + nh > x$, which shows that the set $\{1/a^n \mid n \in \mathbb{Z}_+\}$ is unbounded above since x was arbitrary.

Now we show the main result. Let $A = \{a^n \mid n \in \mathbb{Z}_+\}$ so that we must show that $\inf A = 0$. First we show by induction that 0 is a lower bound of A. For $n = 1$ we clearly have $a^n = a^1 = a \geq 0$. Then, if $a^n \geq 0$, we have $a^{n+1} = a^n \cdot a \geq 0 \cdot a = 0$ since $a > 0$. This completes the induction so that clearly 0 is indeed a lower bound of A.

Now consider any real $x > 0$ so that $1/x > 0$ also. Then, by what was shown above, we know that there is an $n \in \mathbb{Z}_+$ such that $1/a^n > 1/x > 0$. We then have $a^n = 1/(1/a^n) < 1/(1/x) = x$ by Exercise 4.2 part (j). This shows that x is *not* a lower bound of A since obviously $a^n \in A$. It then follows that 0 is the *greatest* lower bound of A since $x > 0$ was arbitrary, because, by the contrapositive, x being a lower bound of A implies that $x \leq 0$. Hence $0 = \inf A$ as desired. □

Exercise 4.9

(a) Show that every nonempty subset of $\mathbb{Z}$ that is bounded above has a largest element.

(b) If $x \notin \mathbb{Z}$, show that there is exactly one $n \in \mathbb{Z}$ such that $n < x < n + 1$.

(c) If $x - y > 1$, show there is at least one $n \in \mathbb{Z}$ such that $y < n < x$.

(d) If $y < x$, show there is a rational number z such that $y < z < x$.

Solution:

Lemma 4.9.1. *The set of integers $\mathbb{Z}$ is an inductive set that has no lower or upper bounds in $\mathbb{R}$.*

Proof. First we show that $\mathbb{Z}$ is inductive. Clearly $1 \in \mathbb{Z}$ since $1 \in \mathbb{Z}_+ \subset \mathbb{Z}$. Now suppose that $n \in \mathbb{Z}$ so that clearly $n+1 \in \mathbb{Z}$ by Exercise 4.5 part (d) since $1 \in \mathbb{Z}$.

Next, consider any $x \in \mathbb{R}$. Then we know that $\mathbb{Z}_+$ has no upper bound so that there is an $n \in \mathbb{Z}_+$ such that $n > x$, and clearly $n \in \mathbb{Z}$ since $\mathbb{Z}_+ \subset \mathbb{Z}$. By the same token there is an $m \in \mathbb{Z}_+$ such that $m > -x$. But then we have $-m < -(-x) = x$ by Exercise 4.2 part (d), and $-m \in \mathbb{Z}_-$ so that also $-m \in \mathbb{Z}$ since $\mathbb{Z}_- \subset \mathbb{Z}$. Since x was arbitrary, this shows that $\mathbb{Z}$ is not bounded above or below. □

Lemma 4.9.2. *There is no integer n such that $0 < n < 1$.*

Proof. Suppose to the contrary that $n \in \mathbb{Z}$ and $0 < n < 1$. Let $S = \{k \in \mathbb{Z} \mid 0 < k < 1\}$ so that clearly $n \in S$ so that $S \neq \varnothing$. Also since $0 < k$ and $k \in \mathbb{Z}$ for any $k \in S$, clearly $S \subset \mathbb{Z}_+$. Thus S is a nonempty subset of positive integers so that it has a smallest element m by the well-ordering property. Since $m \in S$ we have $0 < m < 1$ and hence $m^2 = m \cdot m < 1 \cdot m = m < 1$ by property (6) since $m > 0$. By the same property clearly $0 = 0 \cdot m < m \cdot m = m^2$ as well so that $0 < m^2 < 1$. Also, clearly $m^2 = m \cdot m \in \mathbb{Z}$ by Exercise 4.5 part (e) since $m \in \mathbb{Z}$, and so $m^2 \in S$. However, this cannot be since m is the smallest element of S and yet $m^2 < m$. Therefore we have a contradiction, which proves the result. □

Corollary 4.9.3. *For any integer n, there is no integer a such that $n < a < n+1$.*

Proof. Consider any $n \in \mathbb{Z}$ and suppose to the contrary that there is an $a \in \mathbb{Z}$ such that $n < a < n+1$. First, we have $n - a \in \mathbb{Z}$ by Exercise 4.5 part (d) since $a, n \in \mathbb{Z}$. Also, $n < a$ clearly implies that $0 < a - n$. Similarly, $a < n+1$ means that $a - n < 1$. But then we have that $a - n$ is an integer where $0 < a - n < 1$, which contradicts Lemma 4.9.2. Thus it must be the case that there is no such integer a. □

Main Problem.

(a)

Proof. Suppose that A is a nonempty subset of $\mathbb{Z}$ and that it is bounded above by $\alpha \in \mathbb{R}$. Since $A \neq \varnothing$, there is an $a \in A$, so define $A' = \{n - a + 1 \mid n \in A\}$. First we claim that $\alpha' = \alpha - a + 1$ is an upper bound of A' So consider any $n' \in A'$ so that $n' = n - a + 1$ for some $n \in A$. Since α is an upper bound of A we have

$$\begin{aligned} n &\leq \alpha \\ n - a &\leq \alpha - a \\ n - a + 1 &\leq \alpha - a + 1 \\ n' &\leq \alpha' , \end{aligned}$$

which shows that α' is an upper bound of A' since n' was an arbitrary element. We also have that there is an $N' \in \mathbb{Z}_+$ such that $\alpha' < N'$ since $\mathbb{Z}_+$ has no upper bound.

Now let $B' = A' \cap \mathbb{Z}_+$. Then, for any $n' \in B'$, we have that $n' \in A'$ so that $n' \leq \alpha' < N'$. Since also clearly $n' \in \mathbb{Z}_+$, we have that $n' \in S_{N'} = \{k \in \mathbb{Z}_+ \mid k < N'\} = \{1, \ldots, N' - 1\}$. Hence $B' \subset S_{N'}$ since n' was arbitrary. We also have that $1 \in A'$ since $a \in A$ and $a - a + 1 = 1$. Hence $1 \in B'$ since clearly also $1 \in \mathbb{Z}_+$ since it is inductive. Thus B' is a nonempty subset of $S_{N'}$ so that it has a largest element b' by Exercise 4.4 part (a).

Since $b' \in B'$, we have that $b' \in A'$ so that there is a $b \in A$ such that $b' = b - a + 1$. We claim that b is the largest element of A. We already know that $b \in A$ so we need only show that it is also an upper bound of A. So consider any $n \in A$ so that clearly $n' = n - a + 1 \in A'$. Now, it follows from Exercise 4.5 part (d) that $n' \in \mathbb{Z}$ since $n, a, 1 \in \mathbb{Z}$. Thus we have the following:

Case: $n' \in \mathbb{Z}_+$. Then, clearly $n' \in A' \cap \mathbb{Z}_+ = B'$ so that $n' \leq b'$ since b' is the largest element of B'.

Case: $n' \in \mathbb{Z}_- \cup \{0\}$. Then $n' < 1 \leq b'$ since $1 \in B'$ and b' is the largest element of B'.

Thus in either case $n' \leq b'$ is true so that

$$\begin{aligned} n' &\leq b' \\ n - a + 1 &\leq b - a + 1 \\ n - a &\leq b - a \\ n &\leq b\,, \end{aligned}$$

which shows that b is an upper bound and thus the largest element of A since n was arbitrary. □

(b)

Proof. Suppose an $x \in \mathbb{R}$ where $x \notin \mathbb{Z}$ and let $A = \{n \in \mathbb{Z} \mid n < x\}$. It follows from Lemma 4.9.1 that there is an $m \in \mathbb{Z}$ where $m < x$ since $\mathbb{Z}$ has no lower bounds. Hence by definition $m \in A$ so that $A \neq \varnothing$. Clearly also x is an upper bound of A so that A is a nonempty subset of $\mathbb{Z}$ that is bounded above. It then follows from part (a) that A has a largest element n, where clearly $n < x$ since $n \in A$.

Now, suppose for the moment that $n + 1 \leq x$. Then, since $\mathbb{Z}$ is inductive (again by Lemma 4.9.1) and $n \in \mathbb{Z}$, we have that $n + 1 \in \mathbb{Z}$ as well. But $x \notin \mathbb{Z}$ so that it must be that $n + 1 \neq x$, and hence $n + 1 < x$. Then $n + 1 \in A$ so that $n + 1 \leq n$ since n is the largest element of A. However, this contradicts the obvious fact that $n + 1 > n$ so that it must be that $n + 1 \leq x$ is not true. Hence $n + 1 > x$ and thus we have shown that $n < x < n + 1$.

Lastly, suppose that there is an integer m such that $m < x < m + 1$. Then $m \in A$ so that $m \leq n$ since n is the largest element of A. Suppose for a moment that $m < n$. Then we would have $m < n < x < m + 1$ so that n is an integer between m and $m + 1$, which violates Corollary 4.9.3. Thus is has to be that $m = n$ (since $m \leq n$), which shows that n is the unique integer such that $n < x < n + 1$. □

(c)

Proof. Suppose that $x, y \in \mathbb{R}$ and $x - y > 1$. If $x \in \mathbb{Z}$ then let $n = x - 1$ so that clearly $n \in \mathbb{Z}$ by Exercise 4.5 part (d). First, we have

$$\begin{aligned} x - y &> 1 \\ x &> 1 + y \\ x - 1 &> y \\ n &> y\,. \end{aligned}$$

We also clearly have $n = x - 1 < x$ so that $y < n < x$.

On the other hand, if $x \notin \mathbb{Z}$, then we know from part (b) that there is a unique integer n such that $n < x < n + 1$. We also have that

$$\begin{aligned} x &< n + 1 \\ 1 < x - y &< n + 1 - y \\ 0 &< n - y \\ y &< n \end{aligned}$$

so that again $y < n < x$.

Hence in both cases we have found an integer n such that $y < n < x$, which proves the result. □

(d)

Proof. Suppose that $x, y \in \mathbb{R}$ where $y < x$. Then $0 < x - y$ so that $1/(x-y)$ exists.. Since $\mathbb{Z}_+$ is unbounded above there is a $b \in \mathbb{Z}_+$ where $b > 1/(x-y)$. Hence

$$\begin{aligned} b &> \frac{1}{x-y} \\ b(x-y) &> 1 && \text{(since } x - y > 0\text{)} \\ bx - by &> 1 \,. \end{aligned}$$

It then follows from part (c) that there is an integer a such that $by < a < bx$. We then have that $y < a/b < x$ since $b > 0$ (since $b \in \mathbb{Z}_+$). This shows the result since clearly a/b is rational because $a, b \in \mathbb{Z}$. □

Exercise 4.10

Show that every positive number a has exactly one positive square root, as follows:

(a) Show that if $x > 0$ and $0 \le h < 1$, then

$$\begin{aligned} (x+h)^2 &\le x^2 + h(2x+1)\,, \\ (x-h)^2 &\ge x^2 - h(2x)\,. \end{aligned}$$

(b) Let $x > 0$. Show that if $x^2 < a$, then $(x+h)^2 < a$ for some $h > 0$; and if $x^2 > a$, then $(x-h)^2 > a$ for some $h > 0$.

(c) Given $a > 0$, let B be the set of all real numbers x such that $x^2 < a$. Show that B is bounded above and contains at least one positive number. Let $b = \sup B$; show that $b^2 = a$.

(d) Show that if b and c are positive and $b^2 = c^2$, then $b = c$.

Solution:

Lemma 4.10.1. *If $x \in \mathbb{R}$ and $x^2 < 1$, then $x < 1$ also.*

Proof. Suppose that $x \ge 1$. If $x = 1$ then clearly $x^2 = 1^1 = 1$. On the other hand, if $x > 1$ then clearly $x^2 = x \cdot x > 1 \cdot x = x > 1$ by property (6) since $x > 1 > 0$. Thus in either case $x^2 \ge 1$ so that we have shown that $x \ge 1$ implies that $x^2 \ge 1$. It then follows that $x^2 < 1$ implies $x < 1$ by the contrapositive. □

Lemma 4.10.2. *If $0 < y < x$ then $0 < y^2 < x^2$.*

Proof. Supposing that $0 < y < x$, we have $0 = 0 \cdot y < y \cdot y = y^2 = y \cdot y < x \cdot y = y \cdot x < x \cdot x = x^2$ all by property (6) since both x and y are positive. □

Main Problem.

(a)

Proof. First, we know that $0 \le h < 1$. If $h = 0$ then clearly $h = 0 = 0^2 = h^2$ so that $0 \le h^2 \le h$ is true. If $h \ne 0$ then $0 < h < 1$ so that $0 = 0 \cdot h < h \cdot h = h^2 < 1 \cdot h = h$ by property (6) since $h > 0$ so that again $0 \le h^2 \le h$ is true.

We then have

$$\begin{aligned}
(x+h)^2 &= (x+h)(x+h) \\
&= x^2 + 2xh + h^2 \\
&\leq x^2 + 2xh + h && \text{(since } h^2 \leq h\text{)} \\
&= x^2 + h(2x+1)\,.
\end{aligned}$$

Also

$$\begin{aligned}
(x-h)^2 &= (x-h)(x-h) \\
&= x^2 - 2xh + h^2 \\
&\geq x^2 - 2xh + 0 && \text{(since } h^2 \geq 0\text{)} \\
&= x^2 - h(2x)\,,
\end{aligned}$$

which show the desired results. □

(b) We modify this result so that the h in the second part is not just positive but also $h < x$. In fact, without this stipulation, the theorem becomes obvious since any arbitrarily large h will suffice. Because then then $x - h$ is arbitrarily large in magnitude (but negative) so that $(x-h)^2$ can be made arbitrarily large so that of course $(x-h)^2 > a$. Adding the stipulation that $0 < h < x$ makes the theorem more useful and is necessary for it to be of use in part (c) below.

Proof. Suppose that $x > 0$. Then clearly $2x > 2 \cdot 0 = 0$ as well. Also it then follows that $2x + 1 > 1 > 0$.

If $x^2 < a$ then clearly $0 < a - x^2$. Hence we have that $0 < (a - x^2)/(2x+1)$ by Exercise 4.2 parts (i) and (h) since both $a - x^2$ and $2x + 1$ are positive. So let $y = \min(1, (a - x^2)/(2x+1))$ so that clearly both $y \leq 1$ and $y \leq (a - x^2)/(2x+1)$. Since $0 < 1$ and $0 < (a - x^2)/(2x+1)$, we have that $0 < y$ so that it follows from Exercise 4.9 part (d) that there is a rational h such that $0 < h < y$. Hence $0 < h < y \leq 1$ so that, by part (a), we have

$$\begin{aligned}
(x+h)^2 &\leq x^2 + h(2x+1) \\
&< x^2 + \left(\frac{a - x^2}{2x+1}\right)(2x+1) && \text{(since } h < y \leq (a-x^2)/(2x+1) \text{ and } 2x+1 > 0\text{)} \\
&= x^2 + (a - x^2) \\
&= a\,.
\end{aligned}$$

If $x^2 > a$ then clearly $x^2 - a > 0$. Then we have again that $(x^2 - a)/(2x)$ is positive since we showed previously that $2x$ is. So let $y = \min(1, (x^2 - a)/(2x), x)$ so that clearly $y \leq 1$, $y \leq (x^2 - a)/(2x)$, and $y \leq x$. Since both 1, $(x^2 - a)/(2x)$, and x are all positive it follows that $0 < y$ so that there is a rational h such that $0 < h < y$ by Exercise 4.9 part (d). Therefore $0 > -h > -y$. Since $0 < h < y \leq 1$ we have by part (a) that

$$\begin{aligned}
(x-h)^2 &\geq x^2 - h(2x) \\
&> x^2 - \left(\frac{x^2 - a}{2x}\right)(2x) && \text{(since } -h > -y \geq -(x^2-a)/(2x) \text{ and } 2x > 0\text{)} \\
&= x^2 - (x^2 - a) \\
&= a\,,
\end{aligned}$$

which show the desired results since clearly $0 < h < y \leq x$. □

(c)

Proof. Suppose that $a > 0$ and let $B = \{x \in \mathbb{R} \mid x^2 < a\}$.

If $a < 1$ then $0 < a < 1$ so that $a^2 = a \cdot a < 1 \cdot a = a$ so that a itself is in B (and of course a is positive). Now consider any $x \in B$ so that $x^2 < a$. Then $x^2 < a < 1$ so that also $x < 1$ by Lemma 4.10.1. Since $x \in B$ was arbitrary, this shows that 1 is an upper bound of B.

If $a \geq 1$ then $(1/2)^2 = 1/2^2 = 1/4 < 1 \leq a$ so that $1/2 \in B$ (and of course $1/2$ is positive). Now consider any $x \in B$ so that $x^2 < a$. If $x \leq 1$ then $x \leq 1 \leq a$. On the other hand, if $x > 1$ then $x^2 = x \cdot x > 1 \cdot x = x$ since $x > 1 > 0$ so that $x < x^2 < a$. Thus in both cases $x \leq a$ so that a is an upper bound of B since x was arbitrary.

Therefore in each case B contains a positive element (so that $b \neq \varnothing$) and B is bounded above. It then follows that B has a least upper bound b (so that $b = \sup B$). Clearly since B has a positive element x, it follows that $0 < x \leq b$ so that b is positive.

Now suppose that $b^2 < a$. Then by definition $b \in B$ so that b has to be the largest element of b since it is the least upper bound. Since we know that b is positive and $b^2 < a$, it follows from part (b) that there is an $h > 0$ where $(b+h)^2 < a$ and hence $b + h \in B$. However, since $h > 0$, it follows that $b < b + h$, which contradicts the fact that b is the greatest element of B. Hence it cannot be that $b^2 < a$.

So suppose that $b^2 > a$. Then again by part (b) there is an h where $0 < h < b$ such that $(b-h)^2 > a$. Now, since $h > 0$, it follows that $b - h < b$ so that $n - h$ is not an upper bound of B (since then b would not be the least upper bound). Hence there is an $x \in B$ such that $b - h < x$, noting that $x^2 < a$ by the definition of B. Since $h < b$, we have that $0 < b - h < x$ so that $(b - h)^2 < x^2 < a$ by Lemma 4.10.2. But this contradicts the established fact that $(b - h)^2 > a$ so that it cannot be that $b^2 > a$.

Thus the only possibility remaining is that $b^2 = a$ as desired. □

(d)

Proof. Suppose that b and c are positive and that $b^2 = c^2$. If it were the case that $b < c$ then $0 < b < c$ so that $0 < b^2 < c^2$ by Lemma 4.10.2 so that clearly $b^2 \neq c^2$. As this is a contradiction, it has to be that $b \geq c$. An analogous argument shows that $b > c$ also leads to a contradiction so that $b \leq c$. Hence it must be that $b = c$ as desired. □

Exercise 4.11

Given $m \in \mathbb{Z}$, we say that m is ***even*** if $m/2 \in \mathbb{Z}$, and m is ***odd*** otherwise.

(a) Show that if m is odd, $m = 2n + 1$ for some $n \in \mathbb{Z}$. [Hint: Choose n so that $n < m/2 < n + 1$.]

(b) Show that if p and q are odd, so are $p \cdot q$ and p^n, for any $n \in \mathbb{Z}_+$.

(c) Show that if $a > 0$ is rational, then $a = m/n$ for some $m, n \in \mathbb{Z}_+$ where not both n and m are even. [Hint: Let n be the smallest element of the set $\{x \mid x \in \mathbb{Z}_+ \text{ and } x \cdot a \in \mathbb{Z}_+\}$.]

(d) *Theorem:* $\sqrt{2}$ *is irrational.*

Solution:

Lemma 4.11.1. *If* $n, m \in \mathbb{Z}$ *and* $n < m$, *then* $n + 1 \leq m$ *and* $n \leq m - 1$.

Proof. Suppose that $n+1>m$ so that $n<m<n+1$, which violates Corollary 4.9.3 since $m \in \mathbb{Z}$. Thus it has to be that $n+1 \leq m$. From this it immediately follows that $n = n+1-1 \leq m-1$ by simply subtracting 1 from both sides of the previous inequality. □

Lemma 4.11.2. *An integer m is even if and only if $m = 2n$ for some integer n.*

Proof. (⇒) Supposing that m is even, then $n = m/2 \in \mathbb{Z}$. Then clearly $m = 2n$.

(⇐) Now suppose that $m = 2n$ for some integer n. Then clearly $m/2 = n$ is an integer so that m is even by definition. □

Lemma 4.11.3. *An integer a is odd if and only if a^2 is also odd.*

Proof. (⇒) Suppose that a is odd so that $a = 2n+1$ for some integer n (this is shown in part (a) below, which does not depend on this lemma). Then

$$a^2 = a \cdot a = (2n+1)(2n+1) = 4n^2 + 2n + 2n + 1 = 4n^2 + 4n + 1 = 2\left[2(n^2+n)\right] + 1$$

noting that clearly $2(n^2+n)$ is an integer since n is. Hence a^2 is odd again by what will be shown in part (a).

(⇐) We prove this by contrapositive, so suppose that a is not odd so that it must be even. Therefore $a = 2n$ for some integer n by Lemma 4.11.2. Then $a^2 = a \cdot a = (2n)(2n) = 4n^2 = 2(2n^2)$ so that a^2 is even since clearly $2n^2$ is an integer since n is. Thus a^2 is not odd. □

Main Problem.

(a) Here we show the converse as well, i.e. we show that m is odd if and only if $m = 2n+1$ for some $n \in \mathbb{Z}$.

Proof. (⇒) Suppose that m is odd so that by definition $m/2 \notin \mathbb{Z}$. It then follows from Exercise 4.9 part (b) that there is a unique integer n such that $n < m/2 < n+1$. We then have that $2n < m < 2(n+1) = 2n+2$ since obviously $2 > 0$. Hence by Lemma 4.11.1 we have that $2n+1 \leq m$ and also $m \leq 2n+2-1 = 2n+1$. Therefore it has to be that $m = 2n+1$ as desired.

(⇐) Now suppose that there is an $n \in \mathbb{Z}$ such that $m = 2n+1$. Then we have that

$$\frac{m}{2} = \frac{2n+1}{2} = n + \frac{1}{2}\,.$$

We then clearly have that $n = n+0 < n+1/2 < n+1$ since $0 < 1/2 < 1$ so that $m/2 = n+1/2$ cannot be an integer by Corollary 4.9.3. Hence m is odd by definition. □

(b)

Proof. Suppose that p and q are odd so that $p = 2k+1$ and $q = 2m+1$ for some $k, m \in \mathbb{Z}$ by part (a). We then have that

$$p \cdot q = (2k+1)(2m+1) = 4km + 2m + 2k + 1 = 2(2km + m + k) + 1$$

so that $p \cdot q$ is odd by what was shown in part (a) since clearly $2km + m + k \in \mathbb{Z}$ by Exercise 4.5 since k and m are integers.

Now we show by induction on n that p^n is odd for any $n \in \mathbb{Z}_+$. First, for $n = 1$ we clearly have $p^n = p^1 = p$ is odd by supposition. Then, if we assume that p^n is odd, we have that the product $p^{n+1} = p^n \cdot p$ is odd as well by what was just shown since both p^n and p are odd. This completes the induction. □

(c)

Proof. Suppose that $a > 0$ is rational. Then $a = p/q$ for some integers p and q. Clearly it cannot be that $q = 0$, and if $q < 0$ then $q = -b$ for some $b \in \mathbb{Z}_+$. Then we have $a = p/q = p/(-b) = (-p)/b$ so that $ab = -p$. Furthermore, since a and b are both positive, we have that $ab = -p$ is positive by Exercise 4.2 part (h). Thus clearly $-p \in \mathbb{Z}_+$ since $p \in \mathbb{Z}$.

Now, let $X = \{x \in \mathbb{Z}_+ \mid ax \in \mathbb{Z}_+\}$. Since we just showed that $b \in \mathbb{Z}_+$ and $ab = -p \in \mathbb{Z}_+$ it follows that $b \in X$. Since clearly $X \subset \mathbb{Z}_+$ and X is nonempty (since $b \in X$), it has a smallest element n by the well-ordering property. Letting $m = an$, we clearly have that $m \in \mathbb{Z}_+$ since $n \in X$. Then, we have $a = m/n$, noting again that $m, n \in \mathbb{Z}_+$.

To show that not both m and n are even, suppose to the contrary that they *are* both even. Then by Lemma 4.11.2 we have that $m = 2k$ and $n = 2l$ for some $k, l \in \mathbb{Z}$. Clearly then $k = m/2$ and $l = n/2$ so that both k and l are positive by Exercise 4.2 part (h) since m and n (and $1/2$) are. Hence $k, l \in \mathbb{Z}_+$. We have $a = m/n = 2k/2l = k/l$ so that $al = k$, which implies that $l \in X$ since l and $al = k$ are both in $\mathbb{Z}_+$. However, we also have that $l = n/2 < n$ since $n > 0$, which contradicts the fact that n is the smallest element of X. Thus it has to be the case that not both m and n are even. □

(d) This is one of the most famous proofs in all of mathematics, and is often used as an example of mathematical proofs since it can be understood by most laymen.

Proof. Obviously we take $\sqrt{2}$ to be the unique positive real number such that $(\sqrt{2})^2 = 2$ as was shown to exist in Exercise 4.10. Suppose to the contrary that $\sqrt{2}$ is rational so that $\sqrt{2} = a/b$ for $a, b \in \mathbb{Z}_+$ where not both a and b are even by part (c) since $\sqrt{2} > 0$. We therefore have that $2 = (\sqrt{2})^2 = (a/b)^2 = a^2/b^2$ so that $2b^2 = a^2$. Since b^2 is an integer (clearly, since b is and $b^2 = b{\cdot}b$) it follows from Lemma 4.11.2 that a^2 is even. This means that a itself is even by Lemma 4.11.3. Hence $a = 2n$ for some integer n so that $a^2 = (2n)^2 = 4n^2$. From before, we then have $2b^2 = a^2 = 4n^2$ so that clearly $b^2 = 2n^2$, from which it follows as before that b^2 and therefore b itself is even by Lemmas 4.11.2 and 4.11.3. However, this is a contradiction since we previously established that a and b cannot both be even! So it has to be that $\sqrt{2}$ is not rational and is therefore irrational as desired. □

§5 Cartesian Products

Exercise 5.1

Show that there is a bijective correspondence of $A \times B$ with $B \times A$.

Solution:

Proof. We define a function $f : A \times B \to B \times A$. For any element $(a, b) \in A \times B$ we set $f(a, b) = (b, a)$, noting that of course $a \in A$ and $b \in B$. It should be obvious then that $f(a, b) = (b, a) \in B \times A$ so that $B \times A$ can be the range of f.

First we show that f is injective. To this end consider (a_1, b_1) and (a_2, b_2) in $A \times B$ where $(a_1, b_1) \neq (a_2, b_2)$. Of course we have that $f(a_1, b_1) = (b_1, a_1)$ and $f(a_2, b_2) = (b_2, a_2)$. Since $(a_1, b_1) \neq (a_2, b_2)$ clearly either $a_1 \neq a_2$ or $b_1 \neq b_2$. In either case it should be clear that $f(a_1, b_1) = (b_1, a_1) \neq (b_2, a_2) = f(a_2, b_2)$, which shows that f is injective since (a_1, b_1) and (a_2, b_2) were arbitrary.

It is very easy to that f is also surjective since, for any $(b, a) \in B \times A$, clearly $(a, b) \in A \times B$ and $f(a, b) = (b, a)$. Hence f is a bijection as desired. Note that if $A \times B = \varnothing$ then $f = \varnothing$ as well, which is vacuously a bijective function since it must be that $B \times A = \varnothing$ as well (because either $A = \varnothing$ or $B = \varnothing$). ☐

Exercise 5.2

(a) Show that if $n > 1$ there is a bijective correspondence of

$$A_1 \times \cdots \times A_n \qquad \text{with} \qquad (A_1 \times \cdots \times A_{n-1}) \times A_n \,.$$

(b) Given the indexed family $\{A_1, A_2, \ldots\}$, let $B_i = A_{2i-1} \times A_{2i}$ for each positive integer i. Show that there is a bijective correspondence of $A_1 \times A_2 \times \cdots$ with $B_1 \times B_2 \times \cdots$

Solution:

Lemma 5.2.1. *If $n \in \mathbb{Z}_+$ is even, then $n/2 \in \mathbb{Z}_+$. If $n \in \mathbb{Z}_+$ is odd, then $(n+1)/2 \in \mathbb{Z}_+$.*

Proof. First, suppose that $n \in \mathbb{Z}_+$ is even. Then by definition $n/2$ is an integer. However, since both n and $1/2$ are positive, it follows from Exercise 4.2 part (h) that $n \cdot (1/2) = n/2$ is positive also so that $n/2 \in \mathbb{Z}_+$.

Now, suppose that $n \in \mathbb{Z}_+$ is odd so that $n = 2k + 1$ for some integer k by Exercise 4.11a. Then

$$\frac{n+1}{2} = \frac{(2k+1)+1}{2} = \frac{2k+2}{2} = \frac{2(k+1)}{2} = k+1\,,$$

which is clearly an integer since k is. Moreover, we have $n + 1 > n > 0$ since $n \in \mathbb{Z}_+$ and again $1/2 > 0$ so that $(n+1) \cdot (1/2) = (n+1)/2$ is positive by Exercise 4.2 part (h). Thus $(n+1)/2 \in \mathbb{Z}_+$. ☐

Main Problem.

(a)

Proof. For brevity, let $X = A_1 \times \cdots \times A_n$ and $Y = (A_1 \times \cdots \times A_{n-1}) \times A_n$. Suppose that $n > 1$ so that X and Y make sense. We construct a bijective function $f : X \to Y$. For any $\mathbf{x} = (x_1, \ldots, x_n) \in X$ we have that $x_i \in A_i$ for $1 \le i \le n$. So set $f(x) = ((x_1, \ldots, x_{n-1}), x_n)$, which is clearly an element of Y.

To see that f is injective consider $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ in X where $\mathbf{x} \ne \mathbf{y}$. It then follows that there must be an $i \in \{1, \ldots, n\}$ where $x_i \ne y_i$. Let $\mathbf{x}' = (x_1, \ldots, x_{n-1})$ and $\mathbf{y}' = (y_1, \ldots, y_{n-1})$ so that clearly $f(\mathbf{x}) = (\mathbf{x}', x_n)$ and $f(\mathbf{y}) = (\mathbf{y}', y_n)$. Now, if $i = n$, then clearly $f(\mathbf{x}) = (\mathbf{x}', x_n) \ne (\mathbf{y}', y_n) = f(\mathbf{y})$ since $x_n = x_i \ne y_i = y_n$. On the other hand, if $i \ne n$ then it has to be that $i < n$, and hence $i \le n - 1$. It then follows that $\mathbf{x}' = (x_1, \ldots, x_{n-1}) \ne (y_1, \ldots, y_{n-1}) = \mathbf{y}'$ so that then $f(\mathbf{x}) = (\mathbf{x}', x_n) \ne (\mathbf{y}', y_n) = f(\mathbf{y})$ again. Since $\mathbf{x}$ and $\mathbf{y}$ were arbitrary, this shows that f is indeed injective.

Now consider any $\mathbf{y} = ((x_1, \ldots, x_{n-1}), x_n) \in Y$ and let $\mathbf{x} = (x_1, \ldots, x_n)$. It should be obvious that both $\mathbf{x} \in X$ and $f(\mathbf{x}) = \mathbf{y}$ so that f is surjective. Hence f is a bijective function as desired. ☐

(b)

Proof. First let $A = A_1 \times A_2 \times \cdots$ and $B = B_1 \times B_2 \times \cdots$. We construct a bijective $f : A \to B$. So, for any $\mathbf{a} \in A$, we have that $\mathbf{a} = (a_1, a_2, \ldots)$, where $a_i \in A_i$ for any $i \in \mathbb{Z}_+$. Then, for any $i \in \mathbb{Z}_+$, define $b_i = (a_{2i-1}, a_{2i})$ so that clearly $b_i \in A_{2i-1} \times A_{2i} = B_i$. We then have that $\mathbf{b} = (b_1, b_2, \ldots) \in B_1 \times B_2 \times \cdots = B$. So set $f(\mathbf{a}) = \mathbf{b}$ so that f is a function from A to B.

To show that f is injective, consider $\mathbf{a} = (a_1, a_2, \ldots)$ and $\mathbf{a}' = (a'_1, a'_2, \ldots)$ in A where $\mathbf{a} \neq \mathbf{a}'$. For each $i \in \mathbb{Z}_+$, define $b_i = (a_{2i-1}, a_{2i})$ and $b'_i = (a'_{2i-1}, a'_{2i})$ as above and set $\mathbf{b} = (b_1, b_2, \ldots)$ and $\mathbf{b}' = (b'_1, b'_2, \ldots)$ so that clearly $f(\mathbf{a}) = \mathbf{b}$ and $f(\mathbf{a}') = \mathbf{b}'$. Since $\mathbf{a} \neq \mathbf{a}'$, it follows that there must be an $i \in \mathbb{Z}_+$ where $a_i \neq a'_i$.

Case: i is even. Then let $j = i/2$ so that $j \in \mathbb{Z}_+$ by Lemma 5.2.1. We also clearly have that $i = 2j$ so that $b_j = (a_{2j-1}, a_{2j}) \neq (a'_{2j-1}, a'_{2j}) = b'_j$ since $a_{2j} = a_i \neq a'_i = a'_{2j}$.

Case: i is odd. Then let $j = (i+1)/2$ so that $j \in \mathbb{Z}_+$ by Lemma 5.2.1. We then clearly have that $i = 2j - 1$ so that $b_j = (a_{2j-1}, a_{2j}) \neq (a'_{2j-1}, a'_{2j}) = b'_j$ since $a_{2j-1} = a_i \neq a'_i = a'_{2j-1}$.

Hence in all cases we have that there is a $j \in \mathbb{Z}_+$ where $b_j \neq b'_j$. It then follows that $f(\mathbf{a}) = \mathbf{b} = (b_1, b_2, \ldots) \neq (b'_1, b'_2, \ldots) = \mathbf{b}' = f(\mathbf{a}')$ so that f is injective since $\mathbf{a}$ and $\mathbf{a}'$ were arbitrary.

Lastly, to show that f is surjective, consider any $\mathbf{b} \in B$ so that $\mathbf{b} = (b_1, b_2, \ldots)$ where $b_i \in B_i = A_{2i-1} \times A_{2i}$ for every $i \in \mathbb{Z}_+$. Then, for any $i \in \mathbb{Z}_+$, $b_i = (a'_i, a''_i)$ where $a'_i \in A_{2i-1}$ and $a''_i \in A_{2i}$. So consider any $j \in \mathbb{Z}_+$. If j is even, then $i = j/2 \in \mathbb{Z}_+$ by Lemma 5.2.1. Clearly also $j = 2i$. So, define $a_j = a''_i$ so that $a_j = a_{2i} = a''_i \in A_{2i} = A_j$. On the other hand, if j is odd, then $i = (j+1)/2 \in \mathbb{Z}_+$ again by Lemma 5.2.1. Then clearly $j = 2i - 1$. So, here let $a_j = a'_i$ so that $a_j = a_{2i-1} = a'_i \in A_{2i-1} = A_j$. Hence $a_j \in A_j$ for all $j \in \mathbb{Z}_+$ so that $\mathbf{a} = (a_1, a_2, \ldots) \in A$. Then, for any $i \in \mathbb{Z}_+$, we have $b_i = (a'_i, a''_i) = (a_{2i-1}, a_{2i}) \in A_{2i-1} \times A_{2i} = B_i$ so that by definition $f(\mathbf{a}) = \mathbf{b} = (b_1, b_2, \ldots)$. This shows that f is surjective since $\mathbf{b}$ was arbitrary.

This completes the proof that f is bijective so that the desired result follows. □

Exercise 5.3

Let $A = A_1 \times A_2 \times \cdots$ and $B = B_1 \times B_2 \times \cdots$.

(a) Show that if $B_i \subset A_i$ for all i, then $B \subset A$. (Strictly speaking, if we are given a function mapping the index set $\mathbb{Z}_+$ into the union of the sets B_i, we must change its range before it can be considered as a function mapping $\mathbb{Z}_+$ into the union of the sets A_i. We shall ignore this technicality when dealing with cartesian products)

(b) Show the converse of (a) holds if B is nonempty.

(c) Show that if A is nonempty, each A_i is nonempty. Does the converse hold? (We will return to this question in the exercises of §19.)

(d) What is the relation between the set $A \cup B$ and the cartesian product of the sets $A_i \cup B_i$? What is the relation between the set $A \cap B$ and the cartesian product of the sets $A_i \cap B_i$?

Solution:

(a)

Proof. Suppose that $\mathbf{b} \in B$ so that $\mathbf{b} = (b_1, b_2, \ldots)$ where $b_i \in B_i$ for every $i \in \mathbb{Z}_+$. Consider any such $i \in \mathbb{Z}_+$ so that $b_i \in B_i$. Then also $b_i \in A_i$ since $B_i \subset A_i$. Since i was arbitrary, $b_i \in A_i$ for every $i \in \mathbb{Z}_+$ so that $\mathbf{b} = (b_1, b_2, \ldots) \in A_1 \times A_2 \times \cdots = A$. Since $\mathbf{b}$ was arbitrary, this shows that $B \subset A$. Note that we ignore the function range technicality issue mentioned above. □

(b)

Proof. Suppose that $B \subset A$. Since $B \neq \varnothing$, there is a $\mathbf{b}' \in B$ so that $\mathbf{b}' = (b_1', b_2', \ldots)$ where $b_i' \in B_i$ for every $i \in \mathbb{Z}_+$. Now consider any $i \in \mathbb{Z}_+$ and $b_0 \in B_i$. Then define

$$b_j = \begin{cases} b_0 & j = i \\ b_j' & j \neq i \end{cases}$$

for any $j \in \mathbb{Z}_+$. Clearly we have that $b_j \in B_j$ for any $j \in \mathbb{Z}_+$ so that $\mathbf{b} = (b_1, b_2, \ldots) \in B_1 \times B_2 \times \cdots = B$. It then follows that also $\mathbf{b} \in A$ since $B \subset A$. Hence $b_j \in A_j$ for every $j \in \mathbb{Z}_+$. In particular, we have $b_0 = b_i \in A_i$. Since b_0 was arbitrary, this shows that $B_i \subset A_i$, and since i was arbitrary, this shows the desired result. ☐

(c)

Proof. Suppose that A is nonempty so that there is an $\mathbf{a} \in A$. Then, since $A = A_1 \times A_2 \times \cdots$, it follows that $\mathbf{a} = (a_1, a_2, \ldots)$ where $a_i \in A_i$ for every $i \in \mathbb{Z}_+$. Therefore, for any such $i \in \mathbb{Z}_+$, we have that $a_i \in A_i$ so that $A_i \neq \varnothing$. Hence every A_i is nonempty as desired since i was arbitrary. ☐

Consider the converse. Suppose that each A_i is nonempty (for $i \in \mathbb{Z}_+$). Then there is an $a_i \in A_i$ for every $i \in \mathbb{Z}_+$ so that $\mathbf{a} = (a_1, a_2, \ldots) \in A_1 \times A_2 \times \cdots = A$ so that then $A \neq \varnothing$. While this may seem like an innocuous argument, especially out of the context of axiomatic set theory, it actually requires the Axiom of Choice. The reason is that, in the general case when each A_i may have more than one element, or even an infinite number of elements, we have to choose a specific a_i in each A_i. Since the index set $\mathbb{Z}_+$ is infinite, an infinite number of these choices must be made, which is precisely when the Axiom of Choice is required. If the index set was finite, then the axiom would not be needed.

(d) First, let $C_i = A_i \cup B_i$ for every $i \in \mathbb{Z}_+$, and let $C = C_1 \times C_2 \times \cdots$, so that we are asked to compare C with $A \cup B$.

We claim that $A \cup B \subset C$ but that C is *not* generally a subset of $A \cup B$.

Proof. First consider any $\mathbf{x} \in A \cup B$ so that $\mathbf{x} \in A$ or $x \in B$. If $\mathbf{x} \in A$ then it has to be that $\mathbf{x} = (x_1, x_2, \ldots)$ where $x_i \in A_i$ for every $i \in \mathbb{Z}_+$. Consider then any such $i \in \mathbb{Z}_+$. Then $x_i \in A_i$ so that clearly $x_i \in A_i \cup B_i = C_i$. Since i was arbitrary, we conclude that $\mathbf{x} = (x_1, x_2, \ldots) \in C_1 \times C_2 \times \cdots = C$. An analogous argument shows that $\mathbf{x} \in C$ when $\mathbf{x} \in B$ as well. Hence $A \cup B \subset C$ since $\mathbf{x}$ was arbitrary.

To show that C is *not* a subset of $A \cup B$ in general, consider the following counterexample. Let $A_1 = \varnothing$ and $A_i = \{1\}$ for every $i \in \mathbb{Z}_+$ where $i > 1$. Also let $B_i = \{2\}$ for every $i \in \mathbb{Z}_+$. Now, it follows from the contrapositive of part (c) that $A = \varnothing$ since $A_1 = \varnothing$. We also clearly have $B = B_1 \times B_2 \times \cdots = \{(2, 2, \ldots)\}$ so that $A \cup B = \varnothing \cup B = B = \{(2, 2, \ldots)\}$. Clearly $C_1 = A_1 \cup B_1 = \varnothing \cup \{2\} = \{2\}$ while, for $i > 1$ we have $C_i = A_i \cup B_i = \{1\} \cup \{2\} = \{1, 2\}$. It then follows that, for $a_1 = 2$ and $a_i = 1$ for $i > 1$, we have $\mathbf{a} = (a_1, a_2, \ldots) = (2, 1, 1, \ldots) \in C_1 \times C_2 \times \cdots = C$. However, clearly $\mathbf{a} \notin A \cup B$, which suffices to show that C cannot be a subset of $A \cup B$ in general. ☐

Now let $C_i = A_i \cap B_i$ for every $i \in \mathbb{Z}_+$ so that we are asked to compare $C = C_1 \times C_2 \times \cdots$ and $A \cap B$.

Here we claim that in fact $A \cap B = C$.

Proof. First consider any $\mathbf{x} \in A \cap B$ so that $\mathbf{x} \in A$ and $\mathbf{x} \in B$. It then follows that $\mathbf{x} = (x_1, x_2, \ldots)$ where $x_i \in A_i$ for every $i \in \mathbb{Z}_+$ and $x_i \in B_i$ for every $i \in \mathbb{Z}_+$. Then, for any such $i \in \mathbb{Z}_+$, clearly $x_i \in A_i$ and $x_i \in B_i$ so that $x_i \in A_i \cap B_i = C_i$. We then have that $\mathbf{x} = (x_1, x_2, \ldots) \in C_1 \times C_2 \times \cdots = C$. Since $\mathbf{x}$ was arbitrary, this shows that $A \cap B \subset C$.

Now consider any $\mathbf{x} \in C$ so that $\mathbf{x} = (x_1, x_2, \dots)$ where $x_i \in C_i$ for any $i \in \mathbb{Z}_+$. Then, for any such $i \in \mathbb{Z}_+$, we have $x_i \in C_i = A_i \cap B_i$ so that $x_i \in A_i$ and $x_i \in B_i$. Since i was arbitrary, this shows that both $\mathbf{x} = (x_1, x_2, \dots) \in A_1 \times A_2 \times \cdots = A$ and $\mathbf{x} = (x_1, x_2, \dots) \in B_1 \times B_2 \times \cdots = B$. Hence $\mathbf{x} \in A \cap B$, which shows that $C \subset A \cap B$ since $\mathbf{x}$ was arbitrary.

Therefore it must be that $A \cap B = C$ as desired. □

Exercise 5.4

Let $m, n \in \mathbb{Z}_+$. Let $X \neq \varnothing$.

(a) If $m \leq n$, find an injective map $f : X^m \to X^n$.

(b) Find a bijective map $g : X^m \times X^n \to X^{m+n}$.

(c) Find an injective map $h : X^n \to X^\omega$.

(d) Find a bijective map $k : X^n \times X^\omega \to X^\omega$.

(e) Find a bijective map $l : X^\omega \times X^\omega \to X^\omega$.

(f) If $A \subset B$, find an injective map $m : (A^\omega)^n \to B^\omega$.

NOTE: For part (f), older printings of the text say, "If $A \subset B$, find an injective map $m : X^A \to X^B$." This is assumed to be an error since the meaning of X^A and X^B are not defined in the text (though, for example, X^A would typically mean the set of functions from A to X) as well as the fact that it was changed.

Solution:

(a) If $m \leq n$, find an injective map $f : X^m \to X^n$.

Proof. Suppose that $m \leq n$. Since $X \neq \varnothing$, there is an $x_0 \in X$. Now, for any $\mathbf{x} \in X^m$ we have that $\mathbf{x} = (x_1, \dots, x_m)$ where each $x_i \in X$. Then define

$$y_i = \begin{cases} x_i & 1 \leq i \leq m \\ x_0 & m < i \leq n \end{cases}$$

for $i \in \{1, \dots, n\}$. Clearly $y_i \in X$ for every $i \in \{1, \dots, n\}$ so that $\mathbf{y} = (y_1, \dots, y_n) \in X^n$. Then set $f(\mathbf{x}) = \mathbf{y}$ so that $f : X^m \to X^n$.

To show that f is injective consider $\mathbf{x}$ and $\mathbf{x}'$ in X^m so that $\mathbf{x} = (x_1, \dots, x_m)$ and $\mathbf{x}' = (x_1', \dots, x_m')$ where both x_i and x_i' are of course in X for any $i \in \{1, \dots, m\}$. Also suppose that $\mathbf{x} \neq \mathbf{x}'$ so that it follows that there is an $i \in \{1, \dots, m\}$ where $x_i \neq x_i'$. Let $\mathbf{y} = (y_1, \dots, y_n) = f(\mathbf{x})$ and $\mathbf{y}' = (y_1', \dots, y_n') = f(\mathbf{x}')$. Then, since clearly $1 \leq i \leq m$, we have $y_i = x_i \neq x_i' = y_i'$ by the definition of f. Hence we have $f(\mathbf{x}) = \mathbf{y} \neq \mathbf{y}' = f(\mathbf{x}')$, which shows that f is injective since $\mathbf{x}$ and $\mathbf{x}'$ were arbitrary. □

(b) Find a bijective map $g : X^m \times X^n \to X^{m+n}$.

Proof. Consider any $\mathbf{x} \in X^m \times X^n$ so that $\mathbf{x} = (\mathbf{a}, \mathbf{b})$ where $\mathbf{a} \in X^m$ and $\mathbf{b} \in X^n$. Then we have that $\mathbf{a} = (a_1, \dots, a_m)$ and $\mathbf{b} = (b_1, \dots, b_n)$ where $a_i, b_j \in X$ for every $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$. Then define

$$y_k = \begin{cases} a_k & 1 \leq k \leq m \\ b_{k-m} & m < k \leq m+n \end{cases}$$

for any $k \in \{1, \ldots, m+n\}$, noting that for $m < k \leq m+n$ we have $m+1 \leq k \leq m+n$, and hence $1 \leq k-m \leq n$ so that b_{k-m} is defined. Now set $g(\mathbf{x}) = \mathbf{y} = (y_1, \ldots, y_{m+n})$ so that clearly $g(\mathbf{x}) \in X^{m+n}$ since each $y_k \in X$. Thus g is a function from $X^m \times X^n$ to X^{m+n}.

To show that g is injective, consider any $\mathbf{x} = (\mathbf{a}, \mathbf{b})$ and $\mathbf{x}' = (\mathbf{a}', \mathbf{b}')$ in $X^m \times X^n$ where $\mathbf{x} \neq \mathbf{x}'$. Also let $\mathbf{y} = (y_1, \ldots, y_{m+n}) = g(\mathbf{x})$ and $\mathbf{y}' = \left(y_1', \ldots, y_{m+n}'\right) = g(\mathbf{x}')$. Since $\mathbf{x} \neq \mathbf{x}'$, it must be that $\mathbf{a} \neq \mathbf{a}'$ or $\mathbf{b} \neq \mathbf{b}'$. In the former case we have that $\mathbf{a} = (a_1, \ldots, a_m)$ and $\mathbf{a}' = (a_1', \ldots, a_m')$ since they are both in X^m. Since $\mathbf{a} \neq \mathbf{a}'$ there is an $i \in \{1, \ldots, m\}$ where $a_i \neq a_i'$. Then, since clearly $1 \leq i \leq m$, we have that $y_i = a_i \neq a_i' = y_i'$. In the latter case we have that $\mathbf{b} = (b_1, \ldots, b_n)$ and $\mathbf{b}' = (b_1', \ldots, b_n')$ since they are both in X^n. Then, since $\mathbf{b} \neq \mathbf{b}'$, we have that there is an $i \in \{1, \ldots, n\}$ such that $b_i \neq b_i'$. Let $k = m+i$ so that clearly $k - m = i$. Also $m < m+i = k \leq m+n$ since $0 < 1 \leq i \leq n$ so that $y_k = b_{k-m} = b_i \neq b_i' = b_{k-m}' = y_k'$. Hence in both cases there is a $k \in \{1, \ldots, m+n\}$ such that $y_k \neq y_k'$ so that $g(\mathbf{x}) = \mathbf{y} = (y_1, \ldots, y_{m+n}) \neq \left(y_1', \ldots, y_{m+n}'\right) = \mathbf{y}' = g(\mathbf{x}')$. Since $\mathbf{x}$ and $\mathbf{x}'$ were arbitrary, this shows that g is indeed injective.

Now consider any $\mathbf{y} = (y_1, \ldots, y_{m+n}) \in X^{m+n}$, and define $a_i = y_i$ for any $i \in \{1, \ldots, m\}$ and $b_j = y_{m+j}$ for any $j \in \{1, \ldots, n\}$, noting that y_{m+j} is defined since $0 < 1 \leq j \leq n$ implies that $m < m+j \leq m+n$. Then let $\mathbf{a} = (a_1, \ldots, a_m)$, $\mathbf{b} = (b_1, \ldots, b_n)$, and $\mathbf{x} = (\mathbf{a}, \mathbf{b})$ so that clearly $\mathbf{x} \in X^m \times X^n$. Let $\mathbf{y}' = g(\mathbf{x})$ as defined above so that $\mathbf{y}' = \left(y_1', \ldots, y_{m+n}'\right)$. Consider any $k \in \{1, \ldots, m+n\}$. If $1 \leq k \leq m$ then we have by the definition of g that $y_k' = a_k = y_k$. On the other hand, if $m < k \leq m+n$, then we have $y_k' = b_{k-m} = y_{m+(k-m)} = y_k$. Thus in both cases $y_k' = y_k$ so that clearly $g(\mathbf{x}) = \mathbf{y}' = \left(y_1', \ldots, y_{m+n}'\right) = (y_1, \ldots, y_{m+n}) = \mathbf{y}$ since k was arbitrary. This shows that g is surjective since $\mathbf{y}$ was arbitrary.

Therefore we have shown that g is bijective as desired. □

(c) Find an injective map $h : X^n \to X^\omega$.

Proof. First, we know that $X \neq \varnothing$ so that there is an $x_0 \in X$. So, for any $\mathbf{x} = (x_1, \ldots, x_n) \in X^n$, define

$$y_i = \begin{cases} x_i & 1 \leq i \leq n \\ x_0 & n < i \end{cases}$$

for any $i \in \mathbb{Z}_+$. Then set $h(\mathbf{x}) = \mathbf{y} = (y_1, y_2, \ldots)$ so that clearly $h(\mathbf{x}) \in X^\omega$. Thus h is a function that maps X^n into X^ω.

To show that h is injective, consider $\mathbf{x}$ and $\mathbf{x}'$ in X^n where $\mathbf{x} \neq \mathbf{x}'$. Clearly we have that $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{x}' = (x_1', \ldots, x_n')$, and let $\mathbf{y} = (y_1, y_2, \ldots) = h(\mathbf{x})$ and $\mathbf{y}' = (y_1', y_2', \ldots) = h(\mathbf{x}')$. Since $\mathbf{x} \neq \mathbf{x}'$, there must an $i \in \{1, \ldots, n\}$ where $x_i \neq x_i'$. Then we have $y_i = x_i \neq x_i' = y_i'$ by the definition of h since obviously $1 \leq i \leq n$. It then follows that $h(\mathbf{x}) = \mathbf{y} = (y_1, y_2, \ldots) \neq (y_1', y_2', \ldots) = \mathbf{y}' = h(\mathbf{x}')$, which shows that h is injective since $\mathbf{x}$ and $\mathbf{x}'$ were arbitrary. □

(d) Find a bijective map $k : X^n \times X^\omega \to X^\omega$.

Proof. Consider any $\mathbf{x} = (\mathbf{a}, \mathbf{b}) \in X^n \times X^\omega$ so that clearly $\mathbf{a} = (a_1, \ldots, a_n) \in X^n$ and $\mathbf{b} = (b_1, b_2, \ldots) \in X^\omega$. Then define the sequence

$$y_i = \begin{cases} a_i & 1 \leq i \leq n \\ b_{i-n} & n < i \end{cases}$$

for any $i \in \mathbb{Z}_+$, noting that when $n < i$ we have $n+1 \leq i$ so that $1 \leq i-n$ so that b_{i-n} is defined. We then of course set $k(\mathbf{x}) = \mathbf{y} = (y_1, y_2, \ldots)$ so that clearly $k(\mathbf{x}) \in X^\omega$. Therefore k is a function from $X^n \times X^\omega$ to X^ω.

To show that k is injective consider $\mathbf{x}$ and $\mathbf{x}'$ in $X^n \times X^\omega$ where $\mathbf{x} \neq \mathbf{x}'$. Of course we have $\mathbf{x} = (\mathbf{a}, \mathbf{b})$ and $\mathbf{x}' = (\mathbf{a}', \mathbf{b}')$ where $\mathbf{a}, \mathbf{a}' \in X^n$ while $\mathbf{b}, \mathbf{b}' \in X^\omega$. It then follows that $\mathbf{a} = (a_1, \ldots, a_n)$, $\mathbf{a}' = (a_1', \ldots, a_n')$, $\mathbf{b} = (b_1, b_2, \ldots)$, and $\mathbf{b}' = (b_1', b_2', \ldots)$, where every a_i, a_i', b_j, and b_j' are in X (for $i \in \{1, \ldots, n\}$ and $j \in \mathbb{Z}_+$). Also, let $\mathbf{y} = (y_1, y_2, \ldots) = k(\mathbf{x})$ and $\mathbf{y}' = (y_1', y_2', \ldots) = k(\mathbf{x}')$. Now, since $\mathbf{x} \neq \mathbf{x}'$, we have that either $\mathbf{a} \neq \mathbf{a}'$ or $\mathbf{b} \neq \mathbf{b}'$. If $\mathbf{a} \neq \mathbf{a}'$ then there is an $i \in \{1, \ldots, n\}$ where $a_i \neq a_i'$. We then have that $y_i = a_i \neq a_i' = y_i'$ by the definition of k, since obviously $1 \leq i \leq n$. If, on the other hand, $\mathbf{b} \neq \mathbf{b}'$, then there is an $i \in \mathbb{Z}_+$ such that $b_i \neq b_i'$. Then clearly $n < n+i$ since $0 < i$ so that $y_{n+i} = b_{(n+i)-n} = b_i \neq b_i' = b'_{(n+i)-n} = y'_{n+i}$, noting that clearly $n+i \in \mathbb{Z}_+$. Hence in either case there is an $i \in \mathbb{Z}_+$ such that $y_i \neq y_i'$ so that $k(\mathbf{x}) = \mathbf{y} = (y_1, y_2, \ldots) \neq (y_1', y_2', \ldots) = \mathbf{y}' = k(\mathbf{x}')$. This shows that k is injective since $\mathbf{x}$ and $\mathbf{x}'$ were arbitrary.

Now consider any $\mathbf{y} = (y_1, y_2, \ldots) \in X^\omega$ and set $a_i = y_i$ for any $i \in \{1, \ldots, n\}$ so that clearly $\mathbf{a} = (a_1, \ldots, a_n) \in X^n$. Also, for any $j \in \mathbb{Z}_+$, let $b_j = y_{n+j}$ so that clearly $\mathbf{b} = (b_1, b_2, \ldots) \in X^\omega$. Let $\mathbf{x} = (\mathbf{a}, \mathbf{b})$ so that clearly $x \in X^n \times X^\omega$. Now set $\mathbf{y}' = (y_1', y_2', \ldots) = k(\mathbf{x})$ as defined above. Consider any $i \in \mathbb{Z}_+$. If $1 \leq i \leq n$ then $y_i' = a_i = y_i$ by the definition of k. If $n < i$ then $y_i' = b_{i-n} = y_{n+(i-n)} = y_i$. Hence $y_i' = y_i$ for every $i \in \mathbb{Z}_+$ so that $k(\mathbf{x}) = \mathbf{y}' = (y_1', y_2', \ldots) = (y_1, y_2, \ldots) = \mathbf{y}$, which shows that k is surjective since $\mathbf{y}$ was arbitrary.

This completes the proof that k is bijective. □

(e) Find a bijective map $l : X^\omega \times X^\omega \to X^\omega$.

Proof. Consider any $\mathbf{x} = (\mathbf{a}, \mathbf{b}) \in X^\omega \times X^\omega$ so that clearly $\mathbf{a} = (a_1, a_2, \ldots)$ and $\mathbf{b} = (b_1, b_2, \ldots)$. Now define

$$y_i = \begin{cases} a_{i/2} & i \text{ is even} \\ b_{(i+1)/2} & i \text{ is odd} \end{cases}$$

for any $i \in \mathbb{Z}_+$. Note that $i/2$ and $(i+1)/2$ are in $\mathbb{Z}_+$ if i is even or odd, respectively by Lemma 5.2.1 so that y_i is defined. Clearly we have that $y_i \in X$ for any $i \in \mathbb{Z}_+$ so that $\mathbf{y} = (y_1, y_2, \ldots) \in X^\omega$. Setting $l(\mathbf{x}) = \mathbf{y}$, we then have that l is a function from $X^\omega \times X^\omega$ to X^ω.

To show that l is injective, consider $\mathbf{x} = (\mathbf{a}, \mathbf{b})$ and $\mathbf{x}' = (\mathbf{a}', \mathbf{b}')$ in $X^\omega \times X^\omega$ where $\mathbf{x} \neq \mathbf{x}'$. Also set $\mathbf{y} = (y_1, y_2, \ldots) = l(\mathbf{x})$ and $\mathbf{y}' = (y_1', y_2', \ldots) = l(\mathbf{x}')$. Since $\mathbf{x} \neq \mathbf{x}'$, we have that either $\mathbf{a} \neq \mathbf{a}'$ or $\mathbf{b} \neq \mathbf{b}'$. If $\mathbf{a} \neq \mathbf{a}'$ then there is an $i \in \mathbb{Z}_+$ such that $a_i \neq a_i'$. Then, since clearly $2i$ is even, we have $y_{2i} = a_{(2i)/2} = a_i \neq a_i' = a'_{(2i)/2} = y'_{2i}$. On the other hand, if $\mathbf{b} \neq \mathbf{b}'$ then there is a $j \in \mathbb{Z}_+$ where $b_j \neq b_j'$. Set $k = 2j - 1$, noting that

$$\begin{aligned} 1 &\leq j \\ 2 &\leq 2j \\ 1 &\leq 2j - 1 \\ 1 &\leq k \end{aligned}$$

so that $k \in \mathbb{Z}_+$. Clearly also $(k+1)/2 = j$. Since obviously k is odd, we have $y_k = b_{(k+1)/2} = b_j \neq b_j' = b'_{(k+1)/2} = y_k'$. Hence in both cases we have that there is a $k \in \mathbb{Z}_+$ where $y_k \neq y_k'$ so that $l(\mathbf{x}) = \mathbf{y} = (y_1, y_2, \ldots) \neq (y_1', y_2', \ldots) = \mathbf{y}' = l(\mathbf{x}')$. Since $\mathbf{x}$ and $\mathbf{x}'$ were arbitrary, this shows that l is injective.

Now consider any $\mathbf{y} = (y_1, y_2, \ldots) \in X^\omega$. For any $i \in \mathbb{Z}_+$, define $a_i = y_{2i}$ and $b_i = y_{2i-1}$, noting again that $2i - 1 \in \mathbb{Z}_+$ (and clearly $2i \in \mathbb{Z}_+$). Then set $\mathbf{a} = (a_1, a_2, \ldots)$, $\mathbf{b} = (b_1, b_2, \ldots)$, and $\mathbf{x} = (\mathbf{a}, \mathbf{b})$. Now let $\mathbf{y}' = (y_1', y_2', \ldots) = l(\mathbf{x})$ and consider any $i \in \mathbb{Z}_+$. If i is even then we have by the definition of l that $y_i' = a_{i/2} = y_{2(i/2)} = y_i$. If i is odd then let $j = (i+1)/2$ so that clearly $i = 2j - 1$. Then $y_i' = b_{(i+1)/2} = b_j = y_{2j-1} = y_i$. Hence in either case we have $y_i' = y_i$ so that

$l(\mathbf{x}) = \mathbf{y}' = (y'_1, y'_2, \ldots) = (y_1, y_2, \ldots) = \mathbf{y}$ since i was arbitrary. Since $\mathbf{y}$ was arbitrary this shows that l is surjective.

Thus we have shown that l is bijective as desired. $\square$

(f) If $A \subset B$, find an injective map $m : (A^\omega)^n \to B^\omega$.

Proof. Consider any $\mathbf{x} \in (A^\omega)^n$ so that $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$ where $\mathbf{x}_i \in A^\omega$ for any $i \in \{1, \ldots, n\}$. Then let $x_{ij} = \mathbf{x}_i(j)$ for $i \in \{1, \ldots, n\}$ and $j \in \mathbb{Z}_+$ so that clearly $x_{ij} \in A$, from which it follows that each $x_{ij} \in B$ as well since $A \subset B$. Consider any $k \in \mathbb{Z}_+$. Since $n \neq 0$ (since $n \in \mathbb{Z}_+$), it follows from the Division Theorem from algebra that there are unique integers q and $0 \leq r < n$ where $k = qn + r$. Suppose for a moment that $q < 0$ so that $q + 1 \leq 0$. Then we have that $k = qn + r < qn + n = (q+1)n \leq 0 \cdot n = 0 < k$ (since $k \in \mathbb{Z}_+$) since $r < n$ and $n > 0$ (so that $(q+1)n \leq 0 \cdot n$ since $q + 1 \leq 0$). This is of course a contradiction so that it must be that $q \geq 0$. Then set $i = r + 1 \geq 1$ and $j = q + 1 \geq 1$ so that $i \in \{1, \ldots, n\}$ and $j \in \mathbb{Z}_+$. Set $y_k = x_{ij}$ so that clearly $y_k \in B$ since x_{ij} is. It then follows that $\mathbf{y} = (y_1, y_2, \ldots) \in B^\omega$. Then set $m(\mathbf{x}) = \mathbf{y}$ so that clearly m is a function from $(A^\omega)^n$ to B^ω.

To show that m is injective, consider any $\mathbf{x}$ and $\mathbf{x}'$ in $(A^\omega)^n$ where $\mathbf{x} \neq \mathbf{x}'$. Then $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$ and $\mathbf{x}' = (\mathbf{x}'_1, \ldots, \mathbf{x}'_n)$ where each $\mathbf{x}_i$ and $\mathbf{x}'_i$ are in A^ω for $i \in \{1, \ldots, n\}$. As before set $x_{ij} = \mathbf{x}_i(j)$ and $x'_{ij} = \mathbf{x}'_i(j)$ for $i \in \{1, \ldots, n\}$ and $j \in \mathbb{Z}_+$, and also let $\mathbf{y} = m(\mathbf{x})$ and $\mathbf{y}' = m(\mathbf{x}')$. Now, since $\mathbf{x} \neq \mathbf{x}'$, there is an $i \in \{1, \ldots, n\}$ where $\mathbf{x}_i \neq \mathbf{x}'_i$. It then follows that there is a $j \in \mathbb{Z}_+$ such that $x_{ij} = \mathbf{x}_i(j) \neq \mathbf{x}'_i(j) = x'_{ij}$. Now let $k = (j-1)n + (i-1)$ so that it follows from the definition of m that $y_k = x_{ij}$ and $y'_k = x'_{ij}$ since the quotient q and remainder r are unique by the Division Theorem. Hence $y_k = x_{ij} \neq x'_{ij} = y'_k$ so that clearly $m(\mathbf{x}) = \mathbf{y} = (y_1, y_2, \ldots) \neq (y'_1, y'_2, \ldots) = \mathbf{y}' = m(\mathbf{x}')$. This shows that m is injective as desired since $\mathbf{x}$ and $\mathbf{x}'$ were arbitrary. $\square$

Exercise 5.5

Which of the following subsets of $\mathbb{R}^\omega$ can be expressed as the cartesian product of subsets of $\mathbb{R}$?

(a) $\{\mathbf{x} \mid x_i \text{ is an integer for all } i\}$.
(b) $\{\mathbf{x} \mid x_i \geq i \text{ for all } i\}$.
(c) $\{\mathbf{x} \mid x_i \text{ is an integer for all } i \geq 100\}$.
(d) $\{\mathbf{x} \mid x_2 = x_3\}$.

Solution:

(a) Let $X = \{\mathbf{x} \in \mathbb{R}^\omega \mid x_i \text{ is an integer for all } i\}$ and $Y = \mathbb{Z}^\omega$, noting that $\mathbb{Z} \subset \mathbb{R}$. We claim that $X = Y$.

Proof. Consider any $\mathbf{x} \in X$ so that $x_i \in \mathbb{Z}$ for any $i \in \mathbb{Z}_+$. It is then immediately obvious that $\mathbf{x} \in \mathbb{Z}^\omega = Y$. Hence $X \subset Y$ since $\mathbf{x}$ was arbitrary.

Now consider any $\mathbf{x} \in Y = \mathbb{Z}^\omega$ so that $x_i \in \mathbb{Z}$ for every $i \in \mathbb{Z}_+$. Again it is obvious by the definition of X that $\mathbf{x} \in X$. Hence $Y \subset X$ since $\mathbf{x}$ was arbitrary. This shows that $X = Y$ as desired. $\square$

(b) Let $X = \{\mathbf{x} \in \mathbb{R}^\omega \mid x_i \geq i \text{ for all } i\}$ and define $Y_i = \{x \in \mathbb{R} \mid x \geq i\}$ for $i \in \mathbb{Z}_+$, noting that obviously each $Y_i \subset \mathbb{R}$. Then let $Y = Y_1 \times Y_2 \times \cdots$. We claim that $X = Y$.

Proof. First consider $\mathbf{x} \in X$ so that $x_i \geq i$ for any $i \in \mathbb{Z}_+$. Then, for any $i \in \mathbb{Z}_+$ clearly $x_i \in Y_i$ by definition since $x_i \geq i$ (and also $x_i \in \mathbb{R}$). Hence it follows that $\mathbf{x} = (x_1, x_2, \ldots) \in Y_1 \times Y_2 \times \cdots = Y$. Since $\mathbf{x}$ was arbitrary, this shows that $X \subset Y$.

Now suppose that $\mathbf{x} \in Y$ so that $x_i \in Y_i$ for every $i \in \mathbb{Z}_+$. Consider any such $i \in \mathbb{Z}_+$ so that $x_i \in Y_i$. Then, by definition $x_i \geq i$. Since i was arbitrary, this shows that $\mathbf{x} \in X$ by definition. Hence $Y \subset X$ since $\mathbf{x}$ was arbitrary so that $X = Y$. □

(c) Define $X = \{\mathbf{x} \in \mathbb{R}^\omega \mid x_i \text{ is an integer for all } i \geq 100\}$. Also define $Y_i = \mathbb{R}$ when $i < 100$ and $Y_i = \mathbb{Z}$ when $i \geq 100$ (and $i \in \mathbb{Z}_+$ for both), noting that of course $Y_i \subset \mathbb{R}$ for either case. Let $Y = Y_1 \times Y_2 \times \cdots$, and we claim that $X = Y$.

Proof. Consider any $\mathbf{x} \in X$ so that $x_i \in \mathbb{Z}$ for all $i \geq 100$. Suppose $i \in \mathbb{Z}_+$. If $i < 100$ then clearly $x_i \in \mathbb{R} = Y_i$ since $\mathbf{x} \in \mathbb{R}^\omega$. If $i \geq 100$ then we have that $x_i \in \mathbb{Z} = Y_i$. Hence in either case $x_i \in Y_i$ so that $\mathbf{x} \in Y_1 \times Y_2 \times \cdots = Y$ since i was arbitrary. Since $\mathbf{x}$ was arbitrary, this shows that $X \subset Y$.

Now consider any $\mathbf{x} \in Y$ and any $i \in \mathbb{Z}_+$ where $i \geq 100$. Then $x_i \in Y_i = \mathbb{Z}$ so that x_i is an integer. From this it follows that $\mathbf{x} \in X$ by definition since obviously $\mathbf{x} \in \mathbb{R}^\omega$ (since $x_i \in Y_i = \mathbb{R}$ when $i < 100$). Hence $Y \subset X$ since $\mathbf{x}$ was arbitrary. This completes the proof that $X = Y$. □

(d) We claim that $X = \{\mathbf{x} \in \mathbb{R}^\omega \mid x_2 = x_3\}$ *cannot* be expressed as the cartesian product of subsets of $\mathbb{R}$.

Proof. Suppose to the contrary that there are $X_i \subset \mathbb{R}$ for $i \in \mathbb{Z}_+$ where $X = X_1 \times X_2 \times \cdots$. Let $(a, a, \ldots)$ denote the sequence $(x_1, x_2, \ldots)$ where $x_i = a$ for all $i \in \mathbb{Z}_+$. We then have that $(1, 1, \ldots)$ and $(2, 2, \ldots)$ are both in X since clearly $x_2 = x_3$ in both. Hence we have that 1 and 2 are both in X_i for every $i \in \mathbb{Z}_+$ since $X = X_1 \times X_2 \times \cdots$. Now define

$$y_i = \begin{cases} 1 & i \neq 2 \\ 2 & i = 2 \end{cases}$$

for $i \in \mathbb{Z}_+$. Clearly $\mathbf{y} = (y_1, y_2, \ldots) \in X_1 \times X_2 \times \cdots$ since both 1 and 2 are in each X_i. However, it is also clear that $\mathbf{y} \notin X$ by definition since $y_2 = 2 \neq 1 = y_3$. This contradicts the fact that $X = X_1 \times X_2 \times \cdots$, which shows the desired result. □

§6 Finite Sets

Exercise 6.1

(a) Make a list of all the injective maps

$$f : \{1, 2, 3\} \to \{1, 2, 3, 4\} .$$

Show that none is bijective. (This constitutes a *direct* proof that a set A of cardinality three does not have cardinality four.)

(b) How many injective maps

$$f : \{1, \ldots, 8\} \to \{1, \ldots, 10\}$$

are there? (You can see why one would not wish to try to prove *directly* that there is no bijective correspondence between these sets.)

Solution:

Lemma 6.1.1. *The number of injective mappings (i.e. the cardinality of the set of injective functions) from* $\{1, \ldots, m\}$ *to* $\{1, \ldots, n\}$*, where* $m \leq n$*, is equal to the number of* m*-permutations of* n*, which is*

$$\frac{n!}{(n-m)!}.$$

Proof. We fix n and show this for all $m \leq n$ by induction. First, for $m = 1$, the domain of the mappings is simply $\{1\}$ so that we need only choose a single element to which to map 1. Since there are n elements to choose from (since the range is $\{1, \ldots, n\}$) there are clearly

$$n = \frac{n!}{(n-1)!} = \frac{n!}{(n-m)!}$$

mappings, all of which are trivially injective.

Now suppose that $m < n$ and that there are $n!/(n-m)!$ injective mappings from $\{1, \ldots, m\}$ to $\{1, \ldots, n\}$. Consider any such mapping $(f_1, \ldots, f_m)$. Since this mapping is injective, each f_i is unique so that it uses m of the n available numbers in $\{1, \ldots, n\}$. Thus there are $n - m$ numbers to choose from to which to set f_{m+1} so that the mapping $(f_1, \ldots, f_{m+1})$ is still injective. Hence for each injective mapping $(f_1, \ldots, f_m)$ there are $n - m$ injective mappings from $\{1, \ldots, m+1\}$ to $\{1, \ldots, n\}$. Since there are $n!/(n-m)!$ such mappings by the induction hypothesis, the total number of mappings from $\{1, \ldots, m+1\}$ to $\{1, \ldots, n\}$ will be

$$\frac{n!}{(n-m)!}(n-m) = \frac{n!}{(n-m-1)!} = \frac{n!}{[n-(m+1)]!},$$

which completes the induction. □

Main Problem.

(a) Here we have $n = 4$ and $m = 3$ in Lemma 6.1.1 so that we expect $4!/(4-3)! = 4!/1! = 4! = 24$ injective mappings. Since the domain of each f is a section of the positive integers, these maps can be written simply as 3-tuples. They are enumerated below:

1. $(1, 2, 3)$
2. $(1, 2, 4)$
3. $(1, 3, 2)$
4. $(1, 3, 4)$
5. $(1, 4, 2)$
6. $(1, 4, 3)$
7. $(2, 1, 3)$
8. $(2, 1, 4)$
9. $(2, 3, 1)$
10. $(2, 3, 4)$
11. $(2, 4, 1)$
12. $(2, 4, 3)$
13. $(3, 1, 2)$
14. $(3, 1, 4)$
15. $(3, 2, 1)$
16. $(3, 2, 4)$
17. $(3, 4, 1)$
18. $(3, 4, 2)$
19. $(4, 1, 2)$
20. $(4, 1, 3)$
21. $(4, 2, 1)$
22. $(4, 2, 3)$
23. $(4, 3, 1)$
24. $(4, 3, 2)$

Note that they are all injective since no number is used more than once in each tuple. Also none are surjective since it is easily verified that there is always an element of $\{1, 2, 3, 4\}$ that is not in each tuple. Thus none are a bijection since they are not surjective.

(b) Here we have $n = 10$ and $m = 8$ in Lemma 6.1.1 so that there are $10!/(10-8)! = 10!/2! = 1814400$ injective mappings. That is nearly two million! Certainly a direct proof would be unfeasible by hand, but could be done by computer fairly easily.

Exercise 6.2

Show that if B is not finite and $B \subset A$, then A is not finite.

Solution:

Proof. Suppose that B is not finite and $B \subset A$ but that A *is* finite. Since $B \subset A$, either $B = A$ or B is a proper subset of A. In the former case we clearly have a contradiction since B would be finite since A is and $B = A$. In the latter case we have that there is a bijection from A to $\{1, \ldots, n\}$ for some $n \in \mathbb{Z}_+$ by definition since A is finite. Then, since B is a proper subset of A, it follows from Theorem 6.2 that there is a bijection from B to $\{1, \ldots, m\}$ for some $m < n$. However, then clearly B is finite by definition, which is also a contradiction since we know B is not finite. Hence in either case there is a contradiction so that A must not be finite. □

Exercise 6.3

Let X be the two-element set $\{0, 1\}$. Find a bijective correspondence between X^ω and a proper subset of itself.

Solution:

Proof. Let $Y = \{\mathbf{x} \in X^\omega \mid x_1 = 0\}$, which is clearly a proper subset of X^ω since, for example, $(1, 1, \ldots)$ is in X^ω but not in Y. We construct a bijective function f from X^ω to Y. So consider any $\mathbf{x} \in X^\omega$ and define

$$y_i = \begin{cases} 0 & i = 1 \\ x_{i-1} & i \neq 1 \end{cases}$$

for $i \in \mathbb{Z}_+$, noting that when $i \neq 1$ we have $i > 1$ so that $i - 1 \geq 1$, and thus $y_i = x_{i-1}$ is defined. Now define $f(\mathbf{x}) = \mathbf{y} = (y_1, y_2, \ldots)$ so that clearly f is a function from X^ω to Y, since $y_1 = 0$ for any input $\mathbf{x}$.

To show that f is injective, consider $\mathbf{x}$ and $\mathbf{x}'$ in X^ω where $\mathbf{x} \neq \mathbf{x}'$, and let $\mathbf{y} = f(\mathbf{x})$ and $\mathbf{y}' = f(\mathbf{x}')$. Now, since $\mathbf{x} \neq \mathbf{x}'$, there is an $i \in \mathbb{Z}_+$ where $x_i \neq x'_i$. Since $i > 0$ (since $i \in \mathbb{Z}_+$) it follows that $i + 1 > 1$ so that $i + 1 \neq 1$. We then have by the definition of f that $y_{i+1} = x_{(i+1)-1} = x_i \neq x'_i = x'_{(i+1)-1} = y'_{i+1}$ so that clearly $f(\mathbf{x}) = \mathbf{y} \neq \mathbf{y}' = f(\mathbf{x}')$. Since $\mathbf{x}$ and $\mathbf{x}'$ were arbitrary, this shows that f is indeed injective.

Now consider any $\mathbf{y} \in Y$ so that $y_1 = 0$. Define $x_i = y_{i+1}$ for any $i \in \mathbb{Z}_+$ and let $\mathbf{x} = (x_1, x_2, \ldots)$. Then $\mathbf{x} \in X^\omega$ since clearly each $x_i = y_{i+1} \in X$. Now let $\mathbf{y}' = f(\mathbf{x})$ and consider any $i \in \mathbb{Z}_+$. If $i = 1$ then clearly $y'_i = y'_1 = 0 = y_1 = y_i$ ($y'_1 = 0$ since the range of f is Y). If $i \neq 1$ then $y'_i = x'_{i-1} = y_{(i-1)+1} = y_i$. Hence $y'_i = y_i$ in both cases so that $f(\mathbf{x}) = \mathbf{y}' = \mathbf{y}$ since i was arbitrary. This shows that f is surjective since $\mathbf{y}$ was arbitrary.

Therefore f is bijective as desired. □

Exercise 6.4

Let A be a nonempty finite simply ordered set.

(a) Show that A has a largest element. [Hint: Proceed by induction on the cardinality of A.]

(b) Show that A has the order type of a section of positive integers.

Solution:

(a)

Proof. We show by induction that, for all $n \in \mathbb{Z}_+$, any simply ordered set with cardinality n has a largest element. This of course shows the result since, by definition, $A \neq \varnothing$ has cardinality n for some $n \in \mathbb{Z}_+$ when A is finite.

First, suppose that A is simply ordered and has cardinality 1 so that clearly $A = \{a\}$ for some element a. It is also clear that a is trivially the largest element of A since it is the only element.

Now suppose that any simply ordered set with cardinality n has a largest element. Suppose that A is simply ordered by $\prec$ and has cardinality $n+1$. Then there is a bijection f from A to $\{1, \ldots, n+1\}$, noting that obviously f^{-1} is also a bijection. Clearly A is nonempty (since the cardinality of A is $n+1 > n > 0$) so that there is an $a \in A$. Let $A' = A - \{a\}$ so that A' has cardinality n by Lemma 6.1. Note also that clearly A' is simply ordered by $\prec$ as well (technically we must restrict $\prec$ to elements of A' so that it is really ordered by $\prec \cap (A' \times A')$). It then follows that A' has a largest element b by the induction hypothesis. Since a and b must be comparable in $\prec$ by the definition of a simple order we have the following:

Case: $a = b$. This is not possible since $b \in A'$ but clearly $a \notin A - \{a\} = A'$.

Case: $a \prec b$. We claim that b is the largest element of A. To see this, consider any $x \in A$ so that either $x = a$ or $x \in A'$. In the former case clearly $x = a \preccurlyeq b$, and in the latter $x \preccurlyeq b$ since b is the largest element of A'. This shows that b is the largest element of A since x was arbitrary.

Case: $b \prec a$. We claim that a is the largest element of A. So consider any $x \in A$ so that $x = a$ or $x \in A'$. In the first case obviously $x \preccurlyeq x = a$, and in the second $x \preccurlyeq b \preccurlyeq a$ since b is the largest element of A'. This shows that a is the largest element of A since x was arbitrary.

Thus in all cases we have shown that A has a largest element, which completes the induction. □

(b)

Proof. We again show this by induction on the (finite) cardinality of the set. First, if A is a simply ordered set with cardinality 1 then clearly $A = \{a\}$ for some a, which is clearly trivially the same order type as the section $\{1\}$.

Now suppose that all simply ordered sets of cardinality n have the order type of a section of positive integers. Consider then a set A simply ordered by $\prec$ that has cardinality $n+1$. Clearly $A \neq \varnothing$ so that it has a largest element a by part (a). Then the set $A' = A - \{a\}$ has cardinality n by Lemma 6.1. Since A' is also clearly simply ordered by $\prec$ (with the appropriate restriction) it follows from the induction hypothesis that it has order type of $\{1, \ldots, m\}$ for some $m \in \mathbb{Z}_+$. Since this also implies that A' has the cardinality of m, it has to be that $m = n$ since this cardinality is unique (by Lemma 6.5). So let f' be the order-preserving bijection from A' to $\{1, \ldots, m\} = \{1, \ldots, n\}$. Now define

$$f(x) = \begin{cases} f'(x) & x \neq a \\ n+1 & x = a \end{cases}$$

for any $x \in A$. It is clear that f is a function from A to $\{1, \ldots, n+1\}$ since obviously $n+1 \in \{1, \ldots, n+1\}$ and the image of f' is $\{1, \ldots, n\} \subset \{1, \ldots, n+1\}$.

Consider next any x and x' in A where $x \prec x'$. Suppose for the moment that $x = a$. Then $x' \preccurlyeq a = x$ since a is the largest element of A. This contradicts the fact that $x \prec x'$ so that it must be that $x \neq a$. Then $f(x) = f'(x)$. If also $x' \neq a$ then clearly $f(x) = f'(x) < f'(x') = f(x')$ since f' preserves order. If $x' = a$ then $f(x') = n+1$ so that $f(x) = f'(x) \leq n < n+1 = f(x')$ since the image of f' is only $\{1, \ldots, n\}$. Hence in all cases $f(x) < f(x')$ so that f preserves order since x and x' were arbitrary. Note that this also shows that f is injective since, for any $x, x' \in A$ where $x \neq x'$, we can assume without loss of generality that $x \prec x'$ (since it must be that $x \prec x'$ or $x' \prec x$) so that $f(x) < f(x')$, and hence $f(x) \neq f(x')$.

Lastly consider any $k \in \{1, \ldots, n+1\}$. If $k = n+1$ then clearly by definition $f(a) = n+1 = k$, noting that obviously $a \in A$. On the other hand, if $k \neq n+1$ then it has to be that $k < n+1$ so $k \leq n$. Then $k \in \{1, \ldots, n\}$, which is the image of f' so that there is an $x \in A'$ where $f'(x) = k$ since f' is bijective (and therefore surjective). Since $x \in A'$ we have that $x \in A$ but $x \neq a$ so that $f(x) = f'(x) = k$. This shows that f is surjective since k was arbitrary.

Thus we have shown that f is an order-preserving bijection from A to $\{1, \ldots, n+1\}$, which completes the induction since by definition A has order type $\{1, \ldots, n+1\}$. □

Exercise 6.5

If $A \times B$ is finite, does it follow that A and B are finite?

Solution:

We claim that in general this does not follow.

Proof. As a counterexample, let $A = \mathbb{Z}_+$ and $B = \varnothing$. Clearly A is infinite by Corollary 6.4 so that not both A and B are finite. It also follows from Exercise 5.3 part (c) that $A \times B = \varnothing$ since B is empty. Hence clearly $A \times B$ is finite. □

If we add the additional stipulation that both A and B are nonempty, then the statement becomes true.

Proof. Since $A \times B$ is finite there is a bijective function $f : A \times B \to \{1, \ldots, n\}$ for some $n \in \mathbb{Z}_+$. We then show that A is finite by first constructing an injective function g from A to $A \times B$. Since $B \neq \varnothing$, there is a $b \in B$. So, for any $x \in A$, set $g(x) = (x, b)$, which is clearly in $A \times B$ so that g is a function from A to $A \times B$. Now consider x and x' in A where $x \neq x'$. Then clearly $g(x) = (x, b) \neq (x', b) = g(x')$. This shows that g is injective since x and x' were arbitrary.

We then have that the composition $f \circ g$ is an injective function from A to $\{1, \ldots, n\}$ by Exercise 2.4 part (b) since f is injective as well (since it is a bijection). Therefore A is finite by Corollary 6.7. An analogous argument uses the fact that $A \neq \varnothing$ to show that B is also finite. □

Exercise 6.6

(a) Let $A = \{1, \ldots, n\}$. Show there is a bijection of $\mathcal{P}(A)$ with the cartesian product X^n, where X is the two-element set $X = \{0, 1\}$.

(b) Show that if A is finite, then $\mathcal{P}(A)$ is finite.

Solution:

(a)

Proof. We construct a bijection $f : \mathcal{P}(A) \to X^n$. So, for any $Y \in \mathcal{P}(A)$ we have that clearly $Y \subset A$. Then set

$$x_i = \begin{cases} 0 & i \notin Y \\ 1 & i \in Y \end{cases}$$

for any $i \in \{1, \ldots, n\} = A$. Now set $f(Y) = \mathbf{x} = (x_1, \ldots, x_n)$, noting that clearly $f(Y) \in X^n$ since each $x_i \in \{0, 1\} = X$. Hence f is a function from $\mathcal{P}(A)$ to X^n.

To show that f is injective consider Y and Y' in $\mathcal{P}(A)$ where $Y \neq Y'$. Also let $\mathbf{x} = f(Y)$ and $\mathbf{x}' = f(Y')$ as defined above. Since $Y \neq Y'$, we can without loss of generality assume that there is an $i \in Y$ where $i \notin Y'$. It then follows that $x_i = 1 \neq 0 = x_i'$ by the definition of f. Hence clearly $f(Y) = \mathbf{x} = (x_1, \ldots, x_n) \neq (x_1', \ldots, x_n') = \mathbf{x}' = f(Y')$, which shows that f is injective since Y and Y' were arbitrary.

Now consider any $\mathbf{x} \in X^n$ and let $Y = \{i \in A \mid x_i = 1\}$. Clearly $Y \subset A$ so that $Y \in \mathcal{P}(A)$. Let $\mathbf{x}' = f(Y)$ and consider any $i \in \{1, \ldots, n\} = A$. If $i \in Y$ then $x_i = 1 = x_i'$ by the definitions of Y and f. It $i \notin Y$ then $x_i \neq 1$ so that it has to be that $x_i = 0$ since $x_i \in X = \{0, 1\}$. Also, by the definition of f, we have that $x_i' = 0 = x_i$. Thus in either case $x_i = x_i'$ so that $\mathbf{x} = \mathbf{x}' = f(Y)$ since i was arbitrary. Since $\mathbf{x}$ was arbitrary, this shows that f is surjective.

Therefore f is a bijection from A to X^n as desired. $\square$

(b)

Proof. First, if $A = \varnothing$ then clearly $\mathcal{P}(A) = \mathcal{P}(\varnothing) = \{\varnothing\}$ is finite. So assume in what follows that $A \neq \varnothing$. Since A is finite and nonempty there is a bijection f from A to $B = \{1, \ldots, n\}$ for some $n \in \mathbb{Z}_+$. Let $X = \{0, 1\}$ so that by part (a) there is a bijection g from $\mathcal{P}(B)$ to X^n. For any $Y \in \mathcal{P}(A)$ clearly the mapping $h(Y) = \left\{i \in B \mid f^{-1}(i) \in Y\right\}$ is a bijection from $\mathcal{P}(A)$ to $\mathcal{P}(B)$. It then follows that $g \circ h$ is bijection from $\mathcal{P}(A)$ to X^n. Since clearly X^n is a finite cartesian product of finite sets, it follows from Corollary 6.8 that X^n is finite so that $\mathcal{P}(A)$ must be as well since there is a bijection between them. $\square$

Exercise 6.7

If A and B are finite, show that the set of all functions $f : A \to B$ is finite.

Solution:

Proof. As is customary, denote the set of all functions from A to B by B^A. First, if $A = \varnothing$, then the only function from A to B is the vacuous function $\varnothing$ so that $B^A = \{\varnothing\}$, which is clearly finite. So assume that $A \neq \varnothing$. Then, since A is finite, there is a bijection f from A to $\{1, \ldots, n\}$ for some $n \in \mathbb{Z}_+$, noting that of course f^{-1} is then a bijection from $\{1, \ldots, n\}$ to A.

We construct a bijection h from B^A to B^n. So, for any $g \in B^A$ set $h(g) = g \circ f^{-1}$, noting that clearly this is a function from $\{1, \ldots, n\}$ to B. Hence h is a function from B^A to B^n.

To show that h is injective consider g and g' in B^A where $g \neq g'$. It then follows that there is an $a \in A$ where $g(a) \neq g'(a)$. Then let $k = f(a)$ so that clearly $f^{-1}(k) = a$ and $k \in \{1, \ldots, n\}$. We then have that $(g \circ f^{-1})(k) = g(f^{-1}(k)) = g(a) \neq g'(a) = g'(f^{-1}(k)) = (g' \circ f^{-1})(k)$ so that it must be that $h(g) = g \circ f^{-1} \neq g' \circ f^{-1} = h(g')$. Since g and g' were arbitrary, this shows that h is injective.

Now consider any function $i \in B^n$ and let $g = i \circ f$ so that clearly g is a function from A to B since $f : A \to \{1, \ldots, n\}$ and $i : \{1, \ldots, n\} \to B$. Hence $g \in B^A$, and $h(g) = g \circ f^{-1} = (i \circ f) \circ f^{-1} = i \circ (f \circ f^{-1}) = i$. Since i was arbitrary, this shows that h is surjective as well.

Hence h is bijection from B^A to B^n. Now, since B^n is a finite cartesian product of finite sets (since B is finite), it is finite by Corollary 6.8. Thus it must be that B^A is also finite since there is bijection between them. $\square$

§7 Countable and Uncountable Sets

Exercise 7.1

Show that $\mathbb{Q}$ is countably infinite.

Solution:

Lemma 7.1.1. *The set $\mathbb{Z} \times \mathbb{Z}$ is countably infinite.*

Proof. First, by Example 7.1, the set of integers $\mathbb{Z}$ is countably infinite so that there is a bijection f from $\mathbb{Z}$ to $\mathbb{Z}_+$. We construct a bijection g from $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{Z}_+ \times \mathbb{Z}_+$. For any $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ define $g(a, b) = (f(a), f(b))$, noting that clearly $g(a, b) \in \mathbb{Z}_+ \times \mathbb{Z}_+$ since $\mathbb{Z}_+$ is the range of f. Hence g is a function from $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{Z}_+ \times \mathbb{Z}_+$.

It is easy to show that g is bijective. First, consider any (a, b) and (a', b') in $\mathbb{Z}\times\mathbb{Z}$ where $(a, b) \neq (a', b')$ so that $a \neq a'$ or $b \neq b'$. If $a \neq a'$ then $f(a) \neq f(a')$ since f is bijective (and therefore injective). Thus we have that $g(a, b) = (f(a), f(b)) \neq (f(a'), f(b')) = g(a', b')$. A similar argument shows the same result when $b \neq b'$. Since (a, b) and (a', b') were arbitrary, this shows that g is injective.

Now consider any $(c, d) \in \mathbb{Z}_+\times\mathbb{Z}_+$ so that $c, d \in \mathbb{Z}_+$. Since f is surjective (since it is a bijection) there are $a, b \in \mathbb{Z}$ where $f(a) = c$ and $f(b) = d$. We then clearly have that $g(a, b) = (f(a), f(b)) = (c, d)$ so that g is surjective (c, d) was arbitrary.

Therefore g is a bijection. Now, we know from Corollary 7.4 that $\mathbb{Z}_+ \times \mathbb{Z}_+$ is countably infinite so that there must be a bijection h from $\mathbb{Z}_+ \times \mathbb{Z}_+$ to $\mathbb{Z}_+$. It then follows that $h \circ g$ is bijection from $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{Z}_+$, which shows the desired result by definition. □

Main Problem.

Proof. First we define a straightforward function f from $\mathbb{Z}\times\mathbb{Z}$ to $\mathbb{Q}$. First consider any $(m, n) \in \mathbb{Z}\times\mathbb{Z}$. If $n \neq 0$ then let $q = m/n$. If $n = 0$ then set $q = 0$. Setting $f(m, n) = q$ we clearly have that f is a function from $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{Q}$. Now consider any rational q so that by definition there are integers m and n where $q = m/n$. It then of course follows that $f(m, n) = m/n = q$, which shows that f is surjective since q was arbitrary.

Now, from Lemma 7.1.1 we know that $\mathbb{Z} \times \mathbb{Z}$ is countably infinite so that there is a bijection g from $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{Z}_+$. Hence g^{-1} is a bijection from $\mathbb{Z}_+$ to $\mathbb{Z} \times \mathbb{Z}$. It then follows that the function $f \circ g^{-1}$ is a surjective function from $\mathbb{Z}_+$ to $\mathbb{Q}$. From this it follows from Theorem 7.1 that $\mathbb{Q}$ is countable. Since $\mathbb{Z}_+$ is a subset of $\mathbb{Q}$, it has to be that $\mathbb{Q}$ is infinite, and hence must be countably infinite. □

Exercise 7.2

Show that the maps f and g of Examples 1 and 2 are bijections.

Solution:

It is claimed in Example 7.1 that the function

$$f(n) = \begin{cases} 2n & n > 0 \\ -2n + 1 & n \leq 0 \end{cases}$$

is a bijection from $\mathbb{Z}$ to $\mathbb{Z}_+$.

Proof. To show that f is injective, consider $n, m \in \mathbb{Z}$ where $n \neq m$.

Case: $n > 0$. Then $f(n) = 2n$, which is clearly even. If $m > 0$, then clearly $f(n) = 2n \neq 2m = f(m)$ since $n \neq m$. If $m \leq 0$ then $f(m) = -2m + 1 = 2(-m) + 1$ is clearly odd so that it must be that $f(n) \neq f(m)$.

Case: $n \leq 0$. Then $f(n) = -2n + 1 = 2(-n) + 1$, which is clearly odd. If $m > 0$ then $f(m) = 2m$ is even so that it has to be that $f(n) \neq f(m)$. If $m \leq 0$ then $f(m) = -2m + 1 \neq -2n + 1 = f(n)$ since $n \neq m$.

Thus in every case $f(n) \neq f(m)$, which shows that f is injective since n and m were arbitrary.

To show that f is surjective, consider any $k \in \mathbb{Z}_+$. If k is even then $k = 2n$ for some $n \in \mathbb{Z}_+$. Hence $n > 0$ (since $k > 0$ and $n = k/2$) so that $f(n) = 2n = k$, noting that $n \in \mathbb{Z}$ since $\mathbb{Z}_+ \subset \mathbb{Z}$. If k is odd then $k = 2m - 1$ for some $m \in \mathbb{Z}_+$. So let $n = 1 - m$ so that clearly n is an integer and

$$\begin{aligned} m &\geq 1 && \text{(since } m \in \mathbb{Z}_+\text{)} \\ -m &\leq -1 \\ 1 - m &\leq 0 \\ n &\leq 0\,. \end{aligned}$$

Thus $f(n) = -2n + 1 = -2(1 - m) + 1 = -2 + 2m + 1 = 2m - 1 = k$. This shows that f is surjective since k was arbitrary. Therefore we have shown that f is a bijection as desired. □

Regarding Example 7.2, the following set is defined:

$$A = \{(x, y) \in \mathbb{Z}_+ \times \mathbb{Z}_+ \mid y \leq x\}\ .$$

Then the function f is defined from $\mathbb{Z}_+ \times \mathbb{Z}_+$ to A by

$$f(x, y) = (x + y - 1, y)$$

for $(x, y) \in \mathbb{Z}_+ \times \mathbb{Z}_+$. It is claimed that f is a bijection.

Proof. First, it is not even clear that the range of f is constrained to A, so let us show this. Consider any $(x, y) \in \mathbb{Z}_+ \times \mathbb{Z}_+$ so that $f(x, y) = (x + y - 1, y)$. Since $x \geq 1$ and $y \geq 1$, we have that $x + y \geq 1 + 1 = 2 > 1$ so that $x + y - 1 > 0$ and hence $x + y - 1 \in \mathbb{Z}_+$. Thus clearly $f(x, y) = (x + y - 1, y) \in \mathbb{Z}_+ \times \mathbb{Z}_+$. We also have

$$\begin{aligned} &1 \leq x \\ &0 \leq x - 1 \\ &y \leq x + y - 1\,. \end{aligned}$$

Therefore it is clear that $f(x, y) = (x + y - 1, y) \in A$ by definition.

To show that f is injective consider (x, y) and (x', y') in $\mathbb{Z}_+ \times \mathbb{Z}_+$ where $f(x, y) = (x + y - 1, y) = (x' + y' - 1, y') = f(x', y')$. Thus $x + y - 1 = x' + y' - 1$ and $y = y'$. Therefore $x + y - 1 = x' + y' - 1 = x' + y - 1$, from which it obviously follows that $x = x'$ as well. Then $(x, y) = (x', y')$, which shows that f is injective since (x, y) and (x', y') were arbitrary.

Now consider any $(z, y) \in A$ so that $(z, y) \in \mathbb{Z}_+ \times \mathbb{Z}_+$ and $y \leq z$. Let $x = z - y + 1$ so that clearly $z = x + y - 1$. We also have

$$\begin{aligned} &y \leq z = x + y - 1 \\ &0 \leq x - 1 \\ &1 \leq x \end{aligned}$$

so that $(x, y) \in \mathbb{Z}_+ \times \mathbb{Z}_+$. Since also we have $f(x, y) = (x + y - 1, y) = (z, y)$, f is surjective since (z, y) was arbitrary. This completes the proof that f is a bijection. □

The function g is then defined from A to $\mathbb{Z}_+$ by

$$g(x,y) = \frac{1}{2}(x-1)x + y$$

for $(x,y) \in A$. This is also claimed to be a bijection.

Proof. First we show that the range of g is indeed $\mathbb{Z}_+$ since this is not obvious. Consider any $(x,y) \in A$ so that $(x,y) \in \mathbb{Z}_+ \times \mathbb{Z}_+$ and $y \leq x$. First, if x is even then $x = 2n$ for some $n \in \mathbb{Z}$. Then $g(x,y) = (x-1)x/2 + y = (2n-1)(2n)/2 + y = (2n-1)n + y$, which is clearly an integer. If x is odd then $x = 2n+1$ for some integer n so that

$$g(x,y) = (x-1)x/2 + y = (2n+1-1)(2n+1)/2 + y = (2n)(2n+1)/2 + y = n(2n+1) + y\,,$$

which is also clearly an integer. We also have that $-y < 0$ since $y > 0$ so that

$$\begin{aligned}
x &\geq 1 \\
x - 1 &\geq 0 \\
\frac{1}{2}(x-1) &\geq 0 && \text{(since } 1/2 > 0\text{)} \\
\frac{1}{2}(x-1)x &\geq 0 > -y && \text{(since } x > 0\text{)} \\
\frac{1}{2}(x-1)x + y &> 0 \\
g(x,y) &> 0\,.
\end{aligned}$$

Since we have shown that $g(x,y) \in \mathbb{Z}$ as well, it follows that $g(x,y) \in \mathbb{Z}_+$.

Consider any $(x,y) \in A$ so that $(x,y) \in \mathbb{Z}_+ \times \mathbb{Z}_+$ and $y \leq x$. Then clearly

$$\begin{aligned}
g(x,y) &= \frac{1}{2}(x-1)x + y \leq \frac{1}{2}(x-1)x + x \\
&< \frac{1}{2}(x-1)x + x + 1 = \frac{1}{2}(x^2 - x + 2x) + 1 \\
&= \frac{1}{2}(x^2 + x) + 1 = \frac{1}{2}x(x+1) + 1 \\
&= \frac{1}{2}(x+1-1)(x+1) + 1 \\
&= g(x+1, 1)\,.
\end{aligned}$$

A simple inductive argument shows that $g(x,y) < g(x+n, 1)$ for any $n \in \mathbb{Z}_+$. This was just shown for $n = 1$. Then, assuming it true for n, we have that $g(x,y) < g(x+n,1) < g((x+n)+1,1) = g(x+(n+1),1)$, which completes the induction.

So consider any (x,y) and (x',y') in A so that (x,y) and (x',y') are in $\mathbb{Z}_+ \times \mathbb{Z}_+$, $y \leq x$, and $y' \leq x'$. Also suppose that $(x,y) \neq (x',y')$ so that either $x \neq x'$ or $y \neq y$. If $x = x'$ then it has to be that $y \neq y'$ so that clearly

$$\begin{aligned}
y &\neq y' \\
\frac{1}{2}(x-1)x + y &\neq \frac{1}{2}(x-1)x + y' \\
\frac{1}{2}(x-1)x + y &\neq \frac{1}{2}(x'-1)x' + y' \\
g(x,y) &\neq g(x',y')\,.
\end{aligned}$$

If $x \neq x'$ then we can assume that $x < x'$. Then let $n = x' - x$ so that clearly $n > 0$ and $x' = x + n$. By what was just shown, we have

$$g(x,y) < g(x+n,1) = g(x',1) = \frac{1}{2}(x'-1)x' + 1 \leq \frac{1}{2}(x'-1)x' + y' = g(x',y')$$

since $1 \leq y'$. Thus $g(x,y) \neq g(x',y')$. Since this is true in both cases, this shows that g is injective since (x,y) and (x',y') were arbitrary.

To show that g is also surjective, consider any $z \in \mathbb{Z}_+$. Define the set $B = \{x \in \mathbb{Z}_+ \mid g(x,1) \leq z\}$. First, we have that $g(1,1) = 1 \leq z$ since $z \in \mathbb{Z}_+$ so that $1 \in B$ and therefore $B \neq \varnothing$. If $z = 1$ then clearly $z = 1 \leq 1 = g(1,1) = g(z,1)$. If $z \neq 1$ then we have

$$\begin{aligned}
2 &\leq z \\
1 &\leq \frac{1}{2}z \\
z - 1 &\leq \frac{1}{2}(z-1)z \\
z &\leq \frac{1}{2}(z-1)z + 1 \\
z &\leq g(z,1)
\end{aligned}$$

Now consider any $x, y \in \mathbb{Z}_+$ where $x < y$. It then follows from what was shown above that $g(x,1) \leq g(x,y) < g(x+1,1)$. From this we clearly have that the function $g(x,1)$ is monotonically increasing in x, i.e. for $x, y \in \mathbb{Z}_+$, $x < y$ implies that $g(x,1) < g(y,1)$. By the contrapositive of this, $g(x,1) \geq g(y,1)$ implies that $x \geq y$. With this in mind, consider any $x \in B$ so $g(x,1) \leq z \leq g(z,1)$. Then this implies that $x \leq z$, which shows that z is an upper bound of B since x was arbitrary.

We have thus shown that B is a nonempty set of integers that is bounded above. It then follows from Exercise 4.9 part (a) that B has a largest element x. Now let $y = z - g(x,1) + 1$, noting that, since $x \in B$,

$$\begin{aligned}
g(x,1) &\leq z \\
0 &\leq z - g(x,1) \\
1 &\leq z - g(x,1) + 1 \\
1 &\leq y
\end{aligned}$$

and hence $y \in \mathbb{Z}_+$ so that $(x,y) \in \mathbb{Z}_+ \times \mathbb{Z}_+$. We also must have that $z < g(x+1,1)$ since otherwise we would have that $x + 1 \in B$, which would violate the definition of x as being the largest element of B. Thus we have

$$\begin{aligned}
z &\leq g(x+1,1) - 1 \\
z &\leq \frac{1}{2}(x+1-1)(x+1) + 1 - 1 \\
z &\leq \frac{1}{2}(x+1)x \\
z &\leq x + \frac{1}{2}(x-1)x \\
z &\leq x + \frac{1}{2}(x-1)x + 1 - 1 \\
z &\leq x + g(x,1) - 1 \\
z - g(x,1) + 1 &\leq x \\
y &\leq x
\end{aligned}$$

so that $(x, y) \in A$.

Lastly, since $y = z - g(x,1) + 1$, we clearly have

$$z = y + g(x,1) - 1 = y + \frac{1}{2}(x-1)x + 1 - 1 = \frac{1}{2}(x-1)x + y = g(x,y)\,.$$

This shows that g is surjective since z was arbitrary, thereby completing the long and arduous proof that g is a bijection. □

Exercise 7.3

Let X be the two-element set $\{0,1\}$. Show there is a bijective correspondence between the set $\mathcal{P}(\mathbb{Z}_+)$ and the cartesian product X^ω.

Solution:

Proof. Similar to Exercise 6.6 part (a), we construct such a bijection f from $\mathcal{P}(\mathbb{Z}_+)$ to X^ω. For any $A \in \mathcal{P}(\mathbb{Z}_+)$ we have that $A \subset \mathbb{Z}_+$. Then, for $i \in \mathbb{Z}_+$, set

$$x_i = \begin{cases} 1 & i \in A \\ 0 & i \notin A \end{cases}$$

and set $f(A) = (x_1, x_2, \ldots)$ so that clearly $f(A) \in X^\omega$.

To show that f is injective consider A and A' in $\mathcal{P}(\mathbb{Z}_+)$ where $A \neq A'$. Without loss of generality, we can assume that there is an $i \in A$ where $i \notin A'$, noting that of course $i \in \mathbb{Z}_+$ since $A \subset \mathbb{Z}_+$. Let $\mathbf{x} = (x_1, x_2, \ldots) = f(A)$ and $\mathbf{x}' = (x_1', x_2', \ldots) = f(A')$. Then $x_i = 1 \neq 0 = x_i'$ by the definition of f since $i \in A$ but $i \notin A'$. Thus clearly $f(A) = \mathbf{x} \neq \mathbf{x}' = f(A')$, which shows that f is injective since A and A' were arbitrary.

Now consider any $\mathbf{x} = (x_1, x_2, \ldots) \in X^\omega$ and define the set $A = \{i \in \mathbb{Z}_+ \mid x_i = 1\}$ so that clearly $A \subset \mathbb{Z}_+$ and hence $A \in \mathcal{P}(\mathbb{Z}_+)$. Let $\mathbf{x}' = (x_1', x_2', \ldots) = f(A)$ and consider $i \in \mathbb{Z}_+$. If $i \in A$ then $x_i' = 1 = x_i$ by the definitions of A and f. If $i \notin A$ then $x_i \neq 1$ since otherwise $i \in A$ by definition. Since $x_i \in X = \{0,1\}$ it clearly must be that $x_i = 0$. We then also have that $x_i' = 0$ by the definition of f, and thus $x_i = 0 = x_i'$. Since $x_i = x_i'$ in both cases and i was arbitrary, it follows that $\mathbf{x} = \mathbf{x}' = f(A)$. This proves that f is surjective since $\mathbf{x}$ was arbitrary.

Hence it has been shown that f is a bijection as desired. □

Exercise 7.4

(a) A real number x is said to be ***algebraic*** (over the rationals) if it satisfies some polynomial equation of positive degree

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$$

with rational coefficients a_i. Assuming that each polynomial equation has only finitely many roots, show that the set of algebraic numbers is countable.

(b) A real number is said to be ***transcendental*** if it is not algebraic. Assuming the reals are uncountable, show that the transcendental numbers are uncountable. (It is a somewhat surprising fact that only two transcendental numbers are familiar to us: e and π. Even proving these two numbers transcendental is highly nontrivial.)

Solution:

(a)

Proof. First consider arbitrary degree $n \in \mathbb{Z}_+$. Then for each $\mathbf{q} = (q_1, \dots, q_n) \in \mathbb{Q}^n$, there is a corresponding polynomial equation in x:

$$x^n + \sum_{i=1}^{n} q_i x^{i-1} = x^n + q_n x^{n-1} + \cdots + q_2 x + q_1 = 0\,,$$

which is assumed to have a finite number of solutions. So let $X_{\mathbf{q}}$ be the finite set real numbers that are solutions. (We note that the polynomial corresponding to the vector $\mathbf{q} = (0, \dots, 0) \in \mathbb{Q}^n$ becomes $0 = 0$ so that any real number x satisfies it. Similarly the polynomial corresponding to $\mathbf{q} = (q_1, 0, \dots, 0) \in \mathbb{Q}^n$ for nonzero q_1 corresponds to the equation $q_1 = 0$, which has no solutions. Of course $X_{\mathbf{q}} = \varnothing$ is still finite in this case. For the infinite-solution case we could simply remove the zero vector from $\mathbb{Q}^n$ without changing the argument in any substantial way. This is also taken care of if we really do assume that *any* polynomial has a finite number of solutions as we are evidently doing here.)

Now, we clearly have that $\mathbb{Q}^n$ is countable by Theorem 7.6 since it is a finite product of countable sets (since it was shown in Exercise 7.1 that $\mathbb{Q}$ is countable). Thus the set $A_n = \bigcup_{\mathbf{q} \in \mathbb{Q}^n} X_{\mathbf{q}}$ is countable by Theorem 7.5 since it is a countable union of finite (and therefore countable) sets. Of course, this is the set of all algebraic numbers from polynomials of degree n. Then $A = \bigcup_{n \in \mathbb{Z}_+} A_n$ is the set of all algebraic numbers, which is also then countable by Theorem 7.5 since each A_n was shown to be countable. □

(b)

Proof. As in part (a), let $A \subset \mathbb{R}$ be the set of algebraic numbers so that clearly, by definition, $T = \mathbb{R} - A$ is the set of transcendental numbers. Note that clearly $\mathbb{R} = A \cup T$ so that, if T were countable, then $\mathbb{R}$ would be too since it is a finite union of countable sets. This of course contradicts the (hitherto unproven) fact that $\mathbb{R}$ is uncountable so that it must be that T is also uncountable as desired. □

Exercise 7.5

Determine, for each of the following sets, whether or not it is countable. Justify your answers.

(a) The set A of all functions $f : \{0, 1\} \to \mathbb{Z}_+$.

(b) The set B_n of all functions $f : \{1, \dots, n\} \to \mathbb{Z}_+$.

(c) The set $C = \bigcup_{n \in \mathbb{Z}_+} B_n$.

(d) The set D of all functions $f : \mathbb{Z}_+ \to \mathbb{Z}_+$.

(e) The set E of all functions $f : \mathbb{Z}_+ \to \{0, 1\}$.

(f) The set F of all functions $f : \mathbb{Z}_+ \to \{0, 1\}$ that are "eventually zero." [We say that f is ***eventually zero*** if there is a positive integer N such that $f(n) = 0$ for all $n \geq N$.]

(g) The set G of all functions $f : \mathbb{Z}_+ \to \mathbb{Z}_+$ that are eventually 1.

(h) The set H of all functions $f : \mathbb{Z}_+ \to \mathbb{Z}_+$ that are eventually constant.

(i) The set I of all two-element subsets of $\mathbb{Z}_+$.

(j) The set J of all finite subsets of $\mathbb{Z}_+$.

Solution:

(a) The set A of all functions $f : \{0,1\} \to \mathbb{Z}_+$.

We claim that A is countable.

Proof. For any $f \in A$, clearly the mapping $g(f) = (f(0), f(1))$ is a bijection from A to ${\mathbb{Z}_+}^2$. Since ${\mathbb{Z}_+}^2$ is a finite cartesian product of countable sets, it follows that it is also countable by Theorem 7.6. Hence there is a bijection $h : {\mathbb{Z}_+}^2 \to \mathbb{Z}_+$. It is then obvious that $h \circ g$ is a bijection from A to $\mathbb{Z}_+$ so that A is countable. $\square$

(b) The set B_n of all functions $f : \{1, \ldots, n\} \to \mathbb{Z}_+$.

We claim that B_n (for some $n \in \mathbb{Z}_+$) is also countable.

Proof. By the definition of ${\mathbb{Z}_+}^n$, $B_n = {\mathbb{Z}_+}^n$, which is clearly a finite cartesian product of countable sets. Thus B_n is countable by Theorem 7.6. $\square$

(c) The set $C = \bigcup_{n \in \mathbb{Z}_+} B_n$.

We claim that C is countable.

Proof. Since n was arbitrary in part (b), we showed that B_n is countable for any $n \in \mathbb{Z}_+$. Thus $C = \bigcup_{n \in \mathbb{Z}_+} B_n$ is a countable union of countable sets, which is itself also countable by Theorem 7.5 as desired. $\square$

(d) The set D of all functions $f : \mathbb{Z}_+ \to \mathbb{Z}_+$.

Clearly $D = {\mathbb{Z}_+}^\omega$, which we claim is uncountable.

Proof. We proceed to show, as in Theorem 7.7, that any function $g : \mathbb{Z}_+ \to D$ is not surjective. So denote

$$g(n) = \mathbf{x}_n = (x_{n1}, x_{n2}, \ldots),$$

where of course each $x_{nm} \in \mathbb{Z}_+$ since $\mathbf{x}_n \in D$ and so is a function from $\mathbb{Z}_+$ to $\mathbb{Z}_+$. Now set

$$y_n = \begin{cases} 0 & x_{nn} \neq 0 \\ 1 & x_{nn} = 0 \end{cases}$$

so that clearly $\mathbf{y} = (y_1, y_2, \ldots)$ is an element of D. Now consider any $n \in \mathbb{Z}_+$. If $x_{nn} = 0$ then $y_n = 1 \neq 0 = x_{nn}$, and if $x_{nn} \neq 0$ then $y_n = 0 \neq x_{nn}$. Thus clearly $g(n) = \mathbf{x}_n \neq \mathbf{y}$ since the nth element of each differs. This shows that g cannot be surjective since $\mathbf{y} \in D$ and n was arbitrary. It then follows from Theorem 7.1 that D is not countable. $\square$

(e) The set E of all functions $f : \mathbb{Z}_+ \to \{0,1\}$.

This is exactly the set X^ω in Theorem 7.7, wherein it was shown to be uncountable.

(f) The set F of all functions $f : \mathbb{Z}_+ \to \{0,1\}$ that are "eventually zero." [We say that f is ***eventually zero*** if there is a positive integer N such that $f(n) = 0$ for all $n \geq N$.]

We claim that F is countable.

Proof. For brevity define $X = \{0,1\}$. First let F_N be the set of all eventually zero functions $f : \mathbb{Z}_+ \to X$ that are zero for $n \geq N$, where of course $N \in \mathbb{Z}_+$. Then clearly $F = \bigcup_{N \in \mathbb{Z}_+} F_N$.

We show that each F_N is countable. So consider any $N \in \mathbb{Z}_+$. If $N = 1$ then clearly $f : \mathbb{Z}_+ \to X$ defined by $f(n) = 0$ for $n \in \mathbb{Z}_+$ (which could be denoted $(0, 0, \ldots)$) is the only element of $F_N = F_1$ so that f_N is clearly finite and therefore countable. If $N > 1$ then for $\mathbf{x} = (x_1, \ldots, x_{N-1}) \in X^{N-1}$ define

$$y_n = \begin{cases} x_n & n < N \\ 0 & n \geq N \end{cases}$$

for $n \in \mathbb{Z}_+$. It then trivial to show that g defined by $g(\mathbf{x}) = \mathbf{y} = (y_1, y_2, \ldots)$ is a bijection from X^{N-1} to F_N. Now, since $X = \{0,1\}$ is finite, X^{N-1} is finite by Corollary 6.8. Since this is in bijective correspondence with F_N, it follows that it must also be finite and therefore countable.

Thus $F = \bigcup_{N \in \mathbb{Z}_+} F_N$ is a countable union of countable sets, and so is countable by Theorem 7.5 as desired □

(g) The set G of all functions $f : \mathbb{Z}_+ \to \mathbb{Z}_+$ that are eventually 1.

Since G is clearly a subset of H in part (h) below, it is countable by Corollary 7.3 since H is.

(h) The set H of all functions $f : \mathbb{Z}_+ \to \mathbb{Z}_+$ that are eventually constant.

We claim that H is countable.

Proof. For $N \in \mathbb{Z}_+$, let H_N be the set of functions $f : \mathbb{Z}_+ \to \mathbb{Z}_+$ such that $f(n)$ is constant for $n \geq N$. Thus clearly $H = \bigcup_{N \in \mathbb{Z}_+} H_N$.

We show that each H_N is countable. So consider $N \in \mathbb{Z}_+$. For any $\mathbf{x} = (x_1, \ldots, x_N) \in {\mathbb{Z}_+}^N$ define

$$y_n = \begin{cases} x_n & n < N \\ x_N & n \geq N \end{cases}$$

for $n \in \mathbb{Z}_+$, and set $g(\mathbf{x}) = \mathbf{y} = (y_1, y_2, \ldots)$. It is then a simple matter to show that g is a bijection from ${\mathbb{Z}_+}^N$ to H_N. Then, since ${\mathbb{Z}_+}^N$ is a finite product of countable sets, it is countable by Theorem 7.6. Hence H_N must also be countable since there is a bijective correspondence between them.

Thus $H = \bigcup_{N \in \mathbb{Z}_+} H_N$ is the countable union of countable sets so that it must also be countable by Theorem 7.5. □

(i) The set I of all two-element subsets of $\mathbb{Z}_+$.

In part (j) below it is shown that the set J of all finite subsets of $\mathbb{Z}_+$ is countable. Since clearly $I \subset J$, it follows that I is also countable by Corollary 7.3.

(j) The set J of all finite subsets of $\mathbb{Z}_+$.

We claim that J is countable.

Proof. First, let J_n denote the set of n-element subsets of $\mathbb{Z}_+$ (for $n \in pints$), and let $J_0 = \{\varnothing\}$ since $\varnothing$ is the only "zero-element" subset of $\mathbb{Z}_+$. Clearly then $J = \bigcup_{n \in \mathbb{Z}_+ \cup \{0\}} J_n$. Obviously J_0 is finite and therefore countable. Next, we show that J_n is countable for any $n \in \mathbb{Z}_+$.

So consider any such $n \in \mathbb{Z}_+$. Clearly ${\mathbb{Z}_+}^n$ is countable by Theorem 7.6 since it is a finite product of countable sets. Hence there is a bijection $f : {\mathbb{Z}_+}^n \to \mathbb{Z}_+$. We now construct an injective function $g : J_n \to {\mathbb{Z}_+}^n$. For any $X \in J_n$, we can choose a bijection $h : X \to \{1, \ldots, n\}$ since it has n elements. Since $X \subset \mathbb{Z}_+$, clearly $h^{-1} \in {\mathbb{Z}_+}^n$, so set $g(X) = h^{-1}$. To show that g is injective, consider X and

X' in J_n where $X \neq X'$. Without loss of generality we can assume that there is an $x \in X$ where $x \notin X'$. Let h and h' be the chosen bijections from X and X', respectively, to $\{1, \ldots, n\}$ so that by definition $g(X) = h^{-1}$ and $g(X') = h'^{-1}$. Now let $k = h(x)$ so that $h^{-1}(k) = x$. It has to be that $h'^{-1}(k) \neq x$ since otherwise x would be in X'. Hence $h^{-1}(k) = x \neq h'^{-1}(k)$, which shows that $g(X) = h^{-1} \neq h'^{-1} = g(X')$. Thus g is injective since X and X' were arbitrary. It then follows that $f \circ g$ is an injective function from J_n to $\mathbb{Z}_+$ so that J_n must be countable by Theorem 7.1.

Since n was arbitrary, this shows that J_n is countable for any $n \in \mathbb{Z}_+$. From this it follows from Theorem 7.5 that $J = \bigcup_{n \in \mathbb{Z}_+ \cup \{0\}} J_n$ is also countable since it is clearly a countable union of countable sets. □

Exercise 7.6

We say that two sets A and B ***have the same cardinality*** if there is a bijection of A with B.

(a) Show that if $B \subset A$ and if there is an injection

$$f : A \to B\,,$$

then A and B have the same cardinality. [Hint: Define $A_1 = A$, $B_1 = B$, and for $n > 1$, $A_n = f(A_{n-1})$ and $B_n = f(B_{n-1})$. (Recursive definition again!) Note that $A_1 \supset B_1 \supset A_2 \supset B_2 \supset A_3 \supset \cdots$. Define a bijection $h : A \to B$ by the rule

$$h(x) = \begin{cases} f(x) & \text{if } x \in A_n - B_n \text{ for some } n, \\ x & \text{otherwise.]} \end{cases}$$

(b) *Theorem (Schroeder-Bernstein theorem).* If there are injections $f : A \to C$ and $g : C \to A$, then A and C have the same cardinality.

Solution:

(a)

Proof. Following the hint, we define two sequences of sets recursively:

$$A_1 = A \qquad\qquad B_1 = B$$

and

$$A_n = f(A_{n-1}) \qquad\qquad B_n = f(B_{n-1})$$

for integer $n > 1$. Now define a function from A to B by

$$h(x) = \begin{cases} f(x) & x \in A_n - B_n \text{ for some } n \in \mathbb{Z}_+ \\ x & \text{otherwise} \end{cases}$$

for any $x \in A$.

First we show that B really is the range of h as this is not readily apparent. So consider any $x \in A$. Clearly if $x \in A_n - B_n$ for some $n \in \mathbb{Z}_+$ then $h(x) = f(x) \in B$ since B is the range of f. On the other hand, if this is not the case then $x \notin A_n - B_n$ for any $n \in \mathbb{Z}_+$, and $h(x) = x$. In particular, $x \notin A_1 - B_1 = A - B$ so that it has to be that $h(x) = x \in B$, for otherwise it would be that $x \in A - B$ since $x \in A$. Hence, in either case, $h(x) \in B$ so that h is indeed a function from A to B.

To show that h is injective, consider any $x, x' \in A$ where $x \neq x'$.

1. Case: $x \in A_n - B_n$ for some $n \in \mathbb{Z}_+$. Then by definition $h(x) = f(x)$.
   (a) Case: $x' \in A_m - B_m$ for some $m \in \mathbb{Z}_+$. Then we clearly have $h(x) = f(x) \neq f(x') = h(x')$ since f is injective and $x \neq x'$.
   (b) Case: $x' \notin A_m - B_m$ for all $m \in \mathbb{Z}_+$. Then $h(x') = x'$. Since $x \in A_n$, we have that $f(x) \in f(A_n) = A_{n+1}$. If it were the case that $f(x) \in B_{n+1} = f(B_n)$, then there would be a $y \in B_n$ such that $f(y) = f(x)$. Of course, since f is injective, it would have to be that $x = y \in B_n$, which we know is not the case since $x \in A_n - B_n$. Hence it has to be that $f(x) \notin B_{n+1}$ so that $f(x) \in A_{n+1} - B_{n+1}$. From this it is clearly that it cannot be that $x' = f(x)$ so that $h(x') = x' \neq f(x) = h(x)$.
2. Case: $x \notin A_n - B_n$ for all $n \in \mathbb{Z}_+$. Then by definition $h(x) = x$.
   (a) Case: $x' \in A_m - B_m$ for some $m \in \mathbb{Z}_+$. This is the same as case 1b above with the roles of x and x' reversed.
   (b) Case: $x' \notin A_m - B_m$ for all $m \in \mathbb{Z}_+$. Then clearly $h(x) = x \neq x' = h(x')$.

Thus in all cases $h(x) \neq h(x')$, which shows that h is injective since x and x' were arbitrary.

To show that h is also surjective, consider any $y \in B$, noting that also $y \in A$ since $B \subset A$.

Case: $y \in A_n - B_n$ for some $n \in \mathbb{Z}_+$. It cannot be that $n = 1$ since then $y \in A_1 - B_1 = A - B$, and we know that $y \in B$. Hence $n > 1$ so that $n - 1 \in \mathbb{Z}_+$. Since $y \in A_n = f(A_{n-1})$, there is an $x \in A_{n-1}$ where $f(x) = y$. Suppose for a moment that $x \in B_{n-1}$ so that $y = f(x) \in f(B_{n-1}) = B_n$, which we know not to be the case. Thus it must be that $x \notin B_{n-1}$ so that $x \in A_{n-1} - B_{n-1}$ and so by definition $h(x) = f(x) = y$.

Case: $y \notin A_n - B_n$ for all $n \in \mathbb{Z}_+$. Then clearly $h(y) = y$ by definition.

This shows that h is surjective since y was arbitrary.

Therefore it has been shown that h is a bijection from A to B, which shows that they have the same cardinality by definition. □

(b)

Proof. Clearly f is a bijection from A to $f(A)$ since f is injective. Also, clearly the function $g \circ f$ is an injective function from C into $f(A)$ since both f and g are injective. Noting that obviously $f(A) \subset C$, it then follows from part (a) that C and $f(A)$ have the same cardinality so that there is a bijection $h : f(A) \to C$. We then have that $h \circ f$ is a bijection from A to C so that they have the same cardinality by definition. □

Exercise 7.7

Show that the sets D and E of Exercise 7.5 have the same cardinality.

Solution:

Throughout what follows let A^B denote the set of all functions from set A to set B.

Lemma 7.7.1. *If there is an injection from A_1 to A_2 with $A_2 \neq \varnothing$, and an injection from B_1 to B_2, then there is also an injection from $A_1^{B_1}$ to $A_2^{B_2}$.*

Proof. Since $A_2 \neq \varnothing$, there is an $a_2 \in A_2$. Since we know they exist, let $f_A : A_1 \to A_2$ and $f_B : B_1 \to B_2$ be injections. We construct an injection $F : A_1^{B_1} \to A_2^{B_2}$. So, for any $g \in A_1^{B_1}$, define

$F(g) = h$, where $h \in A_2^{B_2}$ is defined by

$$h(b) = \begin{cases} (f_A \circ g \circ f_B^{-1})(b) & b \in f_B(B_1) \\ a_2 & b \notin f_B(B_1) \end{cases}$$

for $b \in B_2$, noting that f_B^{-1} is a function with domain $f_B(B_1)$ since it is injective.

To show that F is injective, consider $g_1, g_2 \in A_1^{B_1}$ where $g_1 \neq g_2$. Then there is a $b_1 \in B_1$ where $g_1(b_1) \neq g_2(b_1)$. Let $b_2 = f_B(b_1)$ so that clearly $b_2 \in f_B(B_1)$ and $b_1 = f_B^{-1}(b_2)$. Then clearly

$$\begin{aligned} F(g_1)(b_2) &= (f_A \circ g_1 \circ f_B^{-1})(b_2) = f_A(g_1(f_B^{-1}(b_2))) = f_A(g_1(b_1)) \\ &\neq f_A(g_2(b_1)) = f_A(g_2(f_B^{-1}(b_2))) = (f_A \circ g_2 \circ f_B^{-1})(b_2) \\ &= F(g_2)(b_2) \end{aligned}$$

since $g_1(b_1) \neq g_2(b_1)$ and f_A is injective. Thus $F(g_1) \neq F(g_2)$, which shows that F is injective since g_1 and g_2 were arbitrary. □

Lemma 7.7.2. *For sets A, B, and C, the set $(A^B)^C$ has the same cardinality as the set $A^{B\times C}$.*

Proof. We construct a bijection $F : A^{B\times C} \to (A^B)^C$. So, for any $f \in A^{B\times C}$, we have that $f : B \times C \to A$. Define $g : C \to A^B$ by $g(c) = h$ for any $c \in C$, where $h : B \to A$ is defined by $h(b) = f(b, c)$. Then assign $F(f) = g$.

To show that F is injective, consider $f, f' \in A^{B\times C}$ where $f \neq f'$. Then there is a $(b, c) \in B \times C$ where $f(b, c) \neq f'(b, c)$. Also let $g = F(f)$, $g' = F(f')$, $h = g(c)$, and $h' = g'(c)$. Then, by definition, we have $h(b) = f(b, c) \neq f'(b, c) = h'(b)$ so that $g(c) = h \neq h' = g'(c)$. From this it follows that $F(f) = g \neq g' = F(f')$, which shows that F is injective since f and f' were arbitrary.

Now consider any $g \in (A^B)^C$ and any $(b, c) \in B \times C$. Let $h = g(c) \in A^B$, and then assign $f(b, c) = h(b)$. Clearly then $f : B \times C \to A$ so that $f \in A^{B\times C}$. So let $g' = F(f)$ and consider any $c \in C$. Let $h = g(c)$ and $h' = g'(c)$ so that $h'(b) = f(b, c)$ by the definition of F. Consider any $b \in B$ so that $h(b) = f(b, c) = h'(b)$ by the definition of f. Since b was arbitrary, this shows that $g(c) = h = h' = g'(c)$. Since c was also arbitrary, this shows that $F(f) = g' = g$. Lastly, since g was arbitrary, this shows that F is surjective. □

Main Problem.

Recall that we have $D = {\mathbb{Z}_+}^\omega = {\mathbb{Z}_+}^{\mathbb{Z}_+}$ and $E = X^\omega = X^{\mathbb{Z}_+}$, where we let $X = \{0, 1\}$. We show that these have the same cardinality.

Proof. First consider any $f \in E = X^{\mathbb{Z}_+}$. Then define $g(n) = f(n) + 1$ for $n \in \mathbb{Z}_+$ so that clearly $g \in {\mathbb{Z}_+}^{\mathbb{Z}_+} = D$. Now define the function $h : E \to D$ by $h(f) = g$. It is then trivial to show that h is an injection.

Now, for $n \in \mathbb{Z}_+$, define $x_n = 1$ and $x_i = 0$ when $i \in \mathbb{Z}_+$ and $i \neq n$. Clearly then $\mathbf{x} = (x_1, x_2, \ldots) \in X^{\mathbb{Z}_+}$, and it is easily shown that the function defined by $f(n) = \mathbf{x}$ is an injection from $\mathbb{Z}_+$ to $X^{\mathbb{Z}_+}$ Also clearly the identity function on $\mathbb{Z}_+$ is an injection since it is a bijection. It then follows from Lemma 7.7.1 that there is an injection $f_1 : {\mathbb{Z}_+}^{\mathbb{Z}_+} \to (X^{\mathbb{Z}_+})^{\mathbb{Z}_+}$, noting that clearly $X^{\mathbb{Z}_+} \neq \varnothing$.

We presently have that there is a bijection $f_2 : (X^{\mathbb{Z}_+})^{\mathbb{Z}_+} \to X^{\mathbb{Z}_+\times\mathbb{Z}_+}$ by Lemma 7.7.2, which is of course also an injection. Finally, since $\mathbb{Z}_+ \times \mathbb{Z}_+$ has the same cardinality as $\mathbb{Z}_+$ (by Corollary 7.4), it follows that there is an injection from $\mathbb{Z}_+ \times \mathbb{Z}_+$ to $\mathbb{Z}_+$. Since also the identity function on X is an injection, we have again that there is an injection $f_3 : X^{\mathbb{Z}_+\times\mathbb{Z}_+} \to X^{\mathbb{Z}_+}$ by Lemma 7.7.1. Thus clearly then $f_3 \circ f_2 \circ f_1$ is an injection from ${\mathbb{Z}_+}^{\mathbb{Z}_+} = D$ to $X^{\mathbb{Z}_+} = E$.

Therefore, since there is an injection from E to D as well as from D to E, it follows from Exercise 7.6 part (b) that D and E have the same cardinality as desired. □

Exercise 7.8

Let X denote the two-element set $\{0, 1\}$; let $\mathcal{B}$ be the set of *countable* subsets of X^ω. Show that X^ω and $\mathcal{B}$ have the same cardinality.

Solution:

Again let A^B denote the set of functions from A to B.

Proof. First, for $\mathbf{x} \in X^\omega$, clearly the function that maps $\mathbf{x}$ to the set $\{\mathbf{x}\}$ is an injective function from X^ω to $\mathcal{B}$.

Now we construct an injection $f_1 : \mathcal{B} \to (X^\omega)^\omega$. So consider any $S \in \mathcal{B}$ so that S is a countable subset of X^ω. Then, by Theorem 7.1, we can choose a surjective function $g : \mathbb{Z}_+ \to S$. Note that this does require the Axiom of Choice since we must choose such a surjection for each $S \in \mathcal{B}$, and clearly $\mathcal{B}$ is infinite. Since $S \subset X^\omega$, g can be considered as a function from $\mathbb{Z}_+$ to X^ω so that $g \in (X^\omega)^\omega$, though of course it would no longer necessarily be surjective with this range. So we simply set $f_1(S) = g$

To show that f_1 is injective consider $S, S' \in \mathcal{B}$ where $S \neq S'$. Then, setting $g = f_1(S)$ and $g' = f_1(S')$, we have that $g(\mathbb{Z}_+) = S$ and $g'(\mathbb{Z}_+) = S'$ by definition. Since $S \neq S'$, we have that g and g' have the same domain but different image sets. Clearly this means that $f_1(S) = g \neq g' = f_1(S')$, which shows that f_1 is injective since S and S' were arbitrary.

Hence f_1 is an injection from $\mathcal{B}$ to $(X^\omega)^\omega = (X^{\mathbb{Z}_+})^{\mathbb{Z}_+}$. Now, from Lemma 7.7.2, we have that $(X^{\mathbb{Z}_+})^{\mathbb{Z}_+}$ has the same cardinality as $X^{\mathbb{Z}_+ \times \mathbb{Z}_+}$ so that there is a bijection $f_2 : (X^{\mathbb{Z}_+})^{\mathbb{Z}_+} \to X^{\mathbb{Z}_+ \times \mathbb{Z}_+}$, which is of course also an injection. Finally, since $\mathbb{Z}_+ \times \mathbb{Z}_+$ has the same cardinality as $\mathbb{Z}_+$ (by Corollary 7.4), it follows that there is an injection from $\mathbb{Z}_+ \times \mathbb{Z}_+$ to $\mathbb{Z}_+$. Since also the identity function on X is an injection, we have that there is an injection $f_3 : X^{\mathbb{Z}_+ \times \mathbb{Z}_+} \to X^{\mathbb{Z}_+}$ by Lemma 7.7.1. Then clearly $f_3 \circ f_2 \circ f_1$ is an injection from $\mathcal{B}$ to $X^{\mathbb{Z}_+} = X^\omega$.

Since there is an injection from X^ω to $\mathcal{B}$ and vice-versa, it follows that they have the same cardinality by Exercise 7.6 part (b) as desired. □

Exercise 7.9

(a) The formula

$$(*) \qquad \begin{aligned} h(1) &= 1\,, \\ h(2) &= 2\,, \\ h(n) &= [h(n+1)]^2 - [h(n-1)]^2 \qquad \text{for } n \geq 2 \end{aligned}$$

is not one to which the principle of recursive definition applies. Show that nevertheless there does exist a function $h : \mathbb{Z}_+ \to \mathbb{R}$ satisfying this formula. [Hint: Reformulate $(*)$ so that the principle will apply and require h to be positive.]

(b) Show that the formula $(*)$ of part (a) does not determine h uniquely. [Hint: If h is a positive function satisfying $(*)$, let $f(i) = h(i)$ for $i \neq 3$, and let $f(3) = -h(3)$.]

(c) Show that there is no function $h : \mathbb{Z}_+ \to \mathbb{R}$ satisfying the formula

$$\begin{aligned} h(1) &= 1\,, \\ h(2) &= 2\,, \\ h(n) &= [h(n+1)]^2 + [h(n-1)]^2 \qquad \text{for } n \geq 2. \end{aligned}$$

Solution:

(a) First, notice that $(*)$ does not satisfy the principle of recursive definition because, for $n \geq 2$, $h(n)$ is not defined strictly in terms of values of h for positive integers less than n, since its definition depends on $h(n+1)$. Now we show that there does exists a function satisfying $(*)$.

Proof. Consider the following reformulation:

$$\begin{aligned} h(1) &= 1\,, \\ h(2) &= 2\,, \\ h(n) &= \sqrt{h(n-1) + [h(n-2)]^2} \qquad \text{for } n > 2\,, \end{aligned}$$

where as is convention we take the positive square root for $h(n)$. Clearly for $n \in \{1, 2\}$ we have that $h(n)$ is positive. Now suppose $n > 2$ and that $h(k)$ is positive for $k < n$ so that $h(n-1)$ and $h(n-2)$ are both positive. Then clearly $h(n-1) + [h(n-2)]^2$ is positive so that $h(n) = \sqrt{h(n-1) + [h(n-2)]^2}$ is defined and is positive. Hence $h(n)$ is positive and well-defined for all $n \in \mathbb{Z}_+$ by induction.

Thus, since $h(n)$ depends only on values of h for integers less than n, this satisfies the recursion principle so that a unique h satisfying the above exists. We also claim that this h satisfies $(*)$. Clearly the explicitly defined values of $h(1)$ and $h(2)$ are satisfied. For $n \geq 2$, we have that $n+1 > 2$ so that, by definition,

$$\begin{aligned} h(n+1) = \sqrt{h((n+1)-1) + [h((n+1)-2)]^2} &= \sqrt{h(n) + [h(n-1)]^2} \\ [h(n+1)]^2 &= h(n) + [h(n-1)]^2 \\ h(n) &= [h(n+1)]^2 - [h(n-1)]^2\,, \end{aligned}$$

which is the final constraint of $(*)$ so that it is also satisfied since $n \geq 2$ was arbitrary. □

(b) First note that, for the recursively defined function h from part (a),

$$\begin{aligned} h(3) &= \sqrt{h(2) + [h(1)]^2} = \sqrt{2 + 1^2} = \sqrt{3} \\ h(4) &= \sqrt{h(3) + [h(2)]^2} = \sqrt{\sqrt{3} + 2^2} = \sqrt{\sqrt{3} + 4}\,. \end{aligned}$$

Now define the function f as in the hint, that is $f(i) = h(i)$ for $i \neq 3$ and $f(3) = -h(3)$. Then we clearly have $f(3) = -h(3) = -\sqrt{3}$ while

$$[f(4)]^2 - [f(2)]^2 = [h(4)]^2 - [h(2)]^2 = \left(\sqrt{\sqrt{3} + 4}\right)^2 - 2^2 = \sqrt{3} + 4 - 4 = \sqrt{3}$$

so that $f(n) = -\sqrt{3} \neq \sqrt{3} = [f(n+1)]^2 - [f(n-1)]^2$ for $n = 3$, and hence $(*)$ is violated. So it would seem that the hint as given does not exactly work.

Now we show that the function satisfying $(*)$ is not unique, taking inspiration from the hint.

Proof. We construct a function f, different from h from part (a), that also satisfies $(*)$. We define f using recursion:

$$\begin{aligned} f(1) &= 1\,, \\ f(2) &= 2\,, \\ f(3) &= -\sqrt{3}\,, \\ f(n) &= \sqrt{f(n-1) + [f(n-2)]^2} \qquad \text{for } n > 3\,. \end{aligned}$$

Clearly, since each $f(n)$ is defined only in terms of $f(k)$ for $k < n$ (or without dependence on any values of f), f exists uniquely by the recursion principle so long as each $f(n)$ is well-defined. We show this presently by induction.

Clearly $f(n)$ is defined for $n \in \{1, 2, 3\}$. For $n = 4$ we have $f(n) = f(4) = \sqrt{f(3) + [f(2)]^2} = \sqrt{-\sqrt{3} + 2^2}\sqrt{-\sqrt{3} + 4}$. Now, since $1 < 3$, we have that $\sqrt{3} < 3 < 4$ so that $-\sqrt{3} + 4 = 4 - \sqrt{3} > 0$ and hence the square root, and therefore $f(4)$, is defined and positive. Now consider any $n > 4$ and suppose that $f(n-1)$ is positive. Then clearly $f(n) = \sqrt{f(n-1) + [f(n-2)]^2}$ is defined and positive since $f(n-1) > 0$, noting that even if $f(n-2) \leq 0$, its square is non-negative.. This completes the induction that shows that f is uniquely defined.

Clearly $f \neq h$ since $f(3) = -\sqrt{3} \neq \sqrt{3} = h(3)$. Also obviously $f(n)$ satisfies $(*)$ explicitly for $n \in \{1, 2\}$. For $n = 2$ we have

$$[f(n+1)]^2 - [f(n-1)]^2 = [f(3)]^2 - [f(1)]^2 = [-\sqrt{3}]^2 - 1^2 = 3 - 1 = 2 = f(n)\,.$$

Then, for $n > 2$ we have $n + 1 > 3$ so that, by definition,

$$\begin{aligned} f(n+1) = \sqrt{f((n+1)-1) + [f((n+1)-2)]^2} &= \sqrt{f(n) + [f(n-1)]^2} \\ [f(n+1)]^2 &= f(n) + [f(n-1)]^2 \\ f(n) &= [f(n+1)]^2 - [f(n-1)]^2\,. \end{aligned}$$

Thus the recursive part of $(*)$ holds for $n \geq 2$ so that $(*)$ holds over the whole domain of f as desired. □

(c)

Proof. Suppose that such a function h does exist. Since the recursive property holds for $n \geq 2$, we have

$$\begin{aligned} h(2) &= [h(3)]^2 + [h(1)]^2 \\ 2 &= [h(3)]^2 + 1^2 \\ [h(3)]^2 &= 2 - 1^2 = 1 \\ h(3) &= \pm 1\,. \end{aligned}$$

Similarly, we have

$$\begin{aligned} h(3) &= [h(4)]^2 + [h(2)]^2 \\ \pm 1 &= [h(4)]^2 + 2^2 \\ [h(4)]^2 &= \pm 1 - 2^2 = \pm 1 - 4 \end{aligned}$$

so that either $[h(4)]^2 = 1 - 4 = -3$ or $[h(4)]^2 = -1 - 4 = -5$. In either case we have $[h(4)]^2 < 0$, which is of course impossible since the square of a real number is always non-negative! So it must be that such a function does not exist. □

§8 The Principle of Recursive Definition

Exercise 8.1

Let $(b_1, b_2, \ldots)$ be an infinite sequence of real numbers. The sum $\sum_{k=1}^{n} b_k$ is defined by induction as follows:

$$\sum_{k=1}^{n} b_k = b_1 \qquad \text{for } n = 1,$$

$$\sum_{k=1}^{n} b_k = \left(\sum_{k=1}^{n-1} b_k\right) + b_n \qquad \text{for } n > 1.$$

Let A be the set of real numbers; choose ρ so that Theorem 8.4 applies to define this sum rigorously. We sometimes denote the sum $\sum_{k=1}^{n} b_k$ by the symbol $b_1 + b_2 + \cdots + b_n$.

Solution:

For a function $f : \{1, \ldots, m\} \to A$, define $\rho(f) = f(m) + b_{m+1}$. For clarity, denote the sum function by $s : \mathbb{Z}_+ \to A$ so that $s(n) = \sum_{k=1}^{n} b_k$. Then by Theorem 8.4 there is a unique $s : \mathbb{Z}_+ \to A$ such that

$$s(1) = b_1\,,$$

$$s(n) = \rho(s \restriction \{1, \ldots, n-1\}) \qquad \text{for } n > 1.$$

Then we clearly have that $\sum_{k=1}^{1} b_k = s(1) = b_1$ and

$$\sum_{k=1}^{n} b_k = s(n) = \rho(s \restriction \{1, \ldots, n-1\}) = s(n-1) + b_{(n-1)+1} = \sum_{k=1}^{n-1} b_k + b_n$$

for $n > 1$ as desired.

Exercise 8.2

Let $(b_1, b_2, \ldots)$ be an infinite sequence of real numbers. We define the product $\prod_{k=1}^{n} b_k$ by the equations

$$\prod_{k=1}^{1} b_k = b_1\,,$$

$$\prod_{k=1}^{n} b_k = \left(\prod_{k=1}^{n-1} b_k\right) \cdot b_n \qquad \text{for } n > 1.$$

Use Theorem 8.4 to define the product rigorously. We sometimes denote the product $\prod_{k=1}^{n} b_k$ by the symbol $b_1 b_2 \cdots b_n$.

Solution:

First, for any function $f : \{1, \ldots, m\} \to \mathbb{R}$, define ρ by $\rho(f) = f(m) \cdot b_{m+1}$. Then, by the recursion theorem (Theorem 8.4), there is a unique function $p : \mathbb{Z}_+ \to \mathbb{R}$ such that

$$p(1) = b_1\,,$$

$$p(n) = \rho(p \restriction \{1, \ldots, n-1\}) \qquad \text{for } n > 1.$$

Then we define $\prod_{k=1}^{n} b_k = p(n)$ so that we have $\prod_{k=1}^{1} b_k = p(1) = b_1$ and

$$\prod_{k=1}^{n} b_k = p(n) = \rho(p \restriction \{1, \ldots, n-1\}) = p(n-1) \cdot b_{(n-1)+1} = \left(\prod_{k=1}^{n-1} b_k\right) \cdot b_n$$

for $n > 1$ as desired.

Exercise 8.3

Obtain the definitions of a^n and $n!$ for $n \in \mathbb{Z}_+$ as special cases of Exercise 8.2.

Solution:

Regarding a^n, defined the sequence $(b_1, b_2, \ldots)$ by $b_i = a$ for every $i \in \mathbb{Z}_+$, which we could denote by $(a, a, \ldots)$. Then define $a^n = \prod_{k=1}^{n} b_k$ as it is defined in Exercise 8.2, and we claim that this satisfies the inductive definition given in Exercise 4.6 and Example 8.2.

Proof. First, we clearly have $a^1 = \prod_{k=1}^{1} b_k = b_1 = a$. Next, for $n > 1$, we have

$$*a^n = \prod_{k=1}^{n} b_k = \left(\prod_{k=1}^{n-1} b_k \right) \cdot b_n = a^{n-1} \cdot a \,,$$

which shows that the inductive definition is satisfied. □

Since it does not seem to be given in the book thus far, we reiterate the typical inductive definition for $n!$:

$$\begin{aligned} 1! &= 1 \,, \\ n! &= (n-1)! \cdot n \qquad \text{for } n > 1. \end{aligned}$$

Now, define the sequence $(b_1, b_2, \ldots)$ by $b_i = i$ for $i \in \mathbb{Z}_+$. We then claim that defining $n! = \prod_{k=1}^{n} b_k$ as defined in Exercise 8.2 satisfies this definition.

Proof. First, we have $1! = \prod_{k=1}^{1} b_k = b_1 = 1$. Then we also have

$$n! = \prod_{k=1}^{n} b_k = \left(\prod_{k=1}^{n-1} b_k \right) \cdot b_n = (n-1)! \cdot n$$

for $n > 1$ so that the definition is clearly satisfied. □

Exercise 8.4

The *Fibonacci numbers* of number theory are defined recursively by the formula

$$\begin{aligned} \lambda_1 &= \lambda_2 = 1 \,, \\ \lambda_n &= \lambda_{n-1} + \lambda_{n-2} \qquad \text{for } n > 2. \end{aligned}$$

Define them rigorously by use of Theorem 8.4.

Solution:

First, note that the Fibonacci numbers are all positive integers. So, for any function $f : \{1, \ldots, m\} \to \mathbb{Z}_+$ define

$$\rho(f) = \begin{cases} 1 & m = 1 \\ f(m) + f(m-1) & m > 1 \,, \end{cases}$$

noting that clearly the range of ρ is still $\mathbb{Z}_+$ since that is the range of f. Then, by Theorem 8.4, there is a unique function $F : \mathbb{Z}_+ \to \mathbb{Z}_+$ such that

$$F(1) = 1 \,,$$

$$F(n) = \rho(F \restriction \{1, \ldots, n-1\}) \qquad \text{for } n > 1.$$

We claim that the Fibonacci numbers are $\lambda_n = F(n)$ for $n \in \mathbb{Z}_+$.

Proof. To show that the numbers λ_n satisfy the inductive definition of the Fibonacci numbers, first note that we clearly have $\lambda_1 = F(1) = 1$. We also have that

$$\lambda_2 = F(2) = \rho(F \restriction \{1\}) = 1\,.$$

Lastly, for any $n > 2$, clearly $n > 1$ also and $n - 1 > 1$ so that

$$\lambda_n = F(n) = \rho(F \restriction \{1, \ldots, n-1\}) = F(n-1) + F([n-1]-1) = \lambda_{n-1} + \lambda_{n-2}\,,$$

which shows that the inductive definition is satisfied. □

Exercise 8.5

Show that there is a unique function $h : \mathbb{Z}_+ \to \mathbb{R}_+$ satisfying the formula

$$h(1) = 3\,,$$
$$h(i) = [h(i-1) + 1]^{1/2} \qquad \text{for } i > 1.$$

Solution:

Proof. First, for any function $f : \{1, \ldots, m\} \to \mathbb{R}_+$, define

$$\rho(f) = [f(m) + 1]^{1/2}\,.$$

Consider any $m \in \mathbb{Z}_+$ and any function $f : \{1, \ldots, m\} \to \mathbb{R}_+$. Since $f(m) \in \mathbb{R}_+$, it follows that $f(m) + 1 \in \mathbb{R}_+$ also so that $\rho(f) = [f(m)+1]^{1/2}$ is defined and is positive. Hence ρ is a well-defined function with range $\mathbb{R}_+$ since m and f were arbitrary. It then follows from the principle of recursive definition (Theorem 8.4) that there is a unique function $h : \mathbb{Z}_+ \to \mathbb{R}_+$ such that

$$h(1) = 3\,,$$
$$h(n) = \rho(h \restriction \{1, \ldots, n-1\}) \qquad \text{for } n > 1.$$

It is easy to see that this h satisfies the required property since $h(1) = 3$ and

$$h(i) = \rho(h \restriction \{1, \ldots, i-1\}) = [h(i-1) + 1]^{1/2}$$

for $i > 1$ as desired.

Now we show that such a function is unique. Suppose that g and h both satisfy the inductive formula. We show by induction that $g(i) = h(i)$ for all $i \in \mathbb{Z}_+$, from which it clearly follows that $g = h$. First, we clearly have $g(1) = 3 = h(1)$. Now suppose that $g(i) = h(i)$ for $i \in \mathbb{Z}_+$. Then we have that $i + 1 > 1$ so that $g(i+1) = [g(i)+1]^{1/2} = [h(i)+1]^{1/2} = h(i+1)$ since $g(i) = h(i)$ and we are taking the positive root. This completes the induction. □

Exercise 8.6

(a) Show that there is no function $h : \mathbb{Z}_+ \to \mathbb{R}_+$ satisfying the formula

$$\begin{aligned} h(1) &= 3\,, \\ h(i) &= [h(i-1)-1]^{1/2} \qquad \text{for } i > 1. \end{aligned}$$

Explain why this example does not violate the principle of recursive definition.

(b) Consider the recursion formula

$$h(1) = 3\,,$$
$$h(i) = \begin{cases} [h(i-1)-1]^{1/2} & \text{if } h(i-1) > 1 \\ 5 & \text{if } h(i-1) \le 1 \end{cases} \qquad \text{for } i > 1.$$

Show that there exists a unique function $h : \mathbb{Z}_+ \to \mathbb{R}_+$ satisfying this formula.

Solution:

(a)

Proof. Suppose to the contrary that there is such a function h. Then clearly $h(1) = 3$ and $h(2) = \sqrt{h(1)-1} = \sqrt{3-1} = \sqrt{2}$. Now, since $1 < 2 < 4$, we clearly have $1 < \sqrt{2} < \sqrt{4} = 2$. Thus $0 < \sqrt{2} - 1 < 1$ so that $h(3) = \sqrt{h(2)-1} = \sqrt{\sqrt{2}-1}$ is defined. However, we also have that $0 < h(3) = \sqrt{\sqrt{2}-1} < 1$ since $0 < \sqrt{2} - 1 < 1$, and hence $h(3) - 1 < 0$. We then have that

$$\begin{aligned} h(4) &= \sqrt{h(3)-1} \\ [h(4)]^2 &= h(3) - 1 < 0\,, \end{aligned}$$

which is of course impossible since a square is always non-negative. This contradiction shows that such a function h cannot exist. □

Note that this does not ostensibly violate the principle of recursive definition since $h(n)$ is defined only in terms of values of h less than n for $n > 1$. However, were one to try to show the existence of h rigorously using the principle, one would find that the required function ρ would not be well-defined.

(b)

Proof. First, for any function $f : \{1, \ldots, m\} \to \mathbb{R}_+$, define

$$\rho(f) = \begin{cases} [f(m)-1]^{1/2} & f(m) > 1 \\ 5 & f(m) \le 1\,. \end{cases}$$

Consider any $m \in \mathbb{Z}_+$ and any function $f : \{1, \ldots, m\} \to \mathbb{R}_+$. If $f(m) > 1$ then clearly $f(m) - 1 > 0$ so that $\rho(f) = [f(m)-1]^{1/2}$ is defined and positive. If $f(m) \le 1$ then clearly $\rho(f) = 5$ is also defined and positive. Since m and f were arbitrary, this shows that ρ is a well-defined function with range $\mathbb{R}_+$.

It then follows from the principle of recursive definition (Theorem 8.4) that there is a unique function $h : \mathbb{Z}_+ \to \mathbb{R}_+$ such that

$$\begin{aligned} h(1) &= 3\,, \\ h(n) &= \rho(h \restriction \{1, \ldots, n-1\}) \qquad \text{for } n > 1. \end{aligned}$$

To see that this h satisfies the recursion formula, clearly $h(1) = 3$, and, for $i > 1$, we have

$$h(i) = \rho(h \restriction \{1, \ldots, i-1\}) = \begin{cases} [h(i-1) - 1]^{1/2} & h(i-1) > 1 \\ 5 & h(i-1) \leq 1 \end{cases}$$

as desired.

To show that this function is unique, suppose that g and h both satisfy the recursive formula. We show by induction that $g(n) = h(n)$ for all $n \in \mathbb{Z}_+$ so that clearly $g = h$. First, obviously $g(1) = 3 = h(1)$. Now suppose that $g(n) = h(n)$ for $n \in \mathbb{Z}_+$ so that $n + 1 > 1$. Then, if $g(n) = h(n) > 1$ then we have $g(n+1) = [g(n) - 1]^{1/2} = [h(n) - 1]^{1/2} = h(n+1)$ since $g(n) = h(n)$ and the roots are taken to be positive. Similarly, if $g(n) = h(n) \leq 1$, then $g(n+1) = 5 = h(n+1)$. Thus in either case $g(n+1) = h(n+1)$, which completes the induction. □

Exercise 8.7

Prove Theorem 8.4.

Solution:

The proof follows the same pattern used to prove $(*)$ at the beginning of the section, which culminates in Theorem 8.3. Similar to that approach, two lemmas will be proved first. In what follows, $(*)$ refers to the properties defined in the statement of Theorem 8.4.

Lemma 8.7.1. *Given $n \in \mathbb{Z}_+$, there exists a function $f : \{1, \ldots, n\} \to A$ that satisfies $(*)$ for all i in its domain.*

Proof. We show this by induction on n. First, for $n = 1$, clearly the function $f : \{1\} \to A$ defined by $f(1) = a_0$ satisfies $(*)$. Now suppose that $(*)$ holds for some function $f' : \{1, \ldots, n\} \to A$ for $n \in \mathbb{Z}_+$. Now define $f : \{1, \ldots, n+1\} \to A$ by

$$f(i) = \begin{cases} f'(i) & i \in \{1, \ldots, n\} \\ \rho(f') & i = n + 1 \end{cases}$$

for any $i \in \{1, \ldots, n+1\}$. Note that f is not defined in terms of itself, but in terms of f' and ρ.

First, we clearly have $f' = f \restriction \{1, \ldots, n\}$ since $f(i) = f'(i)$ for all $i \in \{1, \ldots, n\}$. Then, clearly $f(1) = f'(1) = a_0$ since $1 \leq n$ and f' satisfies $(*)$. Consider any $i \in \{1, \ldots, n+1\}$ where $i > 1$. Then we have

$$f(i) = f'(i) = \rho(f' \restriction \{1, \ldots, i-1\}) = \rho(f \restriction \{1, \ldots, i-1\})$$

if $1 < i \leq n$ since f' satisfies $(*)$. Lastly, if $i = n + 1$, then

$$f(i) = \rho(f') = \rho(f \restriction \{1, \ldots, n\}) = \rho(f \restriction \{1, \ldots, i-1\})$$

again. This shows that f satisfies $(*)$, thereby completing the induction. □

Lemma 8.7.2. *Suppose that $f : \{1, \ldots, n\} \to A$ and $g : \{1, \ldots, m\} \to C$ both satisfy $(*)$ for all i in their respective domains. Then $f(i) = g(i)$ for all i in both domains.*

Proof. Suppose that this is not the case and let i be the *smallest* integer (in the domain of both f and g) for which $f(i) \neq g(i)$. Hence $f(j) = g(j)$ for all $1 \leq j < i$ so that clearly $f \restriction \{1, \ldots, i-1\} = g \restriction \{1, \ldots, i-1\}$. Now, it cannot be that $i = 1$ since clearly $f(1) = a_0 = g(1)$. So then it must be that $1 < i$ so that $f(i) = \rho(f \restriction \{1, \ldots, i-1\})$ and $g(i) = \rho(g \restriction \{1, \ldots, i-1\})$ since they both satisfy $(*)$. Since $f \restriction \{1, \ldots, i-1\} = g \restriction \{1, \ldots, i-1\}$, we then clearly have

$$f(i) = \rho(f \restriction \{1, \ldots, i-1\}) = \rho(g \restriction \{1, \ldots, i-1\}) = g(i)$$

in contradiction with the definition of i. Thus the result must be true as desired. □

Main Problem.

Proof. Lemmas 8.7.1 and 8.7.2 show that there exists a unique function $f_n : \{1, \ldots, n\} \to A$ satisfying $(*)$ for every $n \in \mathbb{Z}_+$. We then define $h = \bigcup_{n \in \mathbb{Z}_+} f_n$ and claim that this is the unique function from $\mathbb{Z}_+$ to A satisfying $(*)$.

First we must show that h is a function at all. So consider any $i \in \mathbb{Z}_+$ and suppose that (i, x) and (i, y) are in h. Then there are $n, m \in \mathbb{Z}_+$ where $(i, x) \in f_n$ and $(i, y) \in f_m$ since $h = \bigcup_{n \in \mathbb{Z}_+} f_n$, noting that it must be that $i \leq n$ and $i \leq m$. Since f_n and f_m both satisfy $(*)$ and clearly i is in the domain of both, it follows from Lemma 8.7.2 that $x = f_n(i) = f_m(i) = y$. This shows that h is a function since (i, x) and (i, y) were arbitrary. Also, clearly the domain of h is $\mathbb{Z}_+$ since, for any $i \in \mathbb{Z}_+$, i is in the domain of f_i and so in the domain of h. Lastly, clearly the range of h is A since that is the range of all the f_n functions.

Now we show that h satisfies $(*)$. First we have that 1 is clearly in the domain of h and f_1 so that it has to be that $h(1) = f_1(1) = a_0$ since h is a function, $f_1 \subset h$, and f_1 satisfies $(*)$. Now suppose that $i > 1$. Then clearly i is in the domain of h and f_i so that it has to be that $h(j) = f_i(j)$ for $1 \leq j \leq i$ since h was shown to be a function and $f_i \subset h$. It then follows that $h \restriction \{1, \ldots, i-1\} = f_i \restriction \{1, \ldots, i-1\}$. Thus we have

$$h(i) = f_i(i) = \rho(f_i \restriction \{1, \ldots, i-1\}) = \rho(h \restriction \{1, \ldots, i-1\})$$

since f_i satisfies $(*)$. This completes the proof that h also satisfies $(*)$.

Lastly, we show that h is unique, which is very similar to the proof of Lemma 8.7.2. So suppose that f and g are two functions from $\mathbb{Z}_+$ to A that both satisfy $(*)$. Suppose also that $f \neq g$ so that there is a smallest integer i such that $f(i) \neq g(i)$. Now, it cannot be that $i = 1$ since we have $f(1) = a_0 = g(1)$ since they both satisfy $(*)$. Hence $i > 1$ and, since i is the smallest integer where $f(i) \neq g(i)$, it follows that $f(j) = g(j)$ for all $1 \leq j < i$. Therefore we have that $f \restriction \{1, \ldots, i-1\} = g \restriction \{1, \ldots, i-1\}$ so that

$$f(i) = \rho(f \restriction \{1, \ldots, i-1\}) = \rho(g \restriction \{1, \ldots, i-1\}) = g(i)$$

since f and g both satisfy $(*)$ and $i > 1$. This of course contradicts the definition of i so that it has to be that in fact $f = g$. This shows the uniqueness of h constructed above. □

Exercise 8.8

Verify the following version of the principle of recursive definition: Let A be a set. Let ρ be a function assigning, to every function f mapping a section S_n of $\mathbb{Z}_+$ into A, an element $\rho(f)$ of A. Then there is a unique function $h : \mathbb{Z}_+ \to A$ such that $h(n) = \rho(h \restriction S_n)$ for each $n \in \mathbb{Z}_+$.

Solution:

Denote the above property of h by $(*)$. We show that there is a unique $h : \mathbb{Z}_+ \to A$ satisfying this using the standard principle of recursive definition, Theorem 8.4.

Proof. First, note that $S_1 = \{n \in \mathbb{Z}_+ \mid n < 1\} = \varnothing$ by definition. Note also that $\varnothing$ itself is vacuously a function from $S_1 = \varnothing$ to A, and is the only such function. It then follows that $f \upharpoonright S_1 = f \upharpoonright \varnothing = \varnothing$ for any $f : S_n \to A$ for some $n \in \mathbb{Z}_+$. So then, define $a_0 = \rho(\varnothing)$ so that there is a unique function h such that

$$\begin{aligned} h(1) &= a_0\,, \\ h(i) &= \rho(h \upharpoonright \{1, \ldots, i-1\}) \qquad \text{for } i > 1. \end{aligned}$$

by Theorem 8.4. Denote this property by (+).

We first claim that this h satisfies $(*)$. To see this, consider any $n \in \mathbb{Z}_+$. If $n = 1$ then we have

$$h(n) = h(1) = a_0 = \rho(\varnothing) = \rho(h \upharpoonright \varnothing) = \rho(h \upharpoonright S_1) = \rho(h \upharpoonright S_n)\,.$$

If $n > 1$ then by (+) we have

$$h(n) = \rho(h \upharpoonright \{1, \ldots, n-1\}) = \rho(h \upharpoonright S_n)$$

again. Since n was arbitrary, this shows that $(*)$ is satisfied.

To show that this h satisfying $(*)$ is unique, suppose that another function $f : \mathbb{Z}_+ \to A$ satisfies $(*)$. Then we have

$$h(1) = \rho(h \upharpoonright S_1) = \rho(h \upharpoonright \varnothing) = \rho(\varnothing) = a_0$$

and

$$h(i) = \rho(h \upharpoonright S_i) = \rho(h \upharpoonright \{1, \ldots, i-1\})$$

for $i > 1$. This shows that f also satisfies (+), and, since we know that the function satisfying (+) is unique, it must be that $f = h$ as desired. □

§9 Infinite Sets and the Axiom of Choice

Exercise 9.1

Define an injective map $f : \mathbb{Z}_+ \to X^\omega$, where X is the two-element set $\{0, 1\}$, without using the choice axiom.

Solution:

For any $n \in \mathbb{Z}_+$, define

$$x_i = \begin{cases} 0 & i \neq n \\ 1 & i = n \end{cases}$$

for $i \in \mathbb{Z}_+$. Then set $f(n) = \mathbf{x} = (x_1, x_2, \ldots)$ so that clearly f is a function from $\mathbb{Z}_+$ to X^ω. It is easy to show that f is injective.

Proof. Consider $n, m \in \mathbb{Z}_+$ where $n \neq m$. Then let $\mathbf{x} = f(n)$ and $\mathbf{y} = f(m)$. Then we have that $x_n = 1$ while $y_n = 0$ by the definition of f since $n \neq m$. It thus follows that $f(n) = \mathbf{x} \neq \mathbf{y} = f(m)$, which shows that f is injective since n and m were arbitrary. □

Exercise 9.2

Find if possible a choice function for each of the following collections, without using the choice axiom:

(a) The collection $\mathcal{A}$ of nonempty subsets of $\mathbb{Z}_+$.

(b) The collection $\mathcal{B}$ of nonempty subsets of $\mathbb{Z}$.

(c) The collection $\mathcal{C}$ of nonempty subsets of the rational numbers $\mathbb{Q}$.

(d) The collection $\mathcal{D}$ of nonempty subsets of X^ω, where $X = \{0, 1\}$.

Solution:

Lemma 9.2.1. *If A is a countable set and $\mathcal{A}$ is the collection of nonempty subsets of A then $\mathcal{A}$ has a choice function.*

Proof. Since A is countable, there is an injective function $A \to \mathbb{Z}_+$ by Theorem 7.1. We define a choice function $c : \mathcal{A} \to \bigcup_{B\in\mathcal{A}} B$. Consider any $X \in \mathcal{A}$ so that X is a nonempty subset of A. Then $f(X)$ is a nonempty subset of $\mathbb{Z}_+$ so that it has a unique smallest element n since $\mathbb{Z}_+$ is well-ordered. Now, since $n \in f(X)$, clearly there is an $x \in X$ such that $f(x) = n$. Moreover, it follows from the fact that f is injective that this x is unique. So set $c(X) = x$ so that clearly x is a choice function on $\mathcal{A}$ since $c(X) = x \in X$. □

Main Problem.

(a) Since $\mathbb{Z}_+$ is countable, a choice function can be constructed as in Lemma 9.2.1.

(b) Since $\mathbb{Z}$ is countable (by Example 7.1), a choice function can be constructed as in Lemma 9.2.1.

(c) Since $\mathbb{Q}$ is countable (by Exercise 7.1), a choice function can be constructed as in Lemma 9.2.1.

(d) First, there is an injective function f from the real interval $[0, 1]$ to X^ω. The most straightforward such function is, for each $x \in [0, 1]$ let $0.x_1x_2x_3\ldots$ be a unique binary expansion of x (these can be made unique by avoiding binary expansions that end in all 1's, noting though that the expansion of 1 itself must be $0.111\ldots$). So suppose that c were a choice function on $\mathcal{D}$ (that is presumably constructed without the choice axiom). If X is a nonempty subset of $[0, 1]$ then $f(X)$ is a set in $\mathcal{D}$ so that we can choose $c(f(X)) \in f(X)$. Since f is injective, there is a unique $x \in X$ where $f(x) = c(f(X))$, and so choosing x results in a choice function on the collection of nonempty subsets of $[0, 1]$ since X was arbitrary.

This would allow one to then well-order $[0, 1]$ without using the choice axiom, which evidently nobody has done. As far as I have been able to determine, this has not yet been proven impossible, it is just that nobody has been able to do it. So it would seem that such an explicit construction of a choice function on $\mathcal{D}$ would at least make one famous. Or else it is impossible, which is what we assume to be the case here.

Exercise 9.3

Suppose that A is a set and $\{f_n\}_{n\in\mathbb{Z}_+}$ is a given indexed family of injective functions

$$f_n : \{1, \ldots, n\} \to A\,.$$

Show that A is infinite. Can you define an injective function $f : \mathbb{Z}_+ \to A$ without using the choice axiom?

Solution:

We defer the proof that A is infinite until we define an injective $f : \mathbb{Z}_+ \to A$, which we can do without using the choice axiom by using the principle of recursive definition.

Proof. First, let $a_0 = f_1(1) \in A$. Now consider any function $g : S_n \to A$. If $g = \varnothing$ then set $\rho(\varnothing) = \rho(g) = a_0$. Otherwise let $I_g = \{i \in S_{n+1} \mid f_n(i) \notin g(S_n)\}$. Suppose for the moment that $I_g = \varnothing$. Consider any $x \in f_n(S_{n+1})$ so that there is a $k \in S_{n+1}$ where $f_n(k) = x$. Then it has to be that $x = f_n(k) \in g(S_n)$ since otherwise we would have $k \in I_g$. Since x was arbitrary, this shows that $f_n(S_{n+1}) \subset g(S_n)$. Thus the identity function $h_1 : f_n(S_{n+1}) \to g(S_n)$ is an injection. Clearly g is a surjection from S_n to its image $g(S_n)$ so that there is we can construct a particular injection $h_2 : g(S_n) \to S_n$ by Corollary 6.7. Lastly, f_n is an injection from S_{n+1} to $f_n(S_{n+1})$. Therefore $h = h_2 \circ h_1 \circ f_n$ is an injection from S_{n+1} to S_n. Hence h is a bijection from S_{n+1} to $h(S_{n+1})$, which is clearly a subset of S_n since S_n is the range of h. But, since $S_n \subsetneq S_{n+1}$, clearly $h(S_{n+1}) \subsetneq S_{n+1}$ as well so that h is a bijection from S_{n+1} onto a proper subset of itself. As S_{n+1} is clearly finite, this violates Corollary 6.3 so that we have a contradiction.

So it must be that $I_g \neq \varnothing$ so that it is a nonempty set of positive integers, and hence has a smallest element i. So simply set $\rho(g) = f_n(i)$. Now, it then follows from the principle of recursive definition that there is a unique $f : \mathbb{Z}_+ \to A$ such that

$$\begin{aligned} f(1) &= a_0\,, \\ f(n) &= \rho(f \restriction S_n) \qquad \text{for } n > 1. \end{aligned}$$

We claim that this f is injective.

To see this we first show that $f(n) \notin f(S_n)$ for all $n \in \mathbb{Z}_+$. If $n = 1$ we have that $f(n) = f(1) = a_0 = f_1(1)$ and $f(S_n) = f(\varnothing) = \varnothing$ so that clearly the result holds. If $n > 1$ then $f(n) = \rho(f \restriction S_n) = f_n(i)$ for some $i \in I_{f \restriction S_n}$ since clearly $S_n \neq \varnothing$ so that $f \restriction S_n \neq \varnothing$. Since $i \in I_{f \restriction S_n}$ we have that $f(n) = f_n(i) \notin (f \restriction S_n)(S_n) = f(S_n)$ as desired. This shows that f is injective. For consider any $n, m \in \mathbb{Z}_+$ where $n \neq m$. Without loss of generality we can assume that $n < m$. Then clearly $f(n) \in f(S_m)$ since $n \in S_m$ since $n < m$. However, by what was just shown, we have $f(m) \notin f(S_m)$ so that it has to be that $f(n) \neq f(m)$. This shows f to be injective since n and m were arbitrary.

Lastly, since $f : \mathbb{Z}_+ \to A$ is injective, it follows that f is a bijection from $\mathbb{Z}_+$ to $f(\mathbb{Z}_+) \subset A$. Hence $f(\mathbb{Z}_+)$ is infinite since $\mathbb{Z}_+$ is, and since it is a subset of A, it has to be that A is infinite as well. ☐

Exercise 9.4

There was a theorem in §7 whose proof involved an infinite number of arbitrary choices. Which one was it? Rewrite the proof so as to make explicit use of the choice axiom. (Several of the earlier exercises have used the choice axiom also.)

Solution:

This was the proof of Theorem 7.5, which asserts that a countable union of countable sets is also countable. The following rewritten proof makes explicit use of the choice axiom and so points out where it is needed.

Proof. Let $\{A_n\}_{n \in J}$ be an indexed family of countable sets, where the index set J is $\{1, \ldots, N\}$ or $\mathbb{Z}_+$. Assume that each set A_n is nonempty for convenience since this does not change anything. Now, for each $n \in J$, let B_n be the set of surjective functions from $\mathbb{Z}_+$ to A_n. Since each A_n is countable, it follows from Theorem 7.1 that $B_n \neq \varnothing$. Then, by the axiom of choice, the collection $\{B_n\}_{n \in J}$ has a choice function c such that $c(B_n) \in B_n$ for every $n \in J$.

Now set $f_n = c(B_n)$ for every $n \in J$ so that $f_n \in B_n$ and hence is a surjection from $\mathbb{Z}_+$ into A_n. Since J is countable, there is also a surjection $g : \mathbb{Z}_+ \to J$ by Theorem 7.1. Then define $h : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \bigcup_{n \in J} A_n$ by $h(k, m) = f_{g(k)}(m)$ for $k, m \in \mathbb{Z}_+$.

We now show that h is surjective. So consider any $a \in \bigcup_{n \in J} A_n$ so that $a \in A_n$ for some $n \in J$. Since $g : \mathbb{Z}_+ \to J$ is surjective, there is a $k \in \mathbb{Z}_+$ where $g(k) = n$. Also, since $f_n : \mathbb{Z}_+ \to A_n$ is surjective, there is an $m \in \mathbb{Z}_+$ where $f_n(m) = a$. We then have by definition that

$$h(k, m) = f_{g(k)}(m) = f_n(m) = a\,,$$

which shows that h is surjective since a was arbitrary.

Lastly, since $\mathbb{Z}_+ \times \mathbb{Z}_+$ is countable by Example 7.2, there is a bijection $h' : \mathbb{Z}_+ \to \mathbb{Z}_+ \times \mathbb{Z}_+$. It then follows that $h \circ h'$ is a surjection from $\mathbb{Z}_+$ to $\bigcup_{n \in J} A_n$, which shows that $\bigcup_{n \in J} A_n$ is countable again by Theorem 7.1. □

Exercise 9.5

(a) Use the choice axiom to show that if $f : A \to B$ is surjective, then f has a right inverse $h : B \to A$.

(b) Show that if $f : A \to B$ is injective and A is not empty, then f has a left inverse. Is the axiom of choice needed?

Solution:

(a)

Proof. Suppose that $f : A \to B$ is surjective. Now, by the choice axiom, the collection $\mathcal{A} = \mathcal{P}(A) - \{\varnothing\}$ is a collection of nonempty sets and thus has a choice function c. Consider any $b \in B$ and the set $A_b = \{x \in A \mid f(x) = b\}$. Then $A_b \neq \varnothing$ since f is surjective, and hence $A_b \in \mathcal{A}$ since clearly also $A_b \subset A$ so that $A_b \in \mathcal{P}(A)$. So set $h(b) = c(A_b) \in A_b$ so that $h(b) \in A$ since $A_b \subset A$. Hence h is a function from B to A.

Recall that, by definition, h is a right inverse if and only if $f \circ h = i_B$, which we show presently. So consider any $b \in B$ and let $a = h(b) = c(A_b) \in A_b$ so that $f(a) = b$. Then clearly

$$(f \circ h)(b) = f(h(b)) = f(a) = b\,,$$

which shows that $f \circ h = i_B$ since b was arbitrary. Hence h is a right inverse of f. □

(b)

Proof. Suppose that $f : A \to B$ is injective and $A \neq \varnothing$. Then f is a bijection from A to its image $f(A) \subset B$ and hence its inverse f^{-1} is a function from $f(A)$ to A. Now, since A is nonempty, there is an $a_0 \in A$. So define $h : B \to A$ by

$$h(b) = \begin{cases} f^{-1}(b) & b \in f(A) \\ a_0 & b \notin f(A) \end{cases}$$

for any $b \in B$. Recall that h is a left inverse of f if and only if $h \circ f = i_A$ by definition, which we show now.

So consider any $a \in A$ and let $b = f(a)$ so that clearly $b \in f(A)$. Hence by definition $h(b) = f^{-1}(b) = f^{-1}(f(a)) = a$. Finally, we have

$$(h \circ f)(a) = h(f(a)) = h(b) = a\,.$$

This shows that $h \circ f = i_A$ since a was arbitrary. Therefore h is a left inverse of f as desired. □

Note that this proof does not require the axiom of choice as we did not need to make a choice for each $b \in B$ in order to define h as we did in part A.

Exercise 9.6

Most of the famous paradoxes of naive set theory are associated in some way or another with the concept of the "set of all sets." None of the rules we have given for forming sets allows us to consider such a set. And for good reason – the concept itself is self-contradictory. For suppose that $\mathcal{A}$ denotes the "set of all sets."

(a) Show that $\mathcal{P}(\mathcal{A}) \subset \mathcal{A}$; derive a contradiction.

(b) (*Russell's paradox.*) Let $\mathcal{B}$ be the subset of $\mathcal{A}$ consisting of all sets that are not elements of themselves:

$$\mathcal{B} = \{A \mid A \in \mathcal{A} \text{ and } A \notin A\}\ .$$

(Of course, there may be *no* set A such that $A \in A$; If such is the case, then $\mathcal{B} = \mathcal{A}$.) Is $\mathcal{B}$ an element of itself or not?

Solution:

(a) We show that $\mathcal{P}(\mathcal{A}) \subset \mathcal{A}$ and that a contradiction results.

Proof. Consider any set $A \in \mathcal{P}(\mathcal{A})$. Since A is a set and $\mathcal{A}$ is the set of all sets, clearly $A \in \mathcal{A}$ and hence $\mathcal{P}(\mathcal{A}) \subset \mathcal{A}$ since A was arbitrary. Therefore the identity function $i_{\mathcal{P}(\mathcal{A})}$ is clearly an injection from $\mathcal{P}(\mathcal{A})$ to $\mathcal{A}$. However, this is impossible by Theorem 7.8! Hence we have reached a contradiction. □

(b) We show that the existence of $\mathcal{B}$ is a contradiction by showing that supposing either $\mathcal{B} \in \mathcal{B}$ or $\mathcal{B} \notin \mathcal{B}$ results in a contradiction.

Proof. Suppose that $\mathcal{B} \in \mathcal{B}$ so that by definition we have $\mathcal{B} \in \mathcal{A}$ and $\mathcal{B} \notin \mathcal{B}$, the latter of which clearly contradicts our initial supposition. On the other hand, suppose that $\mathcal{B} \notin \mathcal{B}$. Then, since clearly also $\mathcal{B} \in \mathcal{A}$ since it is a set, it follows that $\mathcal{B} \in \mathcal{B}$ by definition. This again contradicts the initial supposition. Since one or the other ($\mathcal{B} \in \mathcal{B}$ or $\mathcal{B} \notin \mathcal{B}$) must be true, we are then guaranteed to have a contradiction. □

Exercise 9.7

Let A and B be two nonempty sets. If there is an injection of B into A, but no injection of A into B, we say that A has ***greater cardinality*** than B.

(a) Conclude from Theorem 9.1 that every uncountable set has greater cardinality than $\mathbb{Z}_+$.

(b) Show that if A has greater cardinality than B, and B has greater cardinality than C, then A has greater cardinality than C.

(c) Find a sequence $A_1, A_2, \ldots$ of infinite sets, such that for each $n \in \mathbb{Z}_+$, the set A_{n+1} has greater cardinality than A_n.

(d) Find a set that for every n has cardinality greater than A_n.

Solution:

Lemma 9.7.1. *For any set A, $\mathcal{P}(A)$ has greater cardinality than A.*

Proof. Clearly the function that maps $a \in A$ to $\{a\} \in \mathcal{P}(A)$ is an injection. However, we know from Theorem 7.8 that there is no injection from $\mathcal{P}(A)$ to A. Together these show that $\mathcal{P}(A)$ has greater cardinality than A as desired. □

Main Problem.

(a)

Proof. Suppose that A is any uncountable set. Clearly A is not finite for then it would be countable. Hence it is infinite and so there is a injection from $\mathbb{Z}_+$ to A by Theorem 9.1. There also cannot be an injection from A to $\mathbb{Z}_+$, for if there were then A would be countable by Theorem 7.1. This shows that A has greater cardinality than $\mathbb{Z}_+$ by definition. □

(b)

Proof. Since A has greater cardinality than B, there is an injection $f : B \to A$. Likewise, since B has greater cardinality than C, there is an injection $g : C \to B$. It then follows that $f \circ g$ is an injection of C into A. Now suppose that $h : A \to C$ is injective. Then $g \circ h$ would be an injection of A into B, which we know cannot exist since A has greater cardinality than B. Hence it must be that no such injection h exists, which shows that A has greater cardinality than C as desired. □

(c) We define a sequence of sets recursively:

$$A_1 = \mathbb{Z}_+ ,$$
$$A_n = \mathcal{P}(A_{n-1}) \qquad \text{for } n > 1.$$

We show that this meets the requirements.

Proof. First we show that each A_n is infinite by induction. Clearly $A_1 = \mathbb{Z}_+$ is infinite. Now assume that A_n is infinite for $n \in \mathbb{Z}_+$ so that there is an injection $f : \mathbb{Z}_+ \to A_n$ by Theorem 9.1. Then, by Lemma 9.7.1, $A_{n+1} = \mathcal{P}(A_n)$ has greater cardinality than A_n so that there is an injection $g : A_n \to A_{n+1}$. Then $g \circ f$ is an injection from $\mathbb{Z}_+$ to A_{n+1} so that A_{n+1} is infinite as well by Theorem 9.1. This completes the induction.

Finally, for any $n \in \mathbb{Z}_+$ we have that $n + 1 > 1$ so that $A_{n+1} = \mathcal{P}\left(A_{(n+1)-1}\right) = \mathcal{P}(A_n)$. Then clearly A_{n+1} has greater cardinality than A_n by Lemma 9.7.1. This shows the desired result. □

(d) Let $A = \bigcup_{n \in \mathbb{Z}_+} A_n$, which we claim has the required property.

Proof. Consider any $n \in \mathbb{Z}_+$. Clearly $A_n \subset A$ so that the identity function i_{A_n} is an injection of A_n into A. Now suppose for the moment that $g : A \to A_n$ is injective. Since clearly also $A_{n+1} \subset A$, it follows that $g \restriction A_{n+1}$ is then an injection of A_{n+1} into A_n. However this contradicts the proven fact that A_{n+1} has greater cardinality than A_n. Hence it has to be that no such injection g exists, which shows that A has greater cardinality than A_n. Since n was arbitrary, this shows the desired result. □

Exercise 9.8

Show that $\mathcal{P}(\mathbb{Z}_+)$ and $\mathbb{R}$ have the same cardinality. [Hint: You may use the fact that every real number has a decimal expansion, which is unique if expansions that end in an infinite string of 9's are forbidden.]

A famous conjecture of set theory, called the *continuum hypothesis*, asserts that there exists no set having cardinality greater than $\mathbb{Z}_+$ and lesser cardinality than $\mathbb{R}$. The *generalized continuum hypothesis* asserts that, given the infinite set A, there is no set having greater cardinality than A and lesser cardinality than $\mathcal{P}(A)$. Surprisingly enough, both of these assertions have been shown to be independent of the usual axioms of set theory. For a readable expository account, see [Sm].

Solution:

Lemma 9.8.1. *If A and B are sets with the same cardinality, then $\mathcal{P}(A)$ and $\mathcal{P}(B)$ have the same cardinality.*

Proof. Since A and B have the same cardinality there is a bijection $f : A \to B$. We define a bijection $g : \mathcal{P}(A) \to \mathcal{P}(B)$. So, for any $X \in \mathcal{P}(A)$, set $g(X) = f(X)$. Clearly $f(X) \subset B$, since B is the range of f, so that $g(X) = f(X) \in \mathcal{P}(B)$ and hence $\mathcal{P}(B)$ can be the range of g.

To show that g is injective, consider sets X and Y in $\mathcal{P}(A)$ so that $X, Y \subset A$. Also suppose that $X \neq Y$ so, without loss of generality, we can assume that there is an $x \in X$ where $x \notin Y$. Clearly $f(x) \in f(X)$ since $x \in X$. Were it the case that $f(x) \in f(Y)$ then there would be a $y \in Y$ such that $f(y) = f(x)$. But then we would have that $y = x$ since f is injective and hence $x = y \in Y$, which we know not to be the case. Hence $f(x) \notin f(Y)$ so that it has to be that $g(X) = f(X) \neq f(Y) = g(Y)$ since $f(x) \in f(X)$. Since X and Y were arbitrary this shows that g is injective.

To show that g is surjective consider any $Y \in \mathcal{P}(B)$ so that $Y \subset B$. Let $X = f^{-1}(Y)$, noting that f^{-1} is a bijection from B to A since f is bijective. Clearly $X \subset A$ since A is the range of f^{-1} so that $X \in \mathcal{P}(A)$. Now consider any $y \in f(X)$ so that there is an $x \in X$ where $f(x) = y$. Then, since $X = f^{-1}(Y)$, there is a $y' \in Y$ where $x = f^{-1}(y')$, and hence $y = f(x) = f(f^{-1}(y')) = y'$. Thus $y = y' \in Y$ so that $f(X) \subset Y$ since y was arbitrary. Now consider $y \in Y$ and let $x = f^{-1}(y)$ so that clearly $x = f^{-1}(y) \in f^{-1}(Y) = X$. Moreover, $f(x) = f(f^{-1}(y)) = y$ so that $y \in f(X)$. Thus $Y \subset f(X)$ as well since y was arbitrary. This shows that $g(X) = f(X) = Y$, from which we conclude that g is surjective since Y was arbitrary.

Hence $g : \mathcal{P}(A) \to \mathcal{P}(B)$ is a bijection so that $\mathcal{P}(A)$ and $\mathcal{P}(B)$ have the same cardinality by definition. □

Main Problem.

Proof. We show this using the Cantor-Schroeder-Bernstein (CSB) Theorem, which was proven in Exercise 7.6 part (b).

First, we construct an injective function f from $\mathbb{R}$ to $\mathcal{P}(\mathbb{Q})$. For any $x \in \mathbb{R}$ let $Q = \{q \in \mathbb{Q} \mid q < x\}$ so that clearly $Q \subset \mathbb{Q}$ and hence $Q \in \mathcal{P}(\mathbb{Q})$. Therefore setting $f(x) = Q$ means that f is a function from $\mathbb{R}$ to $\mathcal{P}(\mathbb{Q})$. To show that f is injective consider $x, y \in \mathbb{R}$ where $x \neq y$. Without loss of generality we can assume that $x < y$ so that there is a $q \in \mathbb{Q}$ where $x < q < y$ since the rationals are order-dense in the reals. Also set $Q = f(x)$ and $P = f(y)$. Since $q > x$ we have that $q \notin Q$. Analogously, since $q < y$ we have that $q \in P$. Thus it has to be that $f(x) = Q \neq P = f(y)$, which shows that f is injective since x and y were arbitrary.

Now, it was shown in Exercise 7.1 that $\mathbb{Q}$ is countably infinite and thus has the same cardinality as $\mathbb{Z}_+$. From Lemma 9.8.1 it then follows that $\mathcal{P}(\mathbb{Q})$ has the same cardinality as $\mathcal{P}(\mathbb{Z}_+)$ so that there is a bijection $g : \mathcal{P}(\mathbb{Q}) \to \mathcal{P}(\mathbb{Z}_+)$. Clearly then $g \circ f$ is an injection from $\mathbb{R}$ to $\mathcal{P}(\mathbb{Z}_+)$.

Now let $X = \{0, 1\}$, and we construct an injection $h : X^\omega \to \mathbb{R}$. For any sequence $\mathbf{x} = (x_1, x_2, \ldots) \in X^\omega$ set $h(\mathbf{x})$ to the decimal expansion $0.x_1x_2x_3\ldots$, where clearly each x_n is the digit 0 or 1. Clearly

$h(\mathbf{x})$ is a real number so that h is a function from X^ω to $\mathbb{R}$. It is easy to see that h is injective since different sequences will result in different decimal expansions. Since none of the expansions end in an infinite sequence of 9's, clearly the corresponding real numbers will be different.

Now, it was shown in Exercise 7.3 that $\mathcal{P}(\mathbb{Z}_+)$ and X^ω have the same cardinality so that there is a bijection $i : \mathcal{P}(\mathbb{Z}_+) \to X^\omega$. It then follows that $h \circ i$ is an injection of $\mathcal{P}(\mathbb{Z}_+)$ into $\mathbb{R}$. Since we have shown the existence of both injections, the result follows from the CSB Theorem. □

§10 Well-Ordered Sets

Exercise 10.1

Show that every well-ordered set has the least upper bound property.

Solution:

Proof. Suppose that A is a set with well-ordering $<$, and that B is some nonempty subset of A with upper bound $a \in A$. Let C then be the set of upper bounds of B, which is not empty since clearly $a \in C$. Then C is a nonempty subset of A and so has a smallest element c since A is well-ordered. Clearly then c is the least upper bound of B by definition. This shows that A has the least upper bound property since B was arbitrary. □

Exercise 10.2

(a) Show that in a well-ordered set, every element except the largest (if one exists) has an immediate successor.

(b) Find a set in which every element has an immediate successor that is not well-ordered.

Solution:

(a)

Proof. Suppose that A is well-ordered by $<$ and consider any $a \in A$ where a is not the largest element. It then follows that there is some $x \in A$ where $a < x$ since otherwise a would be the largest element of A. Let $X = \{y \in A \mid a < y\}$ so that clearly $X \subset A$ and $x \in X$. Thus X is a nonempty subset of A and so has a smallest element b since $<$ well-orders A. We claim that b is the immediate successor of a. To see this suppose that there is a $z \in A$ such that $a < z < b$, noting that clearly $a < b$ since $b \in X$. Then we would have that $z \in X$ but $z < b$ so that it is not true that $b \leq z$, which contradicts the definition of b as the smallest element of X. So it must be that no such z exists, which shows that b is indeed the immediate successor of a. □

(b) The most natural example of such a set is $\mathbb{Z}$. We show that this has the desired properties.

Proof. First, clearly $\mathbb{Z}$ is not well-ordered since, for example, the set of negative integers is a nonempty subset of $\mathbb{Z}$ but has no smallest element. Also, for any $n \in \mathbb{Z}$, clearly $n + 1$ is the immediate successor of n, which was shown back in Corollary 4.9.3. □

Exercise 10.3

Both $\{1,2\} \times \mathbb{Z}_+$ and $\mathbb{Z}_+ \times \{1,2\}$ are well-ordered in the dictionary order. Do they have the same order type?

Solution:

We claim that they do *not* have the same order type, which we show presently.

Proof. First, clearly $(1,1)$ is the smallest element of both ordered sets. For brevity let $A = \{1,2\} \times \mathbb{Z}_+$, $B = \mathbb{Z}_+ \times \{1,2\}$, and $<_A$ and $<_B$ be the corresponding dictionary orderings, with $<$ being the normal ordering of $\mathbb{Z}_+$.

So assume that they *do* have the same order type so that there is an order-preserving bijection $f : A \to B$. Consider $(2,1) \in A$, which is clearly not the smallest element since $(2,1) \neq (1,1)$. Let $(n,b) = f(2,1) \in B$, which cannot be the smallest element of B since f preserves order, so that $(n,b) \neq (1,1)$. Clearly $b \in \{1,2\}$ so that $b = 1$ or $b = 2$. In the former cases we must have that $n > 1$ so that $n - 1 \in \mathbb{Z}_+$. So set $y = (n-1,2)$. In the latter case set $y = (n,1)$. It is easy to see, and trivial to formally show, that y is the immediate predecessor of (n,b) in either case.

Now let $x = f^{-1}(y)$, noting that f^{-1} is an order-preserving bijection from B to A since f is an order-preserving bijection. It then follows that $x <_A (2,1)$ since $f(x) = y <_B (n,b) = f(2,1)$. If $x = (m,a)$ then it has to be that $m < 2$ so that $m = 1$ (because $m \in \{1,2\}$) since there is no $a \in \mathbb{Z}_+$ where $a < 1$. Thus $x = (1,a)$ for some $a \in \mathbb{Z}_+$. We then have that $a + 1 \in \mathbb{Z}_+$ so that clearly $x = (1,a) <_A (1,a+1) <_A (2,1)$. From this we have, $y = f(1,a) <_B f(1,a+1) <_B f(2,1) = (n,b)$, which contradicts the fact that y is the immediate predecessor of (n,b). So it has to be that they do not have the same order type. □

It is worth noting that, in the theory of ordinal numbers, $A = \{1,2\} \times \mathbb{Z}_+$ has order type $\omega + \omega = \omega \cdot 2$ whereas $B = \mathbb{Z}_+ \times \{1,2\}$ has simply order type ω. This also shows that A and B have different order types since distinct ordinal numbers always have different order types.

Exercise 10.4

(a) Let $\mathbb{Z}_-$ denote the set of negative integers in the usual order. Show that a simple ordered set A fails to be well-ordered if and only if it contains a subset having the same order type as $\mathbb{Z}_-$.

(b) Show that if A is simply ordered and every countable subset of A is well-ordered, then A is well-ordered.

Solution:

(a)

Proof. Let A be a set with simple order $\prec$.

($\Rightarrow$) Suppose that $\prec$ is not a well-ordering of A. Then there exists a nonempty subset B of A such that B has no smallest element. For any $b \in B$ define the set $X_b = \{x \in B \mid x \prec b\}$. Clearly $X_b \subset B$ and $X_b \neq \varnothing$ for any $b \in B$ since otherwise b would be the smallest element of B. Now let c be a choice function on the collection of nonempty subsets of B, which of course exists by the axiom of choice. Since B is nonempty there is a $b_0 \in B$. It then follows from the principle of recursive definition that there is a function $f : \mathbb{Z}_+ \to B$ such that

$$f(1) = b_0 ,$$

$$f(n) = c(X_{f(n-1)}) \qquad \text{for } n > 1.$$

It then is easy to show that $f(n+1) \prec f(n)$ for all $n \in \mathbb{Z}_+$, i.e. that the sequence defined by f is decreasing. If we then simply define $g : \mathbb{Z}_- \to \mathbb{Z}_+$ by $g(n) = -n$ for $n \in \mathbb{Z}_-$, it is clear that $f \circ g$ is an order-preserving bijection from $\mathbb{Z}_-$ to some subset C of B. Clearly also $C \subset A$ since $B \subset A$. Hence the subset C has the same order type as $\mathbb{Z}_-$.

($\Leftarrow$) Now suppose that A has a subset B with the same order type as $\mathbb{Z}_-$. Clearly then B is nonempty and has no smallest element since $\mathbb{Z}_-$ does not. The existence of this B shows that A fails to be well-ordered. □

(b)

Proof. Suppose that A is a set that is simply ordered by $\prec$ such that every countable subset is well-ordered by $\prec$. Consider any nonempty subset $B \subset A$. Suppose for a moment that the restricted $\prec$ does not well-order B. Then it follows from part (a) that B has a subset C with the same order type as $\mathbb{Z}_-$. However, clearly $C \subset A$ (since $B \subset A$) and C is countable (since $\mathbb{Z}_-$ is countable) and thus it should be well-ordered. As this is impossible since C has the same order-type as $\mathbb{Z}_-$ (which is clearly not well-ordered), it has to be that the restricted $\prec$ does in fact well-order B. Hence B has a $\prec$-smallest element, which shows that A is well-ordered since B was arbitrary. □

Exercise 10.5

Show the well-ordering theorem implies the choice axiom.

Solution:

Proof. Suppose that $\mathcal{A}$ is a collection of nonempty sets. Then, by the well-ordering theorem there is a well-ordering $<$ of $\bigcup \mathcal{A}$. We construct a choice function $c : \mathcal{A} \to \bigcup \mathcal{A}$. Consider any set $A \in \mathcal{A}$. Since clearly A is then a nonempty subset of $\bigcup \mathcal{A}$, it follows that it has a unique smallest element a according to $<$ since $\bigcup \mathcal{A}$ is well-ordered by $<$. So simply set $c(A) = a$ so that clearly then $c(A) = a \in A$. This shows that c is in fact a choice function on $\mathcal{A}$. □

Exercise 10.6

Let S_Ω be the minimal uncountable well-ordered set.

(a) Show that S_Ω has no largest element.

(b) Show that for every $\alpha \in S_\Omega$, the subset $\{x \mid \alpha < x\}$ is uncountable.

(c) Let X_0 be the subset of S_Ω consisting of all elements x such that x has no immediate predecessor. Show that X_0 is uncountable.

Solution:

Lemma 10.6.1. *If A is an uncountable set and $B \subset A$ is countable then $A - B$ is uncountable.*

Proof. If we let $C = A - B$, then clearly $A = C \cup B$. If C were countable then $A = C \cup B$ would be countable by Theorem 7.5 since B is also countable. Since we know that A is uncountable it therefore must be that $C = A - B$ is uncountable as well. □

Main Problem.

It is assumed in the following that $<$ is the well-order on S_Ω.

(a)

Proof. Suppose to the contrary that S_Ω does have a largest element α. Then, for any $x \in S_\Omega$, we have that $x \leq \alpha$. Hence either $x \in \{y \in S_\Omega \mid y < \alpha\} = S_\alpha$ or $x = \alpha$. Therefore $S_\Omega = S_\alpha \cup \{\alpha\}$ since clearly both S_α and $\{\alpha\}$ are both subsets of S_Ω. Now since S_α is a section of S_Ω, it is countable. Since $\{\alpha\}$ is also clearly countable, it follows from Theorem 7.5 that their union $S_\alpha \cup \{\alpha\} = S_\Omega$ is countable. But this contradicts the fact that S_Ω is uncountable! Hence it has to be that S_Ω has no largest element as desired. □

(b)

Proof. Consider any $\alpha \in S_\Omega$. Let $T_\alpha = \{x \in S_\Omega \mid \alpha < x\}$ so that we must show that T_α is uncountable. Let $\bar{S}_\alpha = S_\alpha \cup \{\alpha\}$ so that clearly we have that $\bar{S}_\alpha = \{x \in S_\Omega \mid x \leq \alpha\}$. It is then easy to show that $T_\alpha = S_\Omega - \bar{S}_\alpha$. Now, since S_α is a section of S_Ω, it is countable so that clearly $\bar{S}_\alpha = S_\alpha \cup \{\alpha\}$ is also countable by Theorem 7.5. Then, since S_Ω itself is uncountable, it follows that $T_\alpha = S_\Omega - \bar{S}_\alpha$ is also uncountable by Lemma 10.6.1. □

(c)

Proof. First we show that X_0 is not bounded above. Assume the contrary so that $\alpha \in S_\Omega$ is an upper bound of X_0. It then follows that the set $T_\alpha = \{x \in S_\Omega \mid \alpha < x\}$ is such that every element of T_α has an immediate predecessor since otherwise there would be a $\beta \in T_\alpha$ where $\beta \in X_0$ so that α would not be an upper bound of X_0 since then $\alpha < \beta$.

Now, we know from part (a) that S_Ω has no largest element so that it follows from Exercise 10.2 that every element of S_Ω has an immediate successor. Since $T_\alpha \subset S_\Omega$ it follows that each element $x \in T_\alpha$ has an immediate successor y. Moreover we then have that $\alpha < x < y$ so that $y \in T_\alpha$ also. Hence every element of T_α has an immediate successor in T_α.

Now, we know that T_α is uncountable by part (b) so that it has a smallest element β since it is then a nonempty subset of the well-ordered S_Ω. We derive a contradiction by showing that T_α has the same order type as $\mathbb{Z}_+$ and is thus countable. We do this by defining an increasing bijection $f : \mathbb{Z}_+ \to T_\alpha$. First, set $f(1) = \beta$ and then set $f(n)$ to the immediate successor of $f(n-1)$ for $n > 1$, which was shown to exist above. Then the function f uniquely exists by the principle of recursive definition. Clearly we have that $f(n+1) > f(n)$ for all $n \in \mathbb{Z}_+$ since $f(n+1)$ is the immediate successor of $f(n)$. Hence f is increasing and therefore also injective.

To show that f is surjective suppose the contrary so that the set $T_\alpha - f(\mathbb{Z}_+)$ is nonempty. Since clearly this is a subset of the well-ordered S_Ω, it has a smallest element y. Now, we know that $f(1) = \beta$ so that $y \neq \beta$, and in fact $\beta < y$ since β is the smallest element of T_α. Since $y \in T_\alpha$ we know that it has an immediate predecessor x and that $\alpha < \beta \leq x$ so that $x \in T_\alpha$. However, it cannot be that $x \in T_\alpha - f(\mathbb{Z}_+)$ since $x < y$ and y is the smallest element of $T_\alpha - f(\mathbb{Z}_+)$. Thus $x \in f(\mathbb{Z}_+)$ so that there is an $n \in \mathbb{Z}_+$ where $f(n) = x$. But then $f(n+1) = y$ since y is the immediate successor of x. As this contradicts the fact that $y \notin f(\mathbb{Z}_+)$, it must be that f is in fact surjective!

Therefore we have shown that f is a bijection from $\mathbb{Z}_+$ to T_α so that T_α is countable. But we know from part (b) that T_α is uncountable. As mentioned above, this is a contradiction so that it must be that indeed X_0 is not bounded above. From this it immediately follows from the contrapositive of Theorem 10.3 that X_0 must be uncountable. □

It is interesting to note that S_Ω corresponds to the ordinal number ω_1, which is the first uncountable ordinal, and the set X_0 of part (c) corresponds to the set of limit ordinals in ω_1. All of the curious properties deduced here for S_Ω apply to ω_1 too, assuming we allow the choice axiom.

Exercise 10.7

Let J be a well-ordered set. A subset J_0 of J is said to be ***inductive*** if for every $\alpha \in J$,

$$(S_\alpha \subset J_0) \Rightarrow \alpha \in J_0 .$$

Theorem (The principle of transfinite induction). If J is a well-ordered set and J_0 is an inductive subset of J, then $J_0 = J$.

Solution:

Proof. Suppose that J_0 is an inductive subset of the well-ordered set J. Also suppose that $J_0 \neq J$. Since $J_0 \subset J$, it follows that there must be an $x \in J$ such that $x \notin J_0$. Thus the set $J - J_0$ is nonempty. Since clearly this is also a subset of J, it must have a smallest element α since J is well-ordered. Consider any $y \in S_\alpha$ so that $y < \alpha$. Then it cannot be that $y \in J - J_0$ since otherwise α would not be the smallest element of $J - J_0$. Since clearly $y \in J$ (since $S_\alpha \subset J$) it has to be that $y \in J_0$. Since y was arbitrary this shows that $S_\alpha \subset J_0$. It then follows that $\alpha \in J_0$ since J_0 is inductive. However, this contradicts the fact that $\alpha \in J - J_0$ so that our initial supposition that $J_0 \neq J$ must be incorrect. Hence $J_0 = J$ as desired. □

Exercise 10.8

(a) Let A_1 and A_2 be disjoint sets, well-ordered by $<_1$ and $<_2$, respectively. Define an order relation on $A_1 \cup A_2$ by letting $a < b$ either if $a, b \in A_1$ and $a <_1 b$, or if $a, b \in A_2$ and $a <_2 b$, or if $a \in A_1$ and $b \in A_2$. Show that this is a well-ordering.

(b) Generalize (a) to an arbitrary family of disjoint well-ordered sets, indexed by a well-ordered set.

Solution:

(a)

Proof. It is easy but tedious to show that $<$ is actually an order on $A_1 \cup A_2$, so we shall skip that proof and jump straight to the proof that it is a well-ordering.

So consider any nonempty subset A of $A_1 \cup A_2$.

Case: $A_1 \cap A \neq \varnothing$. Then clearly $A_1 \cap A$ is a nonempty subset of A_1 so that it has a smallest element a according to $<_1$ since it is a well-ordering. We then claim that a is the smallest element of A according to $<$. So consider any $x \in A$ so that clearly also $x \in A_1 \cup A_2$. Hence $x \in A_1$ or $x \in A_2$. If $x \in A_1$ then obviously $x \in A_1 \cap A$ so that $a \leq_1 x$ since a is the smallest element of $A_1 \cap A$. Then also $a \leq x$ by definition since a and x are both in A_1. On the other hand, if $x \in A_2$ then we again have that $a < x$ since $a \in A_1$ and $x \in A_2$. Therefore $a \leq x$ no matter what so that a is the smallest element of A since x was arbitrary.

Case: $A_1 \cap A = \varnothing$. Then it has to be that $A_2 \cap A \neq \varnothing$ since A is nonempty and $A = A_1 \cup A_2$. Thus $A_2 \cap A$ is a nonempty subset of A_2 so that it has a smallest element a by $<_2$ since it is a well-ordering. We claim that a is the smallest element of A. So consider any $x \in A$. It has to be that $x \in A_2$ since $A_1 \cap A$ is empty and $A = A_1 \cup A_2$. Therefore $x \in A_2 \cap A$ so that $a \leq_2 x$ since a is the smallest element of $A_2 \cap A$. Then, by definition, $a \leq x$ since both a and x are elements of A_2. This shows that a is the smallest element of A since x was arbitrary.

In either case we have shown that A has a smallest element so that $<$ is a well-ordering of $A_1 \cup A_2$ since A was arbitrary. □

Note that well-ordering a union of two well-ordered sets like this is analogous to the addition of two ordinal numbers. In particular if A_1 has order type α_1 and A_1 has order type α_2 where α_1 and α_2 are an ordinal numbers, then $A_1 \cup A_2$ with the above well-ordering has order type $\alpha_1 + \alpha_2$.

(b) Suppose that J is well-ordered by $<_J$ and $\{A_\alpha\}_{\alpha \in J}$ is a collection of well-ordered sets where A_α is well-ordered by $<_\alpha$ for each $\alpha \in J$. Now define an order $<$ on $A = \bigcup_{\alpha \in J} A_\alpha$ as follows. For any x and y in A there are clearly α and β in J where $x \in A_\alpha$ and $y \in A_\beta$, noting that α and β are unique since the collection is mutually disjoint. So set $x < y$ if and only if either $\alpha = \beta$ and $x <_\alpha y$, or else $\alpha <_J \beta$, noting that these are clearly mutually exclusive. We then claim that $<$ is a well-ordering of A.

Proof. Let B be any nonempty subset of A and I be the set of $\alpha \in J$ such that there is an $x \in B$ where $x \in A_\alpha$. Now, since B is nonempty, there is a $z \in B$. Since $B \subset A = \bigcup_{\alpha \in J} A_\alpha$, there is an $\gamma \in J$ where $z \in A_\gamma$. Then clearly $\gamma \in I$ so that I is a nonempty subset of J. Then I has a smallest element α since it is well-ordered by $<_J$. By the definition of I there is a $w \in B$ where $w \in A_\alpha$. Then clearly $w \in A_\alpha \cap B$ so that it is a nonempty subset of A_α. It then follows that $A_\alpha \cap B$ has a smallest element a according to $<_\alpha$ since it is a well-ordering on A_α. We claim that a is the smallest element of B by $<$.

So consider any $x \in B$ so that there is a $\beta \in J$ where $x \in A_\beta$ since $B \subset A$.

Case: $\beta = \alpha$. Then both a and x are in $A_\alpha \cap B = A_\beta \cap B$ so that $a \leq_\alpha b$ since a is the smallest element of $A_\alpha \cap B$. It then follows from the definition of $<$ that $a \leq x$.

Case: $\beta \neq \alpha$. Clearly then $\beta \in I$ so that $\alpha \leq_J \beta$ since α is the smallest element of J. Since we know that $\beta \neq \alpha$ it must be that $\alpha <_J \beta$. From this it follows that $a < x$ by definition.

Hence in either case it is true that $a \leq x$, which shows that a is the smallest element of B. Since B was an arbitrary nonempty subset of A, this shows that A is well-ordered by $<$. ◻

Exercise 10.9

Consider the subset A of $(\mathbb{Z}_+)^\omega$ consisting of all infinite sequences of positive integers $\mathbf{x} = (x_1, x_2, \ldots)$ that end in an infinite string of 1's. Give A the following order: $\mathbf{x} < \mathbf{y}$ if $x_n < y_n$ and $x_i = y_i$ for $i > n$. We call this the "antidictionary order" on A.

(a) Show that for every n, there is a section of A that has the same order type as $(\mathbb{Z}_+)^n$ in the dictionary order.

(b) Show that A is well-ordered.

Solution:

(a)

Proof. Consider any positive integer n. Define a sequence $\mathbf{a} = (a_1, a_2, \ldots)$ in A by

$$a_i = \begin{cases} 2 & i = n+1 \\ 1 & i \neq n+1 \,. \end{cases}$$

We claim that the section $S_\mathbf{a}$ has the same order type as $(\mathbb{Z}_+)^n$. To show this we construct an order-preserving mapping $f : S_\mathbf{a} \to (\mathbb{Z}_+)^n$. So consider any sequence $\mathbf{x} = (x_1, x_2, \ldots)$ in $S_\mathbf{a}$ so that $\mathbf{x} < \mathbf{a}$. Now define a finite sequence where $y_i = x_{n-i+1}$ for any $i \in \{1, \ldots, n\}$, and set $f(\mathbf{x}) = \mathbf{y} = (y_1, \ldots, y_n)$. Clearly $f(\mathbf{x}) \in (\mathbb{Z}_+)^n$ since $\mathbf{x} \in A \subset (\mathbb{Z}_+)^\omega$.

Here we must digress for a moment and show that, for all $\mathbf{x} = (x_1, x_2, \ldots) \in A$, $\mathbf{x} \in S_{\mathbf{a}}$ if and only if $x_i = 1$ for all $i > n$.

($\Rightarrow$) We show the contrapositive. So suppose that there is an $i > n$ where $x_i \neq 1$. Moreover let i be the greatest such index, which must exist since $\mathbf{x}$ must end in an infinite string of 1's. Clearly then the fact that $x_i \in \mathbb{Z}_+$ and $x_i \neq 1$ means that $x_i > 1$. Now, if $i > n+1$ then we have that $x_i > 1 = a_i$ and $x_j = 1 = a_j$ for all $j > i$ so that clearly $\mathbf{x} > \mathbf{a}$. If $i = n+1$ and $i > 2$ clearly $x_i > 2 = a_{n+1} = a_i$ and $x_j = 1 = a_j$ for all $j > i$ so that again $\mathbf{x} > \mathbf{a}$. Lastly suppose that $i = n+1$ but that $x_i = 2 = a_{n+1} = a_i$. If $x_j = 1 = a_j$ for all $j < i = n+1$ then clearly $\mathbf{x} = \mathbf{a}$. On the other hand if there is a $1 \leq j < n+1 = i$ where $x_j \neq 1$ then let j be the greatest such index. Then we clearly have $x_j > 1 = a_j$ while $x_k = a_k$ for all $k > j$ so that $\mathbf{x} > \mathbf{a}$. Therefore in every one of these exhaustive cases we have that $\mathbf{x} \geq \mathbf{a}$ so that $\mathbf{a} \notin S_{\mathbf{a}}$.

($\Leftarrow$) Now suppose that $x_i = 1$ for every $i > n$. Then we have that $x_{n+1} = 1 < 2 = a_{n+1}$ while $x_j = 1 = a_j$ for all $j > n+1 > n$ so that $\mathbf{x} < \mathbf{a}$ and hence $\mathbf{x} \in S_{\mathbf{a}}$.

Now, returning to the main proof, we first show that f as defined above preserves order. To this end let $\prec$ denote the dictionary order on $(\mathbb{Z}_+)^n$. Now consider any $\mathbf{x} = (x_1, x_2, \ldots)$ and $\mathbf{x}' = (x_1', x_2', \ldots)$ in $S_{\mathbf{a}}$ where $\mathbf{x} < \mathbf{x}'$. Also let $\mathbf{y} = f(\mathbf{x})$ and $\mathbf{y}' = f(\mathbf{x}')$. Then there is an $m \in \mathbb{Z}_+$ where $x_m < x_m'$ and $x_i = x_i'$ for all $i > m$. We also have by what was shown above that $x_i = 1x_i'$ for all $i > n$ since $\mathbf{x}, \mathbf{x}' \in S_{\mathbf{a}}$. So it has to be that $m \leq n$. It then follows from the fact that $1 \leq m \leq n$ that $1 \leq n - m + 1 \leq n$ as well. Thus we have

$$y_{n-m+1} = x_{n-(n-m+1)+1} = x_m < x_m' = x_{n-(n-m+1)+1}' = y_{n-m+1}' \,.$$

For any $1 \leq j < n - m + 1$ we have that $n - j + 1 > m$ so that

$$y_j = x_{n-j+1} = x_{n-j+1}' = y_j' \,.$$

Thus by definition we have that $f(\mathbf{x}) = \mathbf{y} \prec \mathbf{y}' = f(\mathbf{x}')$, which shows that f preserves order since $\mathbf{x}$ and $\mathbf{x}'$ were arbitrary. Note that this also clearly shows that f is injective.

To show that f is also surjective, consider any $\mathbf{y} = (y_1, \ldots, y_n) \in (\mathbb{Z}_+)^n$. Now define a sequence

$$x_i = \begin{cases} y_{n-i+1} & 1 \leq i \leq n \\ 1 & i > n \end{cases}$$

so that clearly $\mathbf{x} = (x_1, x_2, \ldots) \in S_{\mathbf{a}}$ by what was shown above. Now let $\mathbf{y}' = (y_1', \ldots, y_n') = f(\mathbf{x})$. Consider any $1 \leq i \leq n$ and let $j = n - i + 1$ so that also $n - j + 1 = i$, noting also that $1 \leq j \leq n$. Then we have

$$y_i = y_{n-j+1} = x_j = x_{n-i+1} = y_i'$$

by the definition of f. Since i was arbitrary this shows that $f(\mathbf{x}) = \mathbf{y}' = \mathbf{y}$, which shows that f is surjective since $\mathbf{y}$ was arbitrary.

The existence of f therefore shows that $S_{\mathbf{a}}$ and $(\mathbb{Z}_+)^n$ have the same order type. $\square$

(b)

Proof. Consider any nonempty subset B of A. Clearly the sequence $(1, 1, \ldots)$ is the smallest element of A and hence if it is in B then it is also the smallest element of B. So suppose that $(1, 1, \ldots) \notin B$ so that, for every $\mathbf{x} \in B$ there is a unique greatest $n_{\mathbf{x}} \in \mathbb{Z}_+$ where $x_{n_{\mathbf{x}}} > 1$ but $x_i = 1$ for all $i > n_{\mathbf{x}}$. So let $I = \{n_{\mathbf{x}} \mid \mathbf{x} \in B\}$, noting that $B \neq \varnothing$ implies that $I \neq \varnothing$ as well. Thus I is a nonempty subset of $\mathbb{Z}_+$ and hence has a smallest element n. If we then let B_n be the set of sequences $\mathbf{x} \in B$

where $x_n > 1$ but $x_i = 1$ for all $i > n$, then the fact that $n \in I$ clearly implies that $B_n \neq \varnothing$. Also, if we define the sequence

$$a_i = \begin{cases} 2 & i = n+1 \\ 1 & i \neq n+1 \end{cases}$$

as in part (a) then it follows from what was shown there that $B_n \subset S_{\mathbf{a}}$. Moreover it was shown that $S_{\mathbf{a}}$ has the same order type as the dictionary order of $(\mathbb{Z}_+)^n$, which we know to be a well-ordering. Hence $S_{\mathbf{a}}$ must also be a well-ordering so that B_n has a smallest element $\mathbf{b} = (b_1, b_2, \ldots)$ since it is a nonempty subset of $S_{\mathbf{a}}$. We claim that $\mathbf{b}$ is in fact the smallest element of all of B.

So consider any $\mathbf{x} \in B$ so that $n_{\mathbf{x}} \in I$. It then follows that $n \leq n_{\mathbf{x}}$ since it is the smallest element of I. If $n = n_{\mathbf{x}}$ then we have that $\mathbf{x} \in B_n$ so that $\mathbf{b} \leq \mathbf{x}$ since it is the smallest element of B_n. If $n < n_{\mathbf{x}}$ then we have that $b_{n_{\mathbf{x}}} = 1 < x_{n_{\mathbf{x}}}$ but $b_i = 1 = x_i$ for every $i > n_{\mathbf{x}} > n$. This shows that $\mathbf{b} < \mathbf{x}$. Thus in all cases $\mathbf{b} \leq \mathbf{x}$, which shows that $\mathbf{b}$ is the smallest element of B since $\mathbf{x}$ was arbitrary. Since B was arbitrary, this shows that A is well-ordered as desired. □

Note that, in the theory of ordinal numbers, the set $(\mathbb{Z}_+)^n$ (and therefore the corresponding section of A) has order type ω^n. It would seem then that the set A has order type ω^ω.

Exercise 10.10

Theorem. Let J and C be well-ordered sets; assume that there is no surjective function mapping a section of J onto C. Then there exists a unique function $h : J \to C$ satisfying the equation

$$(*) \qquad h(x) = \text{smallest}[C - h(S_x)]$$

for each $x \in J$, where S_x is the section of J by x.

Proof.

(a) If h and k map sections of J, or all of J, into C and satisfy $(*)$ for all x in their respective domains, show that $h(x) = k(x)$ for all x in both domains.

(b) If there exists a function $h : S_\alpha \to C$ satisfying $(*)$, show that there exists a function $k : S_\alpha \cup \{\alpha\} \to C$ satisfying $(*)$.

(c) If $K \subset J$ and for all $\alpha \in K$ there exists a function $h_\alpha : S_\alpha \to C$ satisfying $(*)$, show that there exists a function

$$k : \bigcup_{\alpha \in K} S_\alpha \to C$$

satisfying $(*)$.

(d) Show by transfinite induction that for every $\beta \in J$, there exists a function $h_\beta : S_\beta \to C$ satisfying $(*)$. [Hint: If β has an immediate predecessor α, then $S_\beta = S_\alpha \cup \{\alpha\}$. If not, S_β is the union of all S_α with $\alpha < \beta$.]

(e) Prove the theorem.

Solution:

The following lemma is proof by transfinite induction, which is more straightforward than having to frame everything in terms of inductive sets. Henceforth we use this whenever transfinite induction is required.

Lemma 10.10.1. *(Proof by transfinite induction) Suppose that J is a well-ordered set and $P(x)$ is a proposition with parameter x. Suppose also that if $P(x)$ is true for all $x \in S_\alpha$ (where S_α is a section of J), then $P(\alpha)$ is also true. Then $P(\beta)$ is true for every $\beta \in J$.*

Proof. Let $J_0 = \{x \in J \mid P(x)\}$. We show that J_0 is inductive. So consider any $\alpha \in J$ and suppose that $S_\alpha \subset J_0$. Then, for any $x \in S_\alpha$ we have that $x \in J_0$ so that $P(x)$. It then follows that $P(\alpha)$ is also true since x was arbitrary, and so $\alpha \in J_0$. Since $\alpha \in J$ was arbitrary, this shows that J_0 is inductive. It then follows from Exercise 10.7 that $J_0 = J$. So consider any $\beta \in J$ so that also $\beta \in J_0$ and hence $P(\beta)$ is true. Since β was arbitrary, this shows the desired result. □

Main Problem.

(a)

Proof. First suppose that the domains of h and k are sets H and K where each is either a section of J or J itself. Since this is the case, we can assume without loss of generality that $H \subset K$ and so H is exactly the domain common to both h and k. Now suppose that the hypothesis we are trying to prove is *not* true so that there is an x in both domains (i.e. $x \in H$) where $h(x) \neq k(x)$. We can also assume that x is the smallest such element since $H \subset J$ and J is well-ordered. It then clearly follows that $S_x \subset H$ is a section of J and that $h(y) = k(y)$ for all $y \in S_x$. From this we clearly have that $h(S_x) = k(S_x)$. But then we have

$$h(x) = \text{smallest}[C - h(S_x)] = \text{smallest}[C - k(S_x)] = k(x)$$

since both h and k satisfy $(*)$ and x is in the domain of both. This contradicts the supposition that $h(x) \neq k(x)$ so that it must be that no such x exists and hence h and k are the same in their common domain as desired. □

(b)

Proof. Suppose that $h : S_\alpha \to C$ is such a function satisfying $(*)$. Now let $\bar{S}_\alpha = S_\alpha \cup \{\alpha\}$ and we define $k : \bar{S}_\alpha \to C$ as follows. For any $x \in \bar{S}_\alpha$ set

$$k(x) = \begin{cases} h(x) & x \in S_\alpha \\ \text{smallest}[C - h(S_\alpha)] & x = \alpha \,. \end{cases}$$

We note that clearly S_α and $\{\alpha\}$ are disjoint so that this is unambiguous. We also note that h is not surjective onto C since S_α is a section of J, and hence $C - h(S_\alpha) \neq \varnothing$ and so has a smallest element since C is well-ordered.

Now we show that k satisfies $(*)$. First, clearly $h(S_x) = k(S_x)$ for any $x \leq \alpha$ since $k(y) = h(y)$ by definition for any $y \in S_x \subset S_\alpha$. Now consider any $x \in \bar{S}_\alpha$. If $x = \alpha$ then by definition we have

$$k(x) = \text{smallest}[C - h(S_\alpha)] = \text{smallest}[C - k(S_\alpha)] = \text{smallest}[C - k(S_x)] \,.$$

On the other hand, if $x \in S_\alpha$ then $x < \alpha$ so that

$$k(x) = h(x) = \text{smallest}[C - h(S_x)] = \text{smallest}[C - k(S_x)]$$

since h satisfies $(*)$. Therefore, since x was arbitrary, this shows that k also satisfies $(*)$. □

(c)

Proof. Let

$$k = \bigcup_{\alpha \in K} h_\alpha \,,$$

which we claim is the function we seek.

First we show that k is actually a function from $\bigcup_{\alpha \in K} S_\alpha$ to C. So consider any x in the domain of k. Suppose that (x, a) and (x, b) are both in k so that there are α and β in K where $(x, a) \in h_\alpha$ and $(x, b) \in h_\beta$. Since h_α and h_β both satisfy $(*)$, it follows from part (a) that $a = h_\alpha(x) = h_\beta(x) = b$ since clearly x is in the domain of both. This shows that k is indeed a function since (x, a) and (x, b) were arbitrary. Also clearly the domain of k is $\bigcup_{\alpha \in K} S_\alpha$ since, for any $x \in \bigcup_{\alpha \in K} S_\alpha$, we have that there is an $\alpha \in K$ where $x \in S_\alpha$. Hence x is in the domain of h_α and so in the domain of k. In the other direction, clearly if x is in the domain of k then it is in the domain of h_α for some $\alpha \in K$. Since this domain is S_α, clearly $x \in \bigcup_{\alpha \in K} S_\alpha$. Lastly, obviously the range of k can be C since this is the range of every h_α.

Now we show that k satisfies $(*)$. So consider any $x \in \bigcup_{\alpha \in K} S_\alpha$ so that $x \in S_\alpha$ for some $\alpha \in K$. Clearly we have that $k(y) = h_\alpha(y)$ for every $y \in S_\alpha$ since $h_\alpha \subset k$. It then immediately follows that $k(x) = h(x)$ and $k(S_x) = h_\alpha(S_x)$ since $S_x \subset S_\alpha$. Then, since h_α satisfies $(*)$, we have

$$k(x) = h_\alpha(x) = \text{smallest}[C - h_\alpha(S_x)] = \text{smallest}[C - k(S_x)] \,.$$

Since x was arbitrary, this shows that k satisfies $(*)$ as desired. □

(d)

Proof. Consider any $\beta \in J$ and suppose that, for every $x \in S_\beta$, there is a function $h_x : S_x \to C$ satisfying $(*)$. Now, if β has an immediate predecessor α then we claim that $S_\beta = S_\alpha \cup \{\alpha\}$. First if $x \in S_\beta$ then $x < \beta$ so that $x \leq \alpha$ since α is the immediate predecessor of β. If $x < \alpha$ then $x \in S_\alpha$ and if $x = \alpha$ then $x \in \{\alpha\}$. Hence in either case we have that $x \in S_\alpha \cup \{\alpha\}$. Now suppose that $x \in S_\alpha \cup \{\alpha\}$. If $x \in S_\alpha$ then $x < \alpha < \beta$ so that $s \in S_\beta$. On the other hand if $x \in \{\alpha\}$ then $x = \alpha < \beta$ so that again $x \in S_\beta$. Thus we have shown that $S_\beta \subset S_\alpha \cup \{\alpha\}$ and $S_\alpha \cup \{\alpha\} \subset S_\beta$ so that $S_\beta = S_\alpha \cup \{\alpha\}$. Since $\alpha \in S_\beta$ it follows that there is an $h_\alpha : S_\alpha \to C$ that satisfies $(*)$. Then, by part (b), we have that there is an $h_\beta : S_\beta = S_\alpha \cup \{\alpha\} \to C$ that also satisfies $(*)$.

If β does not have an immediate predecessor then we claim that $S_\beta = \bigcup_{\gamma < \beta} S_\gamma$. So consider any $x \in S_\beta$ so that $x < \beta$. Since x cannot be the immediate predecessor of β, there must be an α where $x < \alpha < \beta$. Then $x \in S_\alpha$ so that, since $\alpha < \beta$, clearly $x \in \bigcup_{\gamma < \beta} S_\gamma$. Now suppose that $x \in \bigcup_{\gamma < \beta} S_\gamma$ so that there is an $\alpha < \beta$ where $x \in S_\alpha$. Then clearly $x < \alpha < \beta$ so that also $x \in S_\beta$. Thus we have shown that $S_\beta \subset \bigcup_{\gamma < \beta} S_\gamma$ and $\bigcup_{\gamma < \beta} S_\gamma \subset S_\beta$ so that $S_\beta = \bigcup_{\gamma < \beta} S_\gamma$. Now, clearly S_β is a subset of J where there is an $h_x : S_x \to C$ satisfying $(*)$ for every $x \in S_\beta$. Then it follows from what was shown in part (c) that there is a function h_β from $\bigcup_{\gamma < S_\beta} S_\gamma = \bigcup_{\gamma < \beta} S_\gamma = S_\beta$ to C that satisfies $(*)$.

Therefore, in either case, we have shown that there is an $h_\beta : S_\beta \to C$ that satisfies $(*)$. The desired result then follows by transfinite induction. □

(e)

Proof. First suppose that J has no largest element. Then we claim that $J = \bigcup_{\alpha \in J} S_\alpha$. For any $x \in J$ there must be a $y \in J$ where $x < y$ since x cannot be the greatest element of J. Hence $x \in S_y$ so that also clearly $\bigcup_{\alpha \in J} S_\alpha$. Then, for any $x \in \bigcup_{\alpha \in J} S_\alpha$, there is an $\alpha \in J$ where $x \in S_\alpha$. Clearly $S_\alpha \subset J$ so that $x \in J$ also. Hence $J \subset \bigcup_{\alpha \in J} S_\alpha$ and $\bigcup_{\alpha \in J} S_\alpha \subset J$ so that $J = \bigcup_{\alpha \in J} S_\alpha$. Since we know from part (d) that there is an $h_\alpha : S_\alpha \to C$ that satisfies $(*)$ for every $\alpha \in J$, it follows from part (c) that there is a function h from $\bigcup_{\alpha \in J} S_\alpha = J$ to C that satisfies $(*)$.

If J does have a largest element β then clearly $J = S_\beta \cup \{\beta\}$. Since we know that there is an $h_\beta : S_\beta \to C$ that satisfies $(*)$ by part (d), it follows from part (b) that there is a function h from $S_\beta \cup \{\beta\} = J$ to C that satisfies $(*)$. Hence the desired function h exists in both cases. part (a) also clearly shows that this function is unique. □

Exercise 10.11

Let A and B be two sets. Using the well-ordering theorem, prove that either they have the same cardinality, or one has cardinality greater than the other. [Hint: If there is no surjection $f : A \to B$, apply the preceding exercise.]

Solution:

Lemma 10.11.1. *For well-ordered sets $A \neq \varnothing$ and B there is an injection from A to B if and only if there is a surjection from B to A.*

Proof. ($\Rightarrow$) Suppose that there is an injection $f : A \to B$ and $A \neq \varnothing$. Then there is an $a \in A$. We then construct a surjection $g : B \to A$ as follows. For any $y \in B$ if $b \in f(A)$ then there is a unique $x \in A$ where $y = f(x)$. It is unique since, if x and x' are in A where $f(x) = y = f(x')$, then $x = x'$ since f is injective. So in this case set $g(y) = x$. If $b \notin f(A)$, then set $g(y) = a$. Clearly g is a function from B to A. To show that g is surjective, consider any $x \in A$ and let $y = f(x)$, which is clearly an element of B. Then, since obviously $y \in f(A)$ and x is the unique $x \in A$ such that $y = f(x)$, we have that $g(y) = x$ by definition. This shows that g is surjective since x was arbitrary.

($\Leftarrow$) Now suppose that $g : B \to A$ is surjective. We then construct an injection $f : A \to B$ as follows. For any $x \in A$ we have that the set $B_x = \{y \in B \mid g(y) = x\}$ is nonempty since g is surjective. Hence B_x has a unique smallest element y since it is a nonempty subset of B and B is well-ordered. So simply set $f(x) = y$. Clearly f is a function from A to B. To show that f is injective, consider $x, x' \in A$ where $x \neq x'$. Then clearly the sets B_x and $B_{x'}$ have to be disjoint for otherwise there would be a $y \in B$ where $g(y) = x$ and $g(y) = x'$, which is impossible if $x \neq x'$ since g is a function. Hence, since $f(x)$ and $f(x')$ are defined to be the smallest elements of B_x and $B_{x'}$, respectively, we have $f(x) \neq f(x')$. This shows that f is injective since x and x' were arbitrary. □

Main Problem.

Proof. First suppose that A and B are each well-ordered, which follows from the well-ordering theorem. Also suppose that A and B do *not* have the same cardinality so that it suffices to show that either B has greater cardinality than A or vice versa. If $A = \varnothing$ then it cannot be that $B = \varnothing$ as well since then they would have the same cardinality ($\varnothing$ would be a trivial bijection between them). Hence $B \neq \varnothing$ so that clearly B has greater cardinality than A. Thus in what follows assume that $A \neq \varnothing$.

Suppose that there is an injection from A to B. Then there cannot be an injection from B to A since, if there were, then A and B would have the same cardinality by the Cantor-Schroeder-Bernstein Theorem (shown in Exercise 7.6 part (b)). Thus B has greater cardinality than A by definition.

On the other hand, if there is no injection from A to B then there is no surjection from B to A by Lemma 10.11.1 since they are both well-ordered and $A \neq \varnothing$. It then clearly follows that no section of B can be a surjection onto A since then any extension of such a function to all of B would also be a surjection onto A. From this we have by Exercise 10.10 that there is a unique function $h : B \to A$ with the property that

$$h(x) = \text{smallest}[A - h(S_x)]\,,$$

where of course S_x is the section of B by x.

We claim that h is injective. So consider any y and y' in B where $y \neq y'$. Without loss of generality we can assume that $y < y'$ (by the well-ordering on B). It then follows that $y \in S_{y'}$ so that clearly $h(y) \in h(S_{y'})$. However, we have that $h(y')$ is the smallest element of $A - h(S_{y'})$ so that obviously $h(y') \notin h(S_{y'})$. Hence it must be that $h(y) \neq h(y')$, which shows that h is injective since y and y' were arbitrary.

Therefore there is an injection from B to A but none from A to B so that A has greater cardinality than B by definition. This shows the desired result since these cases are exhaustive. □

§11 The Maximum Principle

Exercise 11.1

If a and b are real numbers, define $a \prec b$ if $b - a$ is positive and rational. Show this is a strict partial order on $\mathbb{R}$. What are the maximal simply ordered subsets?

Solution:

Lemma 11.1.1. *If B is a maximal simply ordered subset of a nonempty partially ordered set A, then B is nonempty.*

Proof. Since A is nonempty, there is an $a \in A$. Clearly $\varnothing$ is vacuously simply ordered. However, it cannot be maximal since clearly the set $\{a\}$ properly contains $\varnothing$ as a subset but is also clearly vacuously simply ordered by $\prec$. Hence, since B *is* maximal it must be that $B \neq \varnothing$ as desired. □

Main Problem.

First we show that $\prec$ is a strict partial order.

Proof. First consider any $a \in \mathbb{R}$ so that $a - a = 0$, which is not positive and hence it is not true that $a \prec a$. Therefore $\prec$ is nonreflexive. Now consider $a, b, c \in \mathbb{R}$ where $a \prec b$ and $b \prec c$. Then we have that $x = b - a$ and $y = c - b$ are positive and rational. It then clearly follows that

$$c - a = (c - b) + (b - a) = y + x$$

is also rational and positive since both x and y are. Thus $a \prec c$, which shows that $\prec$ is transitive. Since $\prec$ was shown to be nonreflexive and transitive, this shows that it is a strict partial order as desired. □

For any element $x \in \mathbb{R}$, define the set $A_x = \{y \in \mathbb{R} \mid x - y \in \mathbb{Q}\}$. We then claim that the collection $\mathcal{A} = \{A_x\}_{x \in \mathbb{R}}$ is exactly the set of all maximal simply ordered subsets.

Proof. Suppose that $\mathcal{B}$ is the set of maximally simply ordered subsets of $\mathbb{R}$. Then we show that $\mathcal{A} = \mathcal{B}$.

To show that $\mathcal{A} \subset \mathcal{B}$ consider any $X \in \mathcal{A}$ so that $X = A_x$ for some $x \in \mathbb{R}$. Now consider any distinct y and z in $X = A_x$ so that by definition $x - y$ and $x - z$ are both rational so that $z - x = -(x - z)$ is also rational. Then clearly $z - y = (z - x) + (x - y)$ is rational as is $y - z = -(z - y)$. Since y and z are distinct, we have that $z - y$ and $y - z$ are nonzero and that either $y < z$ or $z < y$. In the former case we have that $z - y$ is a positive rational number and in the latter $y - z$ is. Thus either $y \prec z$ or $z \prec y$, which shows that $X = A_x$ is simply ordered since y and z were arbitrary. Now consider any

$y \in A_x$ and $z \notin A_x$ so that $x - y$ is rational but $x - z$ is irrational so that $z - x = -(x - z)$ is also irrational. Since a rational added to an irrational is also irrational (which is trivially easy to prove), it follows that $z - y = (z - x) + (x - y)$ is irrational as is $y - z = -(z - y)$. Hence it cannot be that either $y \prec z$ or $z \prec y$. Since $y \in X$ and $z \notin X$ were arbitrary, this show that X is a maximal simply ordered set so that $X \in \mathcal{B}$. This shows that $\mathcal{A} \subset \mathcal{B}$ since X was arbitrary.

Now suppose that $X \in \mathcal{B}$ so that X is a maximal simply ordered set. It follows from Lemma 11.1.1 that X is nonempty so that there is an $x \in X$, and we claim that in fact $X = A_x$. So consider any $y \in X$. Clearly if $y = x$ then $x - y = x - x = 0 \in \mathbb{Q}$ so that $y \in A_x$. If $y \neq x$ than either $x \prec y$ or $y \prec x$ since X is simply ordered by $\prec$. In the former case we have that $y - x$ is positive and rational so that $x - y = -(y - x)$ is negative and rational, and hence $y \in A_x$. In the latter case we have that $x - y$ is positive and rational so that clearly again $y \in A_x$. Since y was arbitrary this shows that $X \subset A_x$. Now consider any $y \in A_x$ so that $x - y \in \mathbb{Q}$. If $y = x$ then clearly $y \in X$. If $y \neq x$ then either $y - x$ or $x - y$ is positive, and also clearly rational since $x - y$ is rational. Hence either $x \prec y$ or $y \prec x$. It then follows from the fact that X is *maximally* simply ordered that y must be in X since otherwise y would not be comparable with x. Since again y was arbitrary this shows that $A_x \subset X$. Hence $X = A_x$ so that clearly $X \in \{A_x\}_{x \in \mathbb{R}} = \mathcal{A}$. Since X was arbitrary this shows that $\mathcal{B} \subset \mathcal{A}$.

Therefore we have shown that $\mathcal{A} = \mathcal{B}$, which shows that $\mathcal{A}$ is exactly the complete set of maximally simply ordered subsets. □

As an example of a particular maximally well-ordered set we have $\mathbb{Q} = A_0$ itself.

Exercise 11.2

(a) Let $\prec$ be a strict partial order on the set A. Define a relation on A by letting $a \preceq b$ if either $a \prec b$ or $a = b$. Show that this relation has the following properties, which are called the ***partial order axioms***:

  (i) $a \preceq a$ for all $a \in A$.

  (ii) $a \preceq b$ and $b \preceq a \Rightarrow a = b$.

  (iii) $a \preceq b$ and $b \preceq c \Rightarrow a \preceq c$.

(b) Let P be a relation on A that satisfies properties (i)-(iii). Define a relation S on A by letting aSb if aPb and $a \neq b$. Show that S is a strict partial order on A.

Solution:

(a)

Proof. We show that $\preceq$ satisfies the three partial order axioms:

(i) Consider any $a \in A$. Since obviously $a = a$ we have by definition that $a \preceq a$.

(ii) Suppose that $a \preceq b$ and $b \preceq a$. Then either $a \prec b$ or $a = b$, and either $b \prec a$ or $b = a$. So suppose that $a \neq b$ so that it must be that $a \prec b$ and $b \prec a$. Since $\prec$ is a strict partial order, it is transitive so that $a \prec a$ since $a \prec b$ and $b \prec a$. But this contradicts the nonreflexivity of $\prec$. Hence it must be that $a = b$ as desired.

(iii) Suppose that $a \preceq b$ and $b \preceq c$. Hence either $a \prec b$ or $a = b$, and either $b \prec c$ or $b = c$.

Case: $a \prec b$. If $b \prec c$ then clearly $a \prec c$ since $\prec$ is transitive (since it is a strict partial order). If $b = c$ then we have that $a \prec b = c$.

Case: $a = b$. If $b \prec c$ then we have that $a = b \prec c$. If $b = c$ then we have that $a = b = c$.

Hence in all cases and sub-cases we have that $a \prec c$ or $a = c$, and thus $a \preceq c$ by definition. $\square$

(b)

Proof. We show that S satisfies the two strict partial order axioms:

Nonreflexivity. Consider any $a \in A$. Since $a = a$ it follows that it is not true that $a \neq a$ and hence not true that aSa. Thus S is nonreflexive since a was arbitrary.

Transitivity. Suppose that aSb and bSc. Hence by definition aPb and $a \neq b$, and bPc and $b \neq c$. Then, by the transitivity property of the partial order axioms, which is property (iii), we have that aPc. Suppose for a moment that $a = c$. Then we would have aPb and bPa (since bPc and $c = a$). Then by partial order axiom (ii) we have that $a = b$, which contradicts the fact that $a \neq b$. So it must be that $a \neq c$. Thus aPc and $a \neq c$ so that aSc, which shows that S is transitive. $\square$

Exercise 11.3

Let A be a set with a strict partial order $\prec$; let $x \in A$. Suppose that we wish to find a maximal simply ordered subset B of A that contains x. One plausible way of attempting to define B is to let B equal the set of all those elements of A that a *comparable* with x:

$$B = \{y \mid y \in A \text{ and either } x \prec y \text{ or } y \prec x\} \ .$$

But this will not always work. In which of Examples 1 and 2 will this procedure succeed and in which will it not?

Solution:

First, it seems that, as defined above, B does not actually contain x itself! This is because it is not true that $x \prec x$ by the nonreflexivity of the partial order $\prec$. We assume that this was an oversight, which is easily remedied by defining

$$B' = \{y \in A \mid \text{either } x \prec y \text{ or } y \prec x\}$$

and $B = B' \cup \{x\}$.

For Example 1, a circular region in $\mathbb{R}^2$ is clearly

$$C_{\mathbf{x}_0,r} = \left\{\mathbf{x} \in \mathbb{R}^2 \mid |\mathbf{x} - \mathbf{x}_0| < r\right\} \ ,$$

where the point $\mathbf{x}_0 \in \mathbb{R}^2$ is the center of the circle, $r \in \mathbb{R}_+$ is the radius, and $|(x, y)| = \sqrt{x^2 + y^2}$ is the standard vector magnitude. Then the collection $\mathcal{A}$ is the set of all circular regions:

$$\mathcal{A} = \left\{C_{\mathbf{x}_0,r} \mid \mathbf{x}_0 \in \mathbb{R}^2 \text{ and } r \in \mathbb{R}_+\right\} \ .$$

Then let $\mathcal{C} = \left\{C_{(0,0),r} \mid r \in \mathbb{R}_+\right\}$ be the set of circles centered at the origin, which is a maximal simply ordered subset according to the example (and this is not difficult to show). Arbitrarily choose $X = C_{(0,0),1}$, that is the circular region of radius 1 centered at the origin, so that clearly $X \in \mathcal{C}$. Since the partial order in this is example is "is a proper subset of", define

$$\mathcal{B}' = \{Y \in \mathcal{A} \mid Y \subsetneq X \text{ or } X \subsetneq Y\}$$

and $\mathcal{B} = \mathcal{B}' \cup \{X\}$. The question is then whether $\mathcal{B} = \mathcal{C}$. We claim that, for this example, this is not the case.

Proof. Consider the set $C_{(1,0),2}$ and any $\mathbf{x} \in X = C_{(0,0),1}$ so that $|\mathbf{x} - (0,0)| < 1$. Then we have

$$|\mathbf{x} - (1,0)| \leq |\mathbf{x} - (0,0)| + |(0,0) - (1,0)| < 1 + |(-1,0)| = 1 + 1 = 2\,,$$

where we have utilized the ever-useful triangle inequality. Therefore $\mathbf{x} \in C_{(1,0),2}$ so that $X \subset C_{(1,0),2}$ since $\mathbf{x}$ was arbitrary. However, clearly the point $(1,0) \in C_{(1,0),2}$ but we have that $(1,0) \notin C_{(0,0),1} = X$ since $|(1,0) - (0,0)| = |(1,0)| = 1 \geq 1$. This shows that $X \subsetneq C_{(1,0),2}$ so that by definition $C_{(1,0),2} \in \mathcal{B}'$ and therefore $C_{(1,0),2} \in \mathcal{B} = \mathcal{B}' \cup \{X\}$. But clearly $C_{(1,0),2} \notin \mathcal{C}$ since it is not centered at the origin. This shows that $\mathcal{B} \neq \mathcal{C}$ as desired. □

Hence it would seem that this method of attempting to define a maximal simply ordered subset containing X has failed in this example. It is easy to come up with an analogous counterexample that shows the same result of the other example of a maximal simply ordered subset of circles tangent to the y-axis at the origin.

Regarding Example 2, recall that the order $\prec$ is defined by

$$(x_0, y_0) \prec (x_1, y_1)$$

if $y_0 = y_1$ and $x_0 < x_1$ for (x_0, y_0) and (x_1, y_1) in $\mathbb{R}^2$. It is then claimed (which is again easy to show) that maximal simply ordered subsets are horizontal lines in the plane, that is sets

$$L_{y_0} = \left\{(x, y) \in \mathbb{R}^2 \mid y = y_0\right\}$$

for some $y_0 \in \mathbb{R}$. So consider any such $y_0 \in \mathbb{R}$ and let $\mathbf{x} = (0, y_0)$. Now define

$$B' = \left\{\mathbf{y} \in \mathbb{R}^2 \mid \mathbf{x} \prec \mathbf{y} \text{ or } \mathbf{y} \prec \mathbf{x}\right\}$$

and $B = B' \cup \{\mathbf{x}\}$. In contrast to Example 1, we here claim that $B = L_{y_0}$, which is to say that B *does* define the maximal simply ordered subset.

Proof. Consider any $(x, y) \in B = B' \cup \{\mathbf{x}\}$ so that either $(x, y) \in B'$ or $(x, y) = \mathbf{x}$. Clearly if $(x, y) = \mathbf{x} = (0, y_0)$ then $(x, y) \in L_{y_0}$ since $y = y_0$. On the other hand, if $(x, y) \in B'$ then $(x, y) \prec \mathbf{x}$ or $\mathbf{x} \prec (x, y)$. In the former case we have that $(x, y) \prec \mathbf{x} = (0, y_0)$ so that, by definition $y = y_0$ and $x < 0$. Clearly then $(x, y) = (x, y_0) \in L_{y_0}$ by definition. In the latter case we also have $y = y_0$ (though this time $0 < x$) so that again $(x, y) \in L_{y_0}$. Since (x, y) was arbitrary, this shows that $B \subset L_{y_0}$.

Now consider any $(x, y) \in L_{y_0}$ so that $y = y_0$. If $x = 0$ then $(x, y) = (0, y_0) = \mathbf{x}$ so that obviously $(x, y) \in \{\mathbf{x}\}$. If $0 < x$ then $(x, y) = (x, y_0) \prec (0, y_0) = \mathbf{x}$ so that $(x, y) \in B'$. Similarly, if $x < 0$, then $\mathbf{x} = (0, y_0) \prec (x, y_0) = (x, y)$ so that again $(x, y) \in B'$. Hence in all cases either $(x, y) \in B'$ or $(x, y) \in \{\mathbf{x}\}$ so that $(x, y) \in B' \cup \{\mathbf{x}\} = B$. This shows that $L_{y_0} \subset B$ since again (x, y) was arbitrary.

Thus we have shown that $B = L_{y_0}$ as desired. □

So it would seem that, in this example, this naive technique does work!

Exercise 11.4

Given two points (x_0, y_0) and (x_1, y_1) of $\mathbb{R}^2$, define

$$(x_0, y_0) \prec (x_1, y_1)$$

if $x_0 < x_1$ and $y_0 \leq y_1$. Show that the curves $y = x^3$ and $y = 2$ are maximal simply ordered subsets of $\mathbb{R}^2$, and the curve $y = x^2$ is not. Find all maximal simply ordered subsets.

Solution:

First define

$$A = \left\{(x, y) \in \mathbb{R}^2 \mid y = x^3\right\} .$$

We show that it is a maximal simply ordered subset of $\mathbb{R}^2$.

Proof. First we show that A is simply ordered by $\prec$. Consider distinct (x_0, y_0) and (x_1, y_1) in A so that $y_0 = x_0^3$ and $y_1 = x_1^3$. Since they are distinct, it has to be that $x_0 \neq x_1$ or $y_0 \neq y_1$. The latter case actually implies the former since the function $f(x) = x^3$ is a well-defined function. Hence we can assume that $x_0 \neq x_1$, from which we can also assume without loss of generality that $x_0 < x_1$. Since $f(x) = x^3$ is also a monotonically increasing function (which is easy to show), it then follows that $y_0 = x_0^3 < x_1^3 = y_1$. Thus we have that $x_0 < x_1$ and $y_0 \leq y_1$ so that $(x_0, y_0) \prec (x_1, y_1)$ by definition. Since (x_0, y_0) and (x_1, y_1) were arbitrary, this shows that $\prec$ is a simple order on A.

To show that it is maximal suppose that B is any proper superset of A so that there is an $(x, y) \in B$ where $(x, y) \notin A$. Therefore clearly $y \neq x^3$ by definition. Now let $z = x^3$ so that $y \neq x^3 = z$ but $(x, z) \in A$. Clearly it is not true that $x < x$ so that it can neither be that $(x, y) \prec (x, z)$ nor $(x, z) \prec (z, y)$. Hence (x, y) and (x, z) are incomparable in $\prec$. This shows that B is not simply ordered and thus that A is maximal since B was an arbitrary superset. $\square$

Now redefine

$$A = \left\{(x, y) \in \mathbb{R}^2 \mid y = 2\right\} ,$$

which we also show is a maximal simply ordered subset of $\mathbb{R}^2$.

Proof. To show that A is simply ordered consider distinct (x_0, y_0) and (x_1, y_1) in A so that $y_0 = y_1 = 2$. Since these points are distinct and $y_0 = y_2$ is must be that $x_0 \neq x_1$, from which we can assume that $x_0 < x_1$ without loss of generality. But then clearly it is true that $x_0 < x_1$ and $y_0 \leq y_1$ so that $(x_0, y_0) \prec (x_1, y_1)$. Since these points were arbitrary this shows that A is simply ordered by $\prec$.

To show that it is maximal suppose that B is any proper superset of A so that there is an $(x, y) \in B$ where $(x, y) \notin A$. It then follows that $y \neq 2$ so that the point $(x, 2) \in A$ but $(x, 2) \neq (x, y)$. Clearly it can be that neither $(x, 2) \prec (x, y)$ nor $(x, y) \prec (x, 2)$ since it is not true that $x < x$. Hence (x, y) and $(x, 2)$ are incomparable in $\prec$. This shows that B is not simply ordered by $\prec$. Since B was an arbitrary superset this shows that A is maximal. $\square$

Now let

$$A = \left\{(x, y) \in \mathbb{R}^2 \mid y = x^2\right\} .$$

We claim that this subset is not simply ordered by $\prec$ and therefore cannot be a maximal simply ordered subset.

Proof. Consider the clearly distinct points $(-1, 1)$ and $(0, 0)$. Clearly since $0 = 0^2$ and $1 = (-1)^2$ these are both in A. However, since $1 > 0$ it is not true that $-1 < 0$ and $1 \leq 0$, and therefore it is not true that $(-1, 1) \prec (0, 0)$. Similarly since $0 \geq -1$ it is not true that $0 < -1$ and $0 \leq 1$, and therefore it is not true that $(0, 0) \prec (-1, 1)$. Hence the two distinct points are both in A but are not comparable. This suffices to show that A is not simply ordered by $\prec$. $\square$

We now claim that the maximal simply ordered subsets of $\mathbb{R}^2$ as ordered by $\prec$ are exactly the collection of sets of the form

$$A_f = \{(x,y) \in \mathbb{R}^2 \mid y = f(x)\}$$

for some function $f : (a,b) \to \mathbb{R}$, where (a,b) is an open interval of $\mathbb{R}$, noting that it could be that $a = -\infty$ and/or $b = \infty$. The function f must also satisfy the following properties:

(i) It is non-decreasing. Recall that this means that $x < y$ implies that $f(x) \leq f(y)$ for any $x, y \in (a,b)$.

(ii) If $b < \infty$ then its image is unbounded above.

(iii) If $a > -\infty$ then its image is unbounded below.

Now, let $\mathcal{A}$ be the collection of all these subsets and let $\mathcal{B}$ denote the set of all maximal simply ordered subsets. We show that $\mathcal{A} = \mathcal{B}$.

Proof. ($\subset$) First consider any $A_f \in \mathcal{A}$ so that $f : (a,b) \to \mathbb{R}$ with the properties above for some open interval (a,b). To show that A_f is simply ordered by $\prec$ consider any distinct (x,y) and (x',y') in A_f so that $y = f(x)$ and $y' = f(x')$. Since these are distinct it follows that $x \neq x'$ or $f(x) = y \neq y' = f(x')$. In the latter case it also follows that $x \neq x'$ as well for otherwise f would not be a function. Hence we can, without loss of generality, assume that $x < x'$. Since f is non-decreasing it follows that also $y = f(x) \leq f(x') = y'$, and therefore by definition $(x,y) \prec (x',y')$. Since these elements of A_f were arbitrary, it follows that A_f is simply ordered by $\prec$.

To show that A_f is maximal consider any proper superset A of A_f so that there is an $(x,y) \in A$ where $(x,y) \notin A_f$. There are a few possible ways in which (x,y) can fail to be an element of A_f.

Case: $x \in (a,b)$. Then it must be that $y \neq f(x)$ since $(x,y) \notin A_f$. Since it is not true that $x < x$, it has to be that neither $(x,y) \prec (x, f(x))$ nor $(x, f(x)), (x,y)$. Hence (x,y) and $(x, f(x))$ are incomparable elements of A (noting that clearly $(x, (f(x)) \in A_f \subset A)$ so that A is not simply ordered by $\prec$.

Case: $x \geq b$. Note that this is only possible if $b < \infty$ so that $b \in \mathbb{R}$. Thus in this case we have that the image of f is unbounded above by property (ii). Hence there is a $y_u \in reals$ where $y' > y$ and y' is in the image of f. Thus there is also an $x' \in (a,b)$ where $y = f(x')$ so that $(x',y') \in A_f \subset A$. Now, we have $x' < b \leq x$ but $y' > y$ so that it is not true that $y' \leq y$, and hence it cannot be that $(x',y') \prec (x,y)$. Similarly is it is clearly not true that $x < x'$ so that it cannot be that $(x,y) \prec (x',y')$ either. This shows that (x,y) and (x',y') are incomparable elements of A so that A is not simply ordered.

Case: $x \leq a$. An argument analogous to the previous case shows that $a > -\infty$ so that the image of f is unbounded below. From this it follows again that A is not simply ordered.

Thus in all cases A is not simply ordered so that A_f is a maximal simply ordered subset of $\mathbb{R}^2$ since A was an arbitrary proper superset. This shows that $A_f \in \mathcal{B}$ so that $\mathcal{A} \subset \mathcal{B}$ since A_f was arbitrary.

($\supset$) Now consider any $B \in \mathcal{B}$ so that B is a maximal simply ordered set by $\prec$. Define

$$X = \{x \in \mathbb{R} \mid (x,y) \in B \text{ for some } y \in \mathbb{R}\} .$$

We prove that B has the following properties:

(1) If (x_0, y_0) and (x_1, y_1) are in B and $x_0 < x_1$ then $y_0 \leq y_1$.

(2) For every $x \in X$ there is a unique $y \in \mathbb{R}$ where $(x,y) \in B$.

To show show (1) consider (x_0, y_0) and (x_1, y_1) in B and suppose that $x_0 < x_1$. Since B is simply ordered, it must be that either $(x_0, y_0) \prec (x_1, y_1)$ or $(x_1, y_1) \prec (x_0, y_0)$. Since $x_0 < x_1$ it clearly must be that $(x_0, y_0) \prec (x_1, y_1)$ and hence also $y_0 \leq y_1$.

To show (2) consider any $x \in X$. Clearly there is a $y \in \mathbb{R}$ where $(x, y) \in B$ by the definition of X. To show that this y is unique, suppose that (x, y_0) and (x, y_1) are both in B but that $y_0 \neq y_1$ so that (x, y_0) and (x, y_1) are distinct. Since B is simply ordered they must be comparable in $\prec$ but they clearly cannot be since it is not true that $x < x$. As this is a contradiction, it must be that $y_0 = y_1$.

With that out of the way, let b be the least upper bound of X if it is bounded above and $b = \infty$ otherwise. Similarly let a be the greatest lower bound if X is bounded below and $a = -\infty$ otherwise. Now we claim that X is equal to the open interval (a, b).

So consider any $x \in X$ so that then clearly $a \leq x \leq b$ since a and b are lower and upper bounds of X, respectively. Clearly if $b = \infty$ then it cannot be that $x = b$ (since $x \in \mathbb{R}$) so assume that $b \in \mathbb{R}$ and $x = b$. Then $b = x \in X$ so that by property (2) there is a unique $y \in \mathbb{R}$ where $(b, y) \in B$. Clearly then $(b+1, y) \notin B$, since $b+1 \notin X$, so that the set $B' = B \cup \{(b+1, y)\}$ is a proper superset of B. Now consider any $(x', y') \in B$ so that clearly $x' \in X$ and hence $x' \leq b < b + 1$. By property (1) above it also follows that $y' \leq y$, and so we have that $(x', y') \prec (b + 1, y)$. Since (x', y') was arbitrary, this shows that $(b + 1, y)$ is comparable to every element of B and hence B' is simply ordered by $\prec$. But this is not possible since B is maximal and B' is a proper superset. Hence it must be that $x \neq b$. An analogous argument shows that $x \neq a$ as well and hence $a < x < b$. Since x was arbitrary this shows that $X \subset (a, b)$.

Now consider any $x \in (a, b)$ so that $a < x < b$. Since b is the least upper bound of X, it has to be that x is not an upper of X so that there is an $x_g \in X$ where $x < x_g < b$ (clearly the existence of x_g also follows when $b = \infty$ since then X is unbounded above). Clearly then there is also a $y_g \in \mathbb{R}$ where $(x_g, y_g) \in B$. It then follows that the set $Y_g = \{y \in \mathbb{R} \mid (z, y) \in B \text{ for some } x < z < b\}$ is nonempty. By an analogous argument there is an $(x_l, y_l) \in B$ where $a < x_l < x$ so that the set $Y_l = \{y \in \mathbb{R} \mid (z, y) \in B \text{ for some } a < z < x\}$ is nonempty. Now, for any $y \in Y_g$, we have that $(z, y) \in B$ for some $x < z < b$. Therefore $x_l < x < z$ and by property (1) of B we have that $y_l \leq y$. Since y was arbitrary this shows that y_l is a lower bound of Y_g and hence it has a greatest lower bound y_v. So suppose that $x \notin X$ so that there is not a $y \in \mathbb{R}$ where $(x, y) \in B$. Then we have that the set $B \cup \{(x, y_v)\}$ is a proper superset of B. However, consider any $(x', y') \in B$ so that $x' \in X$ but $x' \neq x$.

Case: $x' < x$. Then it has to be that $a < x' < x$ so that $y' \in Y_l$. Then, for any $y \in Y_g$ we again have that $(z, y) \in B$ for some $x < z < b$. Hence $x' < x < z$ so that $y' \leq y$ by property (1) since $(x', y') \in B$ and $(z, y) \in B$. Since y was arbitrary, this shows that y' is a lower bound of Y_g. Since y_v is the greatest lower bound of Y_g, we have that $y' \leq y_v$. Then clearly $(x', y') \prec (x, y_v)$ since also $x' < x$.

Case: $x' > x$. Then it has to be that $x < x' < b$ so that $y' \in Y_g$. It then follows that $y_v \leq y'$ since y_v is the greatest lower bound of Y_g. Hence we have that $(x, y_v) \prec (x', y')$ since $x < x'$ as well.

Therefore in all cases we have that (x, y_v) and (x', y') are comparable in $\prec$. Since (x', y') was arbitrary, this clearly shows that $B \cup \{(x, y_v)\}$ is simply ordered. But this cannot be possible since it is a proper superset and B is maximal! So it has to be that in fact there *is* a $y \in \mathbb{R}$ where $(x, y) \in B$, and hence $x \in X$. Since $x \in (a, b)$ was arbitrary, this shows that $(a, b) \subset X$. This completes the rather long proof that $X = (a, b)$.

Now, by property (2) there is a unique $y \in \mathbb{R}$ for every $x \in X = (a, b)$ where $(x, y) \in B$. So we define a function $f : (a, b) \to \mathbb{R}$ by simply setting $f(x) = y$. Clearly based on the way this function is defined and the fact that $(a, b) = X$ we have that $B = A_f$. We must now show that f has the properties (i) through (iii) above.

Property (i) follows almost immediately from property (1) of B. To see this, consider any $x, y \in (a, b)$ where $x < y$. Then $(x, f(x))$ and $(y, f(y))$ are in B and hence $f(x) \leq f(y)$ by property (1). For property (ii) suppose that $b < \infty$ but that the image of f is bounded above. Hence it image has an upper bound, say $y_u \in \mathbb{R}$, so that clearly $B \cup \{(b+1, y_u)\}$ is a proper superset of B. So consider any $(x, y) \in B$ so that $y = f(x)$ for some $x \in (a, b)$. Then clearly $f(x)$ is in the image of f so that $y = f(x) \leq y_u$ since y_u is an upper bound of the image. Since also we must have $x < b < b + 1$, it follows that $(x, y) \prec (b + 1, y_u)$. Since $(x, y) \in B$ was arbitrary, this shows that $B \cup \{(b + 1, y_u)\}$ is simply ordered, which cannot be possible since it is a proper superset and B is maximal. So it has to be that in fact the image of f is unbounded above when $b < \infty$, which shows property (ii). An analogous argument shows property (iii).

Since f has all of the required properties and $B = A_f$, this shows that $B \in \mathcal{A}$. Clearly then $\mathcal{B} \subset \mathcal{A}$ since B was arbitrary. This shows that $\mathcal{A} = \mathcal{B}$ as desired. ☐

Lastly, note that the example curves $y = x^3$ and $y = 2$ are clearly in $\mathcal{A} = \mathcal{B}$ since they are non-decreasing functions on $\mathbb{R}$, ($\mathbb{R}$ being the same as the open interval $(-\infty, \infty)$), while the curve $y = x^2$ is not since it is decreasing when $x < 0$.

Exercise 11.5

Show that Zorn's Lemma implies the following:

Lemma (Kuratowski). Let $\mathcal{A}$ be a collection of sets. Suppose that for every subcollection $\mathcal{B}$ of $\mathcal{A}$ that is simply ordered by proper inclusion, the union of the elements of $\mathcal{B}$ belongs to $\mathcal{A}$. Then $\mathcal{A}$ has an element that is properly contained in no other element of $\mathcal{A}$.

Solution:

Proof. First, we know that $\subsetneq$ is a strict partial order on $\mathcal{A}$, which is trivial to show. So consider any simply ordered subset $\mathcal{B}$ of $\mathcal{A}$ and let $A = \bigcup \mathcal{B}$ so that we know that $A \in \mathcal{A}$. Clearly for any set $B \in \mathcal{B}$ we have that $B \subset \bigcup \mathcal{B} = A$ so that, since B was arbitrary, A is an upper bound of $\mathcal{B}$ in the strict partial order $\subsetneq$. Since $\mathcal{B}$ was arbitrary, this shows the hypothesis of Zorn's Lemma so that $\mathcal{A}$ has a maximal element A. Then clearly A is not properly contained in any other element of $\mathcal{A}$. ☐

Exercise 11.6

A collection $\mathcal{A}$ of subsets of a set X is said to be of *finite type* provided that a subset B of X belongs to $\mathcal{A}$ if and only if every finite subset of B belongs to $\mathcal{A}$. Show that the Kuratowski lemma implies the following:

Lemma (Tukey, 1940). Let $\mathcal{A}$ be a collection of sets. If $\mathcal{A}$ is of finite type, then $\mathcal{A}$ has an element properly contained in no other element of $\mathcal{A}$.

Solution:

Proof. Suppose that $\mathcal{A}$ is a collection of sets of finite type. Let $\mathcal{B}$ be a subcollection of $\mathcal{A}$ that is simply ordered by $\subsetneq$. Consider next any finite subset B of $\bigcup \mathcal{B}$. Then, for every $b \in B$, $b \in \bigcup \mathcal{B}$ so that we can choose a set $B_b \in \mathcal{B}$ such that $b \in B_b$. Note that this does not require the choice axiom since we need to make only a finite number of choices. Then the set $\mathcal{B}' = \{B_b \mid b \in B\}$ is clearly a finite set of elements of $\mathcal{B}$. Since $\mathcal{B}$ is simply ordered by $\subsetneq$, it follows that $\mathcal{B}'$ is as well and so has a largest element C since it is finite.

Hence, for any $b \in B$, we have that $b \in B_b \subset C$ so that $b \in C$, and so B is a finite subset of C. Since $C \in \mathcal{B}$ and $\mathcal{B} \subset \mathcal{A}$, clearly $C \in \mathcal{A}$. Since $\mathcal{A}$ is of finite type and B is a finite subset of C, it follows that $B \in \mathcal{A}$ also. Since B was an arbitrary finite subset of $\bigcup \mathcal{B}$, it then follows that $\bigcup \mathcal{B}$ is also in $\mathcal{A}$ since it is of finite type. It then follows from the Kuratowski lemma (Exercise 11.5) that $\mathcal{A}$ has an element that is properly contained in no other element of $\mathcal{A}$ as desired. □

Exercise 11.7

Show that the Tukey lemma implies the Hausdorff maximum principle. [Hint: If $\prec$ is a strict partial order on A, let $\mathcal{A}$ be the collection of all subsets of A that are simply ordered by $\prec$. Show that $\mathcal{A}$ is of finite type.]

Solution:

Proof. Following the hint, suppose that the set A has strict partial order $\prec$ and let $\mathcal{A}$ be the collection of all subsets of A that are simply ordered by $\prec$. We show that $\mathcal{A}$ has finite type, i.e. that a subset $B \subset A$ is in $\mathcal{A}$ if and only if every finite subset of B is.

($\Rightarrow$) Suppose that $B \subset A$ is in $\mathcal{A}$ so that it is simply ordered by $\prec$. Clearly any finite subset of B is also simply ordered by $\prec$ so that it is also in $\mathcal{A}$, which shows the result.

($\Leftarrow$) Now suppose that $B \subset A$ and that every finite subset of B is in $\mathcal{A}$. Now consider two distinct element x and y of B. Clearly then the set $\{x, y\}$ is a finite subset of B and hence is in $\mathcal{A}$. Then this means that $\{x, y\}$ is simply ordered by $\prec$ so that clearly x and y are comparable. Since x and y were arbitrary this shows that B is simply ordered by $\prec$ and hence $B \in \mathcal{A}$.

We have thus shown that $\mathcal{A}$ is of finite type so that it has a set C such that is properly contained in no other element of $\mathcal{A}$. Since $C \in \mathcal{A}$, it is simply ordered by $\prec$. It is also maximal since, if D is any proper superset of C then it cannot be that D is simply ordered for then we would have $D \in \mathcal{A}$ and $C \subsetneq D$, which would contradict the definition of C. Hence C is the maximal simply ordered subset of A that shows the maximum principle. □

Exercise 11.8

A typical use of Zorn's lemma in algebra is the proof that every vector space has a basis. Recall that if A is a subset of the vector space V, we say a vector belongs to that *span* of A if it equals a finite linear combination of elements of A. The set A is *independent* if the only finite linear combination of elements of A that equals the zero vector is the trivial one having all coefficients zero. If A is independent and if every vector in V belongs to the span of A, then A is a *basis* for V.

(a) If A is independent and $v \in V$ does not belong to the span of A, show $A \cup \{v\}$ is independent.
(b) Show the collection of all independent sets in V has a maximal element.
(c) Show that V has a basis.

Solution:
(a)

Proof. We show this by contradiction. Suppose that A is independent and $v \in V$ does not belong to the span of A. Also let $B = A \cup \{v\}$ and suppose that B is *not* independent. Then

$$\sum_{i=1}^{n} \beta_i b_i = 0$$

for some nonzero coefficients β_i, where each b_i is in B. Now, it must be that one of the b_i vectors is v and the rest in A since otherwise they would all be in A and then A would not be independent. Hence this can be expressed as

$$\sum_{i=1}^{n-1} \alpha_i a_i + \gamma v = 0$$

for nonzero coefficients α_i and γ and vectors $a_i \in A$. However clearly then we would have

$$v = -\frac{1}{\gamma} \sum_{i=1}^{n-1} \alpha_1 a_i = \sum_{i=1}^{n-1} \left(\frac{-\alpha_i}{\gamma} \right) a_i$$

so that v is a linear combination of vectors in A and hence is in the span of A. This is a contradiction so that it must be that in fact $B = A \cup \{v\}$ is independent as desired. □

(b)

Proof. Let $\mathcal{A}$ be the collection of all independent sets in V. We know that $\subsetneq$ is a strict partial order on $\mathcal{A}$. Now let $\mathcal{B}$ be any subset of $\mathcal{A}$ that is simply ordered by $\subsetneq$. We claim that $\bigcup \mathcal{B}$ is an upper bound of $\mathcal{B}$ that is in $\mathcal{A}$. So first consider any $B \in \mathcal{B}$ and any $b \in B$ so that clearly then $b \in \bigcup \mathcal{B}$. Hence $B \subset \bigcup \mathcal{B}$ since b was arbitrary. Since $B \in \mathcal{B}$ was arbitrary, this shows that $\bigcup \mathcal{B}$ is an upper bound of $\mathcal{B}$ by $\subsetneq$.

Next we show that $\bigcup \mathcal{B}$ is also in $\mathcal{A}$. To this end consider any finite set B of elements of $\bigcup \mathcal{B}$ so that B is a set of vectors in V. Now, for each $b \in B$ we have that $b \in \bigcup \mathcal{B}$ so that we can choose any set $B_b \in \mathcal{B}$ where $b \in B_b$. Note that this does not require the axiom of choice since B is finite. Then, since each B_b is in $\mathcal{B}$, which is simply ordered by $\subsetneq$ and $\{B_b \mid b \in B\}$ is finite, it follows that it has a largest element C so that $B_b \subset C$ for any $b \in B$. Hence $B \subset C$ since each $b \in B_b$ and $B_b \subset C$. Also $C \in \mathcal{A}$ since $C \in \mathcal{B}$ and $\mathcal{B} \subset \mathcal{A}$ so that C is independent. Hence the only linear combination of the vectors in B that is the zero vector must have all zero coefficients since they are all in the independent set C. Since B was an arbitrary set of vectors in $\bigcup \mathcal{B}$, this shows that $\bigcup \mathcal{B}$ is independent and therefore in $\mathcal{A}$.

Since $\mathcal{B}$ was an arbitrary simply ordered subset of $\mathcal{A}$, it follows that every such subset has an upper bound in $\mathcal{A}$. Thus by Zorn's Lemma $\mathcal{A}$ has a maximal element as desired. □

(c)

Proof. Again let $\mathcal{A}$ be the collection of all independent sets in V, which we know has a maximal element A from part (b). We claim that A is a basis for V. Suppose to the contrary that it is not so that, since we know that A is independent (since it is in $\mathcal{A}$), there must be a vector $v \in V$ that is not in the span of A. Then by part (a) we have that $A \cup \{v\}$ is also independent and so in $\mathcal{A}$. We also have that $v \notin A$ since otherwise it would clearly be in the span of A. Hence $A \subsetneq A \cup \{v\}$. However, this contradicts the fact that A is a maximal element of $\mathcal{A}$, so that it must be that in fact A is a basis for V as desired. □

§WO Supplementary Exercises: Well-Ordering

Exercise WO.1

Theorem (General principle of recursive definition). Let J be a well-ordered set; let C be a set. Let $\mathcal{F}$ be the set of all functions mapping sections of J into C. Given a function $\rho : \mathcal{F} \to C$, there is a unique

$h : J \to C$ such that $h(\alpha) = \rho(h \restriction S_\alpha)$ for each $\alpha \in J$. [Hint: Follow the pattern outlined in Exercise 10 of §10.]

Solution:

Following the hint, we follow the pattern of Exercise 10.10. In what follows denote by $(*)$ the property

$$h(\alpha) = \rho(h \restriction S_\alpha)$$

for a function h from J or a section of J to C.

Lemma WO.1.1. *If h and k map sections of J, or all of J, into C and satisfy $(*)$ for all x in their respective domains, then $h(x) = k(x)$ for all x in both domains.*

Proof. First suppose that the domains of h and k are sets H and K where each is either a section of J or J itself. Since this is the case, we can assume without loss of generality that $H \subset K$ and so H is exactly the domain common to both h and k. Now suppose that the hypothesis we are trying to prove is *not* true so that there is an x in both domains (i.e. $x \in H$) where $h(x) \neq k(x)$. We can also assume that x is the smallest such element since $H \subset J$ and J is well-ordered. It then clearly follows that $S_x \subset H$ is a section of J and that $h(y) = k(y)$ for all $y \in S_x$. From this we clearly have that $h \restriction S_x = k \restriction S_x$ so that

$$h(x) = \rho(h \restriction S_x) = \rho(k \restriction S_x) = k(x)$$

since both h and k satisfy $(*)$ and x is in the domain of both. This contradicts the supposition that $h(x) \neq k(x)$ so that it must be that no such x exists and hence h and k are the same in their common domain as desired. □

Lemma WO.1.2. *If there exists a function $h : S_\alpha \to C$ satisfying $(*)$, then there exists a function $k : S_\alpha \cup \{\alpha\} \to C$ satisfying $(*)$.*

Proof. Suppose that $h : S_\alpha \to C$ is such a function satisfying $(*)$. Now let $\bar{S}_\alpha = S_\alpha \cup \{\alpha\}$ and we define $k : \bar{S}_\alpha \to C$ as follows. For any $x \in \bar{S}_\alpha$ set

$$k(x) = \begin{cases} h(x) & x \in S_\alpha \\ \rho(h) & x = \alpha\,. \end{cases}$$

We note that clearly S_α and $\{\alpha\}$ are disjoint so that this is unambiguous. We also note that h is a function from a section of J to C so that $h \in \mathcal{F}$ and $\rho(h) \in C$ is therefore defined.

Now we show that k satisfies $(*)$. First, clearly $h \restriction S_x = k \restriction S_x$ for any $x \leq \alpha$ since $k(y) = h(y)$ by definition for any $y \in S_x \subset S_\alpha$. Now consider any $x \in \bar{S}_\alpha$. If $x = \alpha$ then by definition we have

$$k(x) = \rho(h) = \rho(h \restriction S_\alpha) = \rho(k \restriction S_\alpha) = \rho(k \restriction S_x)$$

since clearly $h = h \restriction S_\alpha$ since S_α is the domain of h. On the other hand, if $x \in S_\alpha$ then $x < \alpha$ so that

$$k(x) = h(x) = \rho(h \restriction S_x) = \rho(k \restriction S_x)$$

since h satisfies $(*)$. Therefore, since x was arbitrary, this shows that k also satisfies $(*)$. □

Lemma WO.1.3. *If $K \subset J$ and for all $\alpha \in K$ there exists a function $h_\alpha : S_\alpha \to C$ satisfying $(*)$, then there exists a function*

$$k : \bigcup_{\alpha \in K} S_\alpha \to C$$

satisfying $(*)$.

Proof. Let

$$k = \bigcup_{\alpha \in K} h_\alpha\,,$$

which we claim is the function we seek.

First we show that k is actually a function from $\bigcup_{\alpha\in K} S_\alpha$ to C. So consider any x in the domain of k. Suppose that (x,a) and (x,b) are both in k so that there are α and β in K where $(x,a) \in h_\alpha$ and $(x,b) \in h_\beta$. Since h_α and h_β both satisfy $(*)$, it follows from Lemma WO.1.1 that $a = h_\alpha(x) = h_\beta(x) = b$ since clearly x is in the domain of both. This shows that k is indeed a function since (x,a) and (x,b) were arbitrary. Also clearly the domain of k is $\bigcup_{\alpha\in K} S_\alpha$ since, for any $x \in \bigcup_{\alpha\in K} S_\alpha$, we have that there is an $\alpha \in K$ where $x \in S_\alpha$. Hence x is in the domain of h_α and so in the domain of k. In the other direction, clearly if x is in the domain of k then it is in the domain of h_α for some $\alpha \in K$. Since this domain is S_α, clearly $x \in \bigcup_{\alpha\in K} S_\alpha$. Lastly, obviously the range of k can be C since this is the range of every h_α.

Now we show that k satisfies $(*)$. So consider any $x \in \bigcup_{\alpha\in K} S_\alpha$ so that $x \in S_\alpha$ for some $\alpha \in K$. Clearly we have that $k(y) = h_\alpha(y)$ for every $y \in S_\alpha$ since $h_\alpha \subset k$. It then immediately follows that $k(x) = h(x)$ and $k \restriction S_x = h_\alpha \restriction S_x$ since $S_x \subset S_\alpha$. Then, since h_α satisfies $(*)$, we have

$$k(x) = h_\alpha(x) = \rho(h_\alpha \restriction S_x) = \rho(k \restriction S_x)\,.$$

Since x was arbitrary, this shows that k satisfies $(*)$ as desired. □

Lemma WO.1.4. *For every $\beta \in J$, there exists a function $h_\beta : S_\beta \to C$ satisfying $(*)$.*

Proof. We show this by transfinite induction. So consider any $\beta \in J$ and suppose that, for every $x \in S_\beta$, there is a function $h_x : S_x \to C$ satisfying $(*)$. Now, if β has an immediate predecessor α then we claim that $S_\beta = S_\alpha \cup \{\alpha\}$. First if $x \in S_\beta$ then $x < \beta$ so that $x \le \alpha$ since α is the immediate predecessor of β. If $x < \alpha$ then $x \in S_\alpha$ and if $x = \alpha$ then $x \in \{\alpha\}$. Hence in either case we have that $x \in S_\alpha \cup \{\alpha\}$. Now suppose that $x \in S_\alpha \cup \{\alpha\}$. If $x \in S_\alpha$ then $x < \alpha < \beta$ so that $s \in S_\beta$. On the other hand if $x \in \{\alpha\}$ then $x = \alpha < \beta$ so that again $x \in S_\beta$. Thus we have shown that $S_\beta \subset S_\alpha \cup \{\alpha\}$ and $S_\alpha \cup \{\alpha\} \subset S_\beta$ so that $S_\beta = S_\alpha \cup \{\alpha\}$. Since $\alpha \in S_\beta$ it follows that there is an $h_\alpha : S_\alpha \to C$ that satisfies $(*)$. Then, by Lemma WO.1.2, we have that there is an $h_\beta : S_\beta = S_\alpha \cup \{\alpha\} \to C$ that also satisfies $(*)$.

If β does not have an immediate predecessor then we claim that $S_\beta = \bigcup_{\gamma<\beta} S_\gamma$. So consider any $x \in S_\beta$ so that $x < \beta$. Since x cannot be the immediate predecessor of β, there must be an α where $x < \alpha < \beta$. Then $x \in S_\alpha$ so that, since $\alpha < \beta$, clearly $x \in \bigcup_{\gamma<\beta} S_\gamma$. Now suppose that $x \in \bigcup_{\gamma<\beta} S_\gamma$ so that there is an $\alpha < \beta$ where $x \in S_\alpha$. Then clearly $x < \alpha < \beta$ so that also $x \in S_\beta$. Thus we have shown that $S_\beta \subset \bigcup_{\gamma<\beta} S_\gamma$ and $\bigcup_{\gamma<\beta} S_\gamma \subset S_\beta$ so that $S_\beta = \bigcup_{\gamma<\beta} S_\gamma$. Now, clearly S_β is a subset of J where there is an $h_x : S_x \to C$ satisfying $(*)$ for every $x \in S_\beta$. Then it follows from Lemma WO.1.3 that there is a function h_β from $\bigcup_{\gamma<S_\beta} S_\gamma = \bigcup_{\gamma<\beta} S_\gamma = S_\beta$ to C that satisfies $(*)$.

Therefore, in either case, we have shown that there is an $h_\beta : S_\beta \to C$ that satisfies $(*)$. The desired result then follows by transfinite induction. □

Main Problem.

Proof. First suppose that J has no largest element. Then we claim that $J = \bigcup_{\alpha\in J} S_\alpha$. For any $x \in J$ there must be a $y \in J$ where $x < y$ since x cannot be the largest element of J. Hence $x \in S_y$ so that also clearly $\bigcup_{\alpha\in J} S_\alpha$. Then, for any $x \in \bigcup_{\alpha\in J} S_\alpha$, there is an $\alpha \in J$ where $x \in S_\alpha$. Clearly $S_\alpha \subset J$ so that $x \in J$ also. Hence $J \subset \bigcup_{\alpha\in J} S_\alpha$ and $\bigcup_{\alpha\in J} S_\alpha \subset J$ so that $J = \bigcup_{\alpha\in J} S_\alpha$. Since we know from Lemma WO.1.4 that there is an $h_\alpha : S_\alpha \to C$ that satisfies $(*)$ for every $\alpha \in J$, it follows from Lemma WO.1.3 that there is a function h from $\bigcup_{\alpha\in J} S_\alpha = J$ to C that satisfies $(*)$.

If J does have a largest element β then clearly $J = S_\beta \cup \{\beta\}$. Since we know that there is an $h_\beta : S_\beta \to C$ that satisfies $(*)$ by Lemma WO.1.4, it follows from Lemma WO.1.2 that there is a function h from $S_\beta \cup \{\beta\} = J$ to C that satisfies $(*)$. Hence the desired function h exists in both cases. Lemma WO.1.1 also clearly shows that this function is unique. □

Exercise WO.2

(a) Let J and E be well-ordered sets; let $h : J \to E$. Show that the following statements are equivalent:

(i) h is order preserving and its image is E or a section of E.

(ii) $h(\alpha) = \text{smallest } [E - h(S_\alpha)]$ for all α.

[Hint: Show that each of these conditions implies that $h(S_\alpha)$ is a section of E; conclude that it must be the section by $h(\alpha)$.]

(b) If E is a well-ordered set, show that no section of E has the order type of E, nor do two different sections of E have the same order type. [Hint: Given J, there is a most one order preserving map of J into E whose image is E or a section of E.]

Solution:

(a)

Proof. First, for any $\alpha \in J$ and $\beta \in E$, let S_α denote the section of J by α, and T_β denote the section of E by β. To avoid ambiguity, also suppose that $<$ is the well-order on J and $\prec$ is the well-order on E. We show that each of these conditions are equivalent to the condition that $h(S_\alpha) = T_{h(\alpha)}$ for every $\alpha \in J$. Call this condition (iii). This of course also shows that the conditions are equivalent to each other.

First we show that (i) implies (iii). So suppose that h is order preserving and its image is E or a section of E. Consider any $\alpha \in J$ and any $y \in h(S_\alpha)$ so that there is an $x \in S_\alpha$ where $y = h(x)$. Then $x < \alpha$ and $y = h(x) \prec h(\alpha)$ since h preserves order. Therefore $y \in T_{h(\alpha)}$ so that $h(S_\alpha) \subset T_{h(\alpha)}$ since y was arbitrary. Now consider $y \in T_{h(\alpha)}$ so that $y \prec h(\alpha)$. Since also clearly $y \in E$ (since $T_{h(\alpha)} \subset E$), y is in the image of h if its image is all of E. If the image of h is some section of E, say T_β, then clearly $h(\alpha) \in T_\beta$ since $h(\alpha)$ is obviously in the image of h. Hence we have $y \prec h(\alpha) \prec \beta$ so that $y \in T_\beta$ and hence in the image of h. Since y is in the image of h in either case, there is an $x \in J$ such that $y = h(x)$. Then $h(x) = y \prec h(\alpha)$ so that $x < \alpha$ since h preserves order. Hence $x \in S_\alpha$ so that $y \in h(S_\alpha)$ since $y = h(x)$. This shows that $T_{h(\alpha)} \subset h(S_\alpha)$ since y was arbitrary. Therefore $h(S_\alpha) = T_{h(\alpha)}$ so that condition (iii) is true since α was arbitrary.

Next we show that (iii) implies (i). So suppose that $h(S_\alpha) = T_{h(\alpha)}$ for all $\alpha \in J$. First, it is easy to see that h preserves order since, if $x, y \in J$ where $x < y$, then we have that $x \in S_y$ so that clearly $h(x) \in h(S_y) = T_{h(y)}$, and hence $h(x) < h(y)$. To show that the image of h, i.e. $h(J)$, is either E or a section of E, consider the set $E - h(J)$.

Case: $E - h(J) = \varnothing$. Then clearly for any $y \in E$ we must have that $y \in h(J)$ since otherwise it would be that $y \in E - h(J)$. Thus $E \subset h(J)$ since y was arbitrary. Also clearly $h(J) \subset E$ since E is the range of h. This shows that $h(J) = E$.

Case: $E - h(J) \neq \varnothing$. Then clearly $E - h(J)$ is a nonempty subset of E so that it has a smallest element β since E is well-ordered, noting that clearly $\beta \notin h(J)$. We claim that $h(J) = T_\beta$. So consider any $y \in h(J)$ so that there is an $x \in J$ where $y = h(x)$. Suppose for a moment that $\beta \preceq y$. Now it cannot be that $\beta = y$ since $y \in h(J)$ but $\beta \notin h(J)$, and so $\beta \prec y$. But then $\beta \in T_y = T_{h(x)} = h(S_x)$ since $x \in J$. Then β is in the image of h since clearly $h(S_x) \subset h(J)$. As this contradicts the fact that $\beta \notin h(J)$, it must be that $y \prec \beta$ so that $y \in T_\beta$. This shows that $h(J) \subset T_\beta$ since y was arbitrary. Suppose now that $y \in T_\beta$ so that $y \prec \beta$. Since β is the smallest element of $E - h(J)$, it follows that $y \notin E - h(J)$. Since clearly $y \in E$ (since $T_\beta \subset E$), it must be that $y \in h(J)$. This shows that $T_\beta \subset h(J)$ since y was arbitrary. Hence we have shown that $h(J) = T_\beta$.

Therefore in every case either the image of h is E or a section of E as desired. This completes the proof of (i).

Now we show that (ii) implies (iii). So suppose that $h(\alpha)$ is the smallest element of $E - h(S_\alpha)$ for every $\alpha \in J$. First we show that h is injective. So consider any $x, y \in J$ where $x \neq y$. We can assume without loss of generality that $x < y$ so that $x \in S_y$ and hence $h(x) \in h(S_y)$. However since we have that $h(y)$ is the smallest element of $E - h(S_y)$, clearly $h(y) \notin h(S_y)$. Therefore we have that $h(x) \neq h(y)$ so that h is injective.

Now consider any $\alpha \in J$ so that clearly $h(\alpha)$ is the smallest element of $E - h(S_\alpha)$. Suppose that $y \in h(S_\alpha)$ so that there is an $x \in S_\alpha$ where $y = h(x)$, and therefore $x < \alpha$. Consider the possibility that $h(\alpha) \preccurlyeq h(x) = y$. It cannot be that $h(\alpha) = h(x) = y$ since $x \neq \alpha$ and h is injective, so it must be that $h(\alpha) \prec h(x)$. It then follows that $h(\alpha) \notin E - h(S_x)$ since $h(x)$ is the smallest element of $E - h(S_x)$. Thus $h(\alpha) \in h(S_x)$ since clearly $h(\alpha) \in E$. It then follows from the fact that h is injective that $\alpha \in S_x$ so that we have $\alpha < x < \alpha$, which is clearly a contradiction. So it must be that $y = h(x) \prec h(\alpha)$ so that $y \in T_{h(\alpha)}$. This shows that $h(S_\alpha) \subset T_{h(\alpha)}$ since y was arbitrary.

Now suppose that $y \in T_{h(\alpha)}$ so that $y \prec h(\alpha)$. Since $h(\alpha)$ is the smallest element of $E - h(S_\alpha)$, it follows that $y \notin E - h(S_\alpha)$. Since clearly $y \in E$, it must be that $y \in h(S_\alpha)$. This shows that $T_{h(\alpha)} \subset h(S_\alpha)$ since y was arbitrary, and hence $h(S_\alpha) = T_{h(\alpha)}$, which shows (iii) since α was arbitrary.

Lastly, we show that (iii) implies (ii). So suppose that $h(S_\alpha) = T_{h(\alpha)}$ for every $\alpha \in J$ and consider any such α. Clearly we have that $h(\alpha) \in E$ but $h(\alpha) \notin T_{h(\alpha)} = h(S_\alpha)$ so that $h(\alpha) \in E - h(S_\alpha)$. Suppose for the moment that $h(\alpha)$ is not the smallest element of $E - h(S_\alpha)$ so that there is a $\beta \in E - h(S_\alpha)$ where $\beta \prec h(\alpha)$. Then $\beta \in T_{h(\alpha)}$ so that it must be that $\beta \notin E - T_{h(\alpha)} = E - h(S_\alpha)$ since $h(S_\alpha) = T_{h(\alpha)}$. Clearly this is a contradiction so that it must be that $h(\alpha)$ really is the smallest element of $E - h(S_\alpha)$, which shows (ii) since α was arbitrary. □

Exercise WO.3

Let J and E be well-ordered sets; suppose there is an order preserving map $k : J \to E$. Using Exercises 1 and 2, show that J has the order type of E or a section of E. [Hint: Choose $e_0 \in E$. Define $h : J \to E$ by the recursion formula

$$h(\alpha) = \text{smallest } [E - h(S_\alpha)] \qquad \text{if} \qquad h(S_\alpha) \neq E\,,$$

and $h(\alpha) = e_0$ otherwise. Show that $h(\alpha) \leq k(\alpha)$ for all α; conclude that $h(S_\alpha) \neq E$ for all α.]

Solution:

Proof. First, if $E = \varnothing$ then it must be that $J = \varnothing$ as well so that they vacuously have the same order type. Otherwise, following the hint, choose $e_0 \in E$ and define $h : J \to E$ by

$$h(\alpha) = \text{smallest } [E - h(S_\alpha)] \qquad \text{if} \qquad h(S_\alpha) \neq E\,,$$

and $h(\alpha) = e_0$ otherwise, noting that this function is uniquely defined by the general principle of recursive definition (Exercise WO.1). We show that $h(\alpha) \leq k(\alpha)$ for all $\alpha \in J$ using transfinite induction (see Lemma 10.10.1). So consider $\alpha \in J$ and assume that $h(x) \leq k(x)$ for all $x \in S_\alpha$.

Since k preserves order we have that $h(x) \leq k(x) < k(\alpha)$ when $x < \alpha$. In particular, this means that $h(x) \neq k(\alpha)$ for all $x \in S_\alpha$ so that $k(\alpha) \in E - h(S_\alpha)$. Hence $E - h(S_\alpha)$ is not empty so that $h(S_\alpha) \neq E$. Thus $h(\alpha)$ is the smallest element of $E - h(S_\alpha)$ and so $h(\alpha) \leq k(\alpha)$ since $k(\alpha) \in E - h(S_\alpha)$. This completes the induction.

Therefore, for any $\alpha \in J$ and any $x < \alpha$ we have $h(x) \leq k(x) < k(\alpha)$ since k preserves order so that $h(x) \neq k(\alpha)$. As in the induction step above, it follows that $h(S_\alpha) \neq E$. Hence, since α was arbitrary,

$$h(\alpha) = \text{smallest } [E - h(S_\alpha)]$$

for all $\alpha \in J$. It then follows from Exercise WO.2 part (a) that h is order preserving and maps J onto E or a section of E. This clearly shows that J has the order type of E or a section of E as desired. □

Exercise WO.4

Use Exercises 1-3 to prove the following:

(a) If A and B are well-ordered sets, then exactly one of the following three conditions holds: A and B have the same order type, or A has the order type of a section of B, or B has the order type of a section of A. [Hint: Form a well-ordered set containing both A and B, as in Exercise 8 of §10; then apply the preceding exercise.]

(b) Suppose that A and B are well-ordered sets that are uncountable, such that every section of A and B is countable. Show that A and B have the same order type.

Solution:

(a)

Proof. First, we can assume that A and B are disjoint since, if not, we can form $A' = \{(x, 1) \mid x \in A\}$ and $B' = \{(x, 2) \mid x \in B\}$, which clearly *are* disjoint and have the same order types as A and B if ordered in the same way. So let $\prec$ be the order on $A \cup B$ as in Exercise 10.8 with all the elements of A before the elements of B. From the exercise, we know that $A \cup B$ is well-ordered by $\prec$. Now, clearly the identity function i_B with $A \cup B$ as the range is an order-preserving function from B to $A \cup B$ so that B is the same order type as $A \cup B$ or a section of $A \cup B$ by Exercise WO.3.

If B has the same order type as $A \cup B$, then there is a an order preserving bijection $g : A \cup B \to B$. Let b be the smallest element of B so that $y = g(b) \in B$. Since b is the smallest element of B, clearly the section $S_b = \{x \in A \cup B \mid x \prec b\} = A$. Also clearly $g(A) = g(S_b) = S_y = \{x \in B \mid x < y\}$ so that A has the same order type as a section of B since g preserves order.

If B has the same order type as a section of $A \cup B$ then there is an order preserving bijection $f : B \to S_\alpha$ for some $\alpha \in A \cup B$. If $\alpha \in A$ then clearly S_α lies entirely in A and is a section of A so that B has the same order type as a section of A. So now suppose that $\alpha \in B$. If α is the smallest element of B then again it has to be that S_α lies in A and is in fact the entirety of A so that B and A have the same order type. If α is not the smallest element of B then S_α contains elements of both A and B. So let b be the smallest element of B so that $b \in S_\alpha$, and let $y \in B$ be such that $f(y) = b$, which exists since f is surjective. We also have that $S_b = A$ since b is the smallest element of B. It then follows that $f(S_y) = S_b = A$ since $f(y) = b$ so that A has the same order type as the section S_y of B since f preserves order.

Hence in all cases one of the desired results always follows. To show that exactly one of these is the case, note that if A and B have the same order type then clearly it cannot be that A has the same order type as a section of B since then B would also have the same order type is its own section, which would violate Exercise WO.2 part (b). Similarly B cannot have the same order type as a section of A since then A would have the same order type as its own section. Now suppose that A has the same order type as a section S_b of B. Then A and B cannot have the same order type since then B would have the same order type as its section S_b. Also B cannot have the same order type as a section S_a of A since then the section S_b, and therefore A, would have the same order type as a smaller section of A. An analogous argument shows the result when B has the same order type as a section of A. □

(b)

Proof. Suppose that A has the same order type as a section of B. Then there would be a bijection from A, an uncountable set, to a section of B, which is countable. A similar contradiction arises if B were to have the same order type as a section of A. By part (a), the only remaining possibility is that A and B have the same order type as desired. □

Exercise WO.5

Let X be a set; let $\mathcal{A}$ be the collection of all pairs $(A, <)$, where A is a subset of X and $<$ is a well-ordering of A. Define

$$(A, <) \prec (A', <')$$

if $(A, <)$ *equals* a section of $(A', <')$.

(a) Show that $\prec$ is a strict partial order on $\mathcal{A}$.

(b) Let $\mathcal{B}$ be a subcollection of $\mathcal{A}$ that is simply ordered by $\prec$. Define B' to be the union of the sets B, for all $(B, <) \in \mathcal{B}$; and define $<'$ to be the union of the relations $<$, for all $(B, <) \in \mathcal{B}$. Show that $(B', <')$ is a well-ordered set.

Solution:

(a)

Proof. For any $(A, <) \in \mathcal{A}$ we have that it is not equal to a section of itself since then it would then clearly have the same order type as its own section, which would violate Exercise WO.2 part (b). Hence it is not true that $(A, <) \prec (A, <)$ by definition, which shows that $\prec$ is nonreflexive.

Now consider $(A, <)$, $(A', <')$, and $(A'', <'')$ in $\mathcal{A}$ where $(A, <) \prec (A', <')$ and $(A', <') \prec (A'', <'')$. Then $(A, <)$ is a section of $(A', <')$. Also $(A', <')$ is a section of $(A'', <'')$ so that clearly any section of $(A', <')$ is also a section of $(A'', <'')$. Since $(A, <)$ is such a section we have that $(A, <)$ is a section of $(A'', <'')$ so that $(A, <) \prec (A'', <'')$. This shows that $\prec$ is transitive.

This completes the proof that $\prec$ is a strict partial order. □

(b)

Proof. First we must show that B' is simply ordered by $<'$.

First consider any $(x, y) \in <'$ so that there is a $(B, <) \in \mathcal{B}$ where $(x, y) \in <$ and $x, y \in B$. Clearly then x and y are in the union B' so that $(x, y) \in B' \times B'$. This shows that $<' \subset B' \times B'$ so that $<'$ is a relation on B'.

Next consider any x and y in B' where $x \neq y$. Then there are well-ordered sets $(B_1, <_1)$ and $(B_2, <_2)$ in $\mathcal{B}$ where $x \in B_1$ and $y \in B_2$. Since $\mathcal{B}$ is simply ordered by $\prec$ we have that $(B_1, <_1) \prec (B_2, <_2)$

or $(B_2, <_2) \prec (B_1, <_1)$. Without loss of generality we can assume the former case (since otherwise we can just swap the roles of x and y). Then $(B_1 <_1)$ is a section of $(B_2, <_2)$ and is thus also a subset so that $x, y \in B_2$, It then follows that x and y are comparable by $<_2$ since $x \neq y$ and $<_2$ is a well-order and therefore a simple order. Thus (x, y) or (y, x) are in $<_2$. Since $<'$ is the union of all relations $<$ where $(B, <) \in \mathcal{B}$, clearly we have that (x, y) or (y, x) are in $<'$ since $<_2$ is such a relation. Thus shows that $<'$ has the comparability property.

Now consider any $x \in B'$ so that there is a $(B, <) \in \mathcal{B}$ where $x \in B$. Consider also any $(B'', <'') \in \mathcal{B}$. Then, since $\mathcal{B}$ is simply ordered, it follows that $(B, <)$ and $(B'', <'')$ are comparable in $\prec$. If $(B, <) \prec (B'', <'')$ then $(B, <)$ is a section of $(B'', <'')$ so that $x \in B''$ as well. Then it cannot be that $x <'' x$ since $<''$ is a simple order. If $(B'', <'') \prec (B, <)$ then $(B'', <'')$ is a section of $(B, <)$. If $x \in B''$ then again it cannot be that $x <'' x$ since $<''$ is a simple order. If $x \notin B''$ then $(x, x) \notin <''$ since it is a relation on B''. Thus in all cases and sub-cases it is not true that $x <'' x$ so that $x <' x$ does not hold since $<''$ was arbitrary and $<'$ is their union. This shows that $<'$ is nonreflexive.

Lastly, suppose that $x <' y$ and $y <' z$. Then it has to be that there is a $(B_1, <_1)$ and $(B_2, <_2)$ in $\mathcal{B}$ where $z <_1 y$ and $y <_2 z$. Then $(B_1, <_1)$ and $(B_2, <_2)$ are comparable in $\prec$ since $\mathcal{B}$ is simply ordered. Hence one is a section of the other so that, in either case, it follows that $x < y$ and $y < z$ where either $< = <_1$ or $< = <_2$. Then clearly $x < z$ since both $<_1$ and $<_2$ are transitive since they are simple orders. Thus $x <' z$ since $<'$ is the union of all the orders in $\mathcal{B}$ and $<$ is such an order. This shows that $<'$ is transitive.

This completes the proof that $<'$ is a simple order on B'. To show that it is a well-order, consider any nonempty subset $A \subset B'$. Then there is an $x \in A$ so that $x \in B'$ as well. It then follows that there is a $(B, <) \in \mathcal{B}$ where $x \in B$. Then clearly $B \cap A$ is a nonempty subset of B since $x \in B$ and $x \in A$. Let b be the $<$-smallest element in $B \cap A$, and we claim that this is the smallest element of A by $<'$. First, obviously $b \in A$ since $b \in B \cap A$. Next consider any $y \in A$ so that $y \in B'$ as well. Then there is a $(B'', <'') \in \mathcal{B}$ where $y \in B''$. Since $\mathcal{B}$ is simply ordered by $\prec$ we have that $(B, <)$ and $(B'', <'')$ are comparable. Hence $(B, <)$ is a section of $(B'', <'')$ or vice-versa.

In the first case we have that both b and y are in B''. If $y \in B$ then also $y \in B \cap A$ so that $b \leq y$ since it is the smallest element of $B \cap A$ by $<$. If $y \notin B$ then $b <'' y$ since B is a section of B'', and therefore $b \leq'' y$ is true. In the second case in which $(B'', <'')$ is a section of $(B, <)$ we have that both b and y are in B and hence in $B \cap A$. Then, again $b \leq y$ since b is the smallest element of $B \cap A$ by $<$. Hence in all cases either $b \leq y$ or $b \leq'' y$. Either way it follows that $b \leq' y$ as well since $<'$ is the union. This shows that b is the smallest element of A by $<'$ as desired. Since A was an arbitrary nonempty subset, this shows that B' is well-ordered by $<'$. □

Exercise WO.6

Use Exercises 1 and 5 to prove the following:

Theorem. The maximum principle is equivalent to the well-ordering theorem.

Solution:

Proof. First suppose that the maximum principle is true and let X be any set. Then let $\mathcal{A}$ be the collection of all pairs $(A, <)$, where $A \subset X$ and $<$ is a well-ordering of A as in Exercise WO.5. Define the relation $\prec$ on $\mathcal{A}$ also as in Exercise WO.5, i.e $(A, <) \prec (A', <')$ if $(A, <)$ is a section of $(A', <')$. It was then shown in that exercise that $\prec$ is a strict partial order on $\mathcal{A}$ so that, by the maximum principle, there is a maximal simply ordered subset $\mathcal{B} \subset \mathcal{A}$. Now let $(B', <')$ be the unions of the corresponding elements of $\mathcal{B}$ so that we know that $<'$ well-orders B' by part (b) of Exercise WO.5.

We claim that $B' = X$. Suppose that this is not the case so that there is a $x \in X$ where $x \notin B'$ (since we know that $B' \subset X$). Then define $B'' = B' \cup \{x\}$ and the relation $<'' = <' \cup \{(b', x) \mid b' \in B'\}$. It is then easy to see (and trivial but tedious to show) that B'' is well-ordered by $<''$. Also, clearly B' is the section of B'' by x so that, for any $B \in \mathcal{B}$, we have $B \preceq B' \prec B''$. Since B was arbitrary, this shows that the set $\mathcal{B} \cup \{B''\}$ is simply ordered by $\prec$ and is a subset of $\mathcal{A}$. Since $x \notin B'$ we have that $x \notin B$ for any $B \in \mathcal{B}$ (since B' is their union) so that $B'' \neq B$ since $x \in B$. It follows that $\mathcal{B} \subsetneq \mathcal{B} \cup \{B''\}$, but this contradicts the maximality of $\mathcal{B}$! So it has to be that in fact $B' = X$ itself so that $<'$ is a well-ordering of X. Since X was an arbitrary set, this shows the well-ordering theorem.

Now suppose the well-ordering theorem and that A is a set with strict partial ordering $\prec$. Then we know that A has a well-ordering, say $<$. Now, for any function f from a section S_x (by $<$) to $\mathcal{P}(A)$, define

$$\rho(f) = \begin{cases} \bigcup f(S_x) \cup \{x\} & \text{if } \prec \text{ is a simple order on } \bigcup f(S_x) \cup \{x\} \\ \bigcup f(S_x) & \text{otherwise}. \end{cases}$$

Then by the general principle of recursive defamation (Exercise WO.1) there is a unique function $h : A \to \mathcal{P}(A)$ such that $h(\alpha) = \rho(h \restriction S_\alpha)$ for all $\alpha \in A$.

First we show that, for $\alpha, \beta \in A$ where $\alpha < \beta$, we have $h(\alpha) \subset h(\beta)$. So consider any $x \in h(\alpha) = \rho(h \restriction S_\alpha)$, and hence either $x \in \bigcup h(S_\alpha) \cup \{\alpha\}$ or $x \in \bigcup h(S_\alpha)$. Either way obviously $x \in \bigcup h(S_\alpha)$ so that there is a set $X \in h(S_\alpha)$ where $x \in X$. Then there is a $\gamma \in S_\alpha$ where $X = h(\gamma)$. Since we have $\alpha < \beta$, clearly also $\gamma \in S_\beta$ and hence $X \in h(S_\beta)$. Then also clearly both $x \in \bigcup h(S_\beta) \cup \{\beta\}$ and $x \in \bigcup h(S_\beta)$ so that for sure $x \in \rho(h \restriction S_\beta) = h(\beta)$. Since x was arbitrary this shows that $h(\alpha) \subset h(\beta)$ as desired.

Next we show by transfinite induction that $h(\alpha)$ is simply ordered by $\prec$ for every $\alpha \in A$. So consider $\alpha \in A$ and suppose that $h(\beta)$ is simply ordered by $\prec$ for every $\beta < \alpha$. If $\bigcup h(S_\alpha) \cup \{\alpha\}$ is simply ordered by $\prec$ then clearly $h(\alpha)$ is since then $h(\alpha) = \rho(h \restriction S_\alpha) = \bigcup h(S_\alpha) \cup \{\alpha\}$. So suppose that this is not the case so that $h(\alpha) = \rho(h \restriction S_\alpha) = \bigcup h(S_\alpha)$. Consider then any $x, y \in h(\alpha) = \bigcup h(S_\alpha)$ where $x \neq y$ so that there are X and Y in $h(S_\alpha)$ where $x \in X$ and $y \in Y$. Then there is a β and γ in S_α where $X = h(\beta)$ and $Y = h(\gamma)$. If $\beta = \gamma$ then x and y are both in $X = h(\beta) = h(\gamma) = Y$, which is simply ordered by the induction hypothesis so that x and y are comparable in $\prec$. If $\beta < \gamma$ then by what was shown above we have that $x \in X = h(\beta) \subset h(\gamma)$ so that x and y are both in $h(\gamma)$, which is simply ordered by the induction hypothesis so that again x and y are comparable. A similar argument shows that x and y are both in $h(\beta)$ and thus are comparable when $\beta > \gamma$. This completes the induction since x and y are comparable in all cases so that $h(\alpha)$ is always simply ordered.

We then claim that the set $B = \bigcup_{\alpha \in A} h(\alpha)$ is a maximal simply ordered (by $\prec$) subset of A, which of course shows the maximum principle. First, it is obviously a subset of A since each $h(\alpha) \in \mathcal{P}(A)$ so and so is a subset of A. To show that that B is simply ordered by $\prec$, consider x and y in B where $x \neq y$ so that there is an α and β in A where $x \in h(\alpha)$ and $y \in h(\beta)$. Without loss of generality we can assume that $\alpha < \beta$ so that $h(\alpha) \subset h(\beta)$ by what was shown below. Then both x and y are in $h(\beta)$, which is simply ordered by what was shown above. Hence x and y are comparable in $\prec$ so that B is simply ordered.

To show that B is maximal, suppose that $B \subsetneq Z$ and $Z \subset A$ is simply ordered by $\prec$. Then there is a $z \in Z$ where $z \notin B$. Now let $x \in \bigcup h(S_z)$ so that there is an $X \in h(S_z)$ where $x \in X$. Then there is an $\alpha \in S_z$ where $x \in X = h(\alpha)$. Hence clearly $x \in B$ so that also $x \in Z$ and so x and z are comparable in $\prec$ since Z is simply ordered. Since x was arbitrary this shows that the set $\bigcup h(S_z) \cup \{z\}$ is simply ordered so that $h(z) = \rho(h \restriction S_z) = \bigcup h(S_z) \cup \{z\}$. However, then we have that $z \in h(z)$ so that $z \in B$, which is a contradiction. So it must be that there is no such set Z and hence B is maximal. $\square$

Exercise WO.7

Use Exercises 1-5 to prove the following:

Theorem. The choice axiom is equivalent to the well-ordering theorem.

Proof. Let X be a set; let c be a fixed choice function for the nonempty subsets of X. If T is a subset of X and $<$ is a relation on T, we say that $(T, <)$ is a ***tower*** in X if $<$ is a well-ordering of T and if for each $x \in T$,

$$x = c(X - S_x(T)) ,$$

where $S_x(T)$ is the section of T by x.

(a) Let $(T_1, <_1)$ and $(T_2, <_2)$ be two towers in X. Show that either these two ordered sets are the same, or one equals a section of the other. [Hint: Switching indices if necessary, we can assume that $h : T_1 \to T_2$ is order preserving and $h(T_1)$ equals either T_2 or a section of T_2. Use Exercise 2 to show that $h(x) = x$ for all x.]

(b) If $(T, <)$ is a tower in X and $T \neq X$, show that there is a tower in X of which $(T, <)$ is a section.

(c) Let $\{(T_k, <_k) \mid k \in K\}$ be the collection of all towers in X. Let

$$T = \bigcup_{k \in K} T_k \qquad \text{and} \qquad < = \bigcup_{k \in K} (<_k) .$$

Show that $(T, <)$ is a tower in X. Conclude that $T = X$.

Solution:

(a)

Proof. Since $(T_1, <_1)$ and $(T_2, <_2)$ are both well-ordered sets, it follows from Exercise WO.4 part (a) that either they have the same order type, T_1 has the same order type as a section of T_2, or vice-versa. We can assume that either they have the same order type of T_1 has the same order type as a section of T_2 since, in the third case, we can just swap the roles of T_1 and T_2. Thus there is an order preserving function $h : T_1 \to T_2$ whose image is either all of T_2 or a section of T_2. Given this, it was shown in the proof of Exercise WO.2 part (a) that $h(S_x(T_1)) = S_{h(x)}(T_2)$ for all $x \in T_1$.

We show that $h(x) = x$ for all $x \in T_1$ by transfinite induction. So suppose that $h(y) = y$ for all $y < x$, i.e. for all $y \in S_x(T_1)$ so that clearly $h(S_x(T_1)) = S_x(T_1)$. Then, since both T_1 and T_2 are towers in X and $h(x) \in T_2$, we have

$$h(x) = c(X - S_{h(x)}(T_2)) = c(X - h(S_x(T_1))) = c(X - S_x(T_1)) = x .$$

This completes the induction. Since $h(x) = x$ for all $x \in T_1$ and h preserves order, it follows that T_1 is equal to T_2 or a section of T_2 as desired. □

(b)

Proof. Since $T \neq X$, it follows that $X - T$ is nonempty. So let $a = c(X - T)$, $T' = T \cup \{a\}$, and $<' = < \cup \{(x, a) \mid x \in T\}$. Then clearly a is the largest element of T' and an upper bound of T so that $T = S_a(T')$, and hence $a = c(X - T) = c(X - S_a(T'))$. Since T is a tower, it then follows that T' is also a tower in X and that T is a section of T' as desired. □

(c)

Proof. First we need to show that $<$ is even a well-ordering of T as this is not obvious. To show that it is a simple order, consider $x, y \in T$ where $x \neq y$. It follows that $x \in T_k$ and $y \in T_l$ for some $k, l \in K$ by the definition of T. Since T_k and T_l are both towers in X, it follows from part (a) that they are equal or one is a section of the other. So, without loss of generality, we can assume that $T_k \subset T_l$ and also $<_k \subset <_l$. It then follows that both x and y are in T_l so that either $x <_l y$ or $y <_l x$ since $x \neq y$ and $<_l$ is a simple order. Then clearly $x < y$ or $y < x$ from the definition of $<$. This shows that $<$ has the comparability property.

Now suppose that there is an $x \in T$ where $x < x$. Then there is a $k \in K$ where $x <_k x$, which violates the fact that $<_k$ is a simple order. Hence it must be that $<$ is nonreflexive.

Lastly consider $x, y, z \in T$ where $x < y$ and $y < z$. Then there $k, l \in K$ where $x <_k y$ and $y <_l z$ and it must then be that $x, y \in T_k$ and $y, z \in T_l$. Again, since these are both towers, they are either equal or one is a section of the other by part (a). So we can assume that $T_k \subset T_l$ and $<_k \subset <_l$ without loss of generality so that we have $x, y, z \in T_l$ and $x <_l y <_l z$. From this clearly $x <_l z$ since it is a simple order and therefore transitive. Hence we have $x < y$, which of course shows that $<$ is transitive as well. This all shows that $<$ is indeed a simple order by definition.

To show that $<$ is a well-ordering, consider any nonempty subset Y of T. Then there is a $b \in Y$ so that also $b \in T$. It follows that there is a $k \in K$ such that $b \in T_k$, and also that $Y \cap T_k$ is a nonempty subset of T_k. It then follows that $Y \cap T_k$ has a smallest element a since T_k is well-ordered by $<_k$. We claim that in fact a is the smallest element of all of Y. To see this, consider any other $x \in Y$ so that also $x \in T$. Hence there is an $l \in K$ where $x \in T_l$. Now, since both T_k and T_l are towers in X, it follows from part (a) that they are equal or one is a section of the other.

Case: T_k and T_l are equal. Then both a and x are in T_k and so both in $Y \cap T_k$. Then $a \leq_k x$ since a is the smallest element of $Y \cap T_k$.

Case: T_k is a section of T_l. Then, if $a \in T_k$ then the argument in the previous case shows that $a \leq_k x$. On the other hand, if $a \notin T_k$, then it has to be that that $x <_l a$ since $x \in T_k$ and T_k is a section of T_l.

Case: T_l is a section of T_k. Then $T_l \subset T_k$ so that both a and x are in T_k and thus in $Y \cap T_k$. Hence again $a \leq_k x$ since a is the smallest element of $Y \cap T_k$.

In all cases $a \leq_m x$ for some $m \in K$ and hence $a \leq x$. Since x was an arbitrary element of Y, this shows that a is in fact the smallest element of Y. Since Y was an arbitrary nonempty subset of T, this shows that T is well-ordered by $<$.

Next we digress for a moment to show, for any $k \in K$ and $x \in T_k$, that $S_x(T_k) = S_x(T)$. So consider such k and x and suppose that $y \in S_x(T_k)$ so that $y <_k x$. Then clearly also $y < x$ by the definition of $<$ and hence $y \in S_x(T)$. This shows that $S_x(T_k) \subset S_x(T)$ since y was arbitrary. Now suppose $y \in S_x(T)$ so that $y < x$. Then there is an $l \in K$ where $y <_l x$. Hence $x, y \in T_l$, and by part (a) either T_l and T_k are equal, or one is a section of the other. If they are equal or T_l is a section of T_k then clearly we have $<_l \subset <_k$ so that $y <_k x$. If T_k is a section of T_l then, since $y <_l x$ and $x \in T_k$, it has to be that also $y \in T_k$ since T_k is a section of T_l. Hence it must be that $y <_k x$. Since this is true in all cases it follows that $y \in S_x(T_k)$, which shows that $S_x(T) \subset S_x(T_k)$. This completes the proof that $S_x(T_k) = S_x(T)$.

With this having been shown, we can easily show that T is a tower in X. For any $x \in T$ there is a $k \in K$ where $x \in T_k$. Since T_k is a tower in X we have

$$x = c(X - S_x(T_k)) = c(X - S_x(T))$$

by what was just shown. Thus suffices to show that T is a tower in X.

Lastly, we claim that $T = X$. To see this, suppose that it is not the case so that by part (b) there is a tower S in X such that T is a section of S. From this we have that $T = S_a(S)$ for some $a \in S$ and that of course $a \notin T$. However, since S is a tower and $\{(T_k, <_k) \mid k \in K\}$ is the collection of *all* towers in X, it follows that there must be a $k \in K$ such that $S = T_k$. Then we have that $a \in S = T_k$ so that of course $a \in T$ by definition, which is a contradiction. So it must be that in fact $T = X$ as desired.

This of course shows that $<$ is a well-ordering of $X = T$ so that the choice axiom implies the well-ordering theorem since X is an arbitrary set. In contrast to the previous proof, it is easy to prove that the well-ordering theorem implies the choice axiom. For a collection of nonempty sets $\mathcal{B}$ define $X = \bigcup \mathcal{B}$. Then X can be well-ordered by the well-ordering theorem. Then we simply define a choice function c on $\mathcal{B}$ in the following way: any $B \in \mathcal{B}$ is clearly a nonempty subset of X and so has a smallest element a since X is well ordered. So simply set $c(B) = a$, from which it is clear that $c(B) \in B$ and so c is a valid choice function. □

Exercise WO.8

Using Exercises 1-4, construct an uncountable well-ordered set, as follows. Let $\mathcal{A}$ be the collection of all pairs $(A, <)$, where A is a subset of $\mathbb{Z}_+$ and $<$ is a well-ordering of A. (We allow A to be empty.) Define $(A, <) \sim (A', <')$ if $(A, <)$ and $(A', <')$ have the same order type. It is trivial to show that this is an equivalence relation. Let $[(A, <)]$ denote the equivalence class of $(A, <)$; let E denote the collection of these equivalence classes. Define

$$[(A, <)] \ll [(A', <')]$$

if $(A, <)$ has the order type of a *section* of $(A', <')$.

(a) Show that the relation $\ll$ is well defined and is a simple order on E. Note that the equivalence class $[(\varnothing, \varnothing)]$ is the smallest element of E.

(b) Show that if $\alpha = [(A, <)]$ is an element of E, then $(A, <)$ has the same order type as the section $S_\alpha(E)$ of E by α. [Hint: Define a map $f : A \to E$ by setting $f(x) = [(S_x(A), \text{restriction } <)]$ for each $x \in A$.]

(c) Conclude that E is well-ordered by $\ll$.

(d) Show that E is uncountable. [Hint: If $h : E \to \mathbb{Z}_+$ is a bijection, then h gives rise to a well-ordering of $\mathbb{Z}_+$.]

Solution:

(a)

Proof. First, to show that $\ll$ is well defined, suppose that $[(A, <)] \ll [(A', <')]$ and that $(B, \prec) \in [(A, <)]$ and $(B', \prec') \in [(A', <')]$. Then $(A, <)$ has the same order type as a section of $(A', <')$ so that there is an order-preserving map h from A onto a section of A'. We also then have that $(B, \prec)$ has the same order type as $(A, <)$ since they are in the same equivalence class. Thus there is an order-preserving bijection $f : B \to A$. Likewise there is an order-preserving bijection from $g : B' \to A'$. It is then trivial to show that $g^{-1} \circ h \circ f$ is bijection from B onto a section of B' that preserves order. Hence $(B, \prec)$ has the same order type as a section of $(B', \prec')$. Since $(B, \prec)$ and $(B', \prec')$ were arbitrary elements in their respective equivalence classes, this shows that $\ll$ is well defined such that it does not matter which representatives we use from the equivalence classes.

Now consider any equivalence class $[(A, <)]$ in E. Then clearly it cannot be that $[(A, <)] \ll [(A, <)]$, since this would mean that A has the same order type as a section of itself, which would contradict what was shown in Exercise WO.2 part (b). Thus $\ll$ is nonreflexive.

Next consider two distinct equivalence classes $[(A, <)]$ and $[(A', <')]$. Then it cannot be that $(A, <)$ and $(A', <')$ have the same order type, for then they would be the same equivalence class. Then, by Exercise WO.4 part (a), it must be that either $(A,<)$ has the same order type as a section of $(A',<')$ or vice-versa. Clearly then, in the former case $[(A,<)] \ll [(A',<')]$, and in the latter case $[(A',<')] \ll [(A,<)]$. This shows that $\ll$ has the comparability property.

Lastly, suppose that $[(A,<)] \ll [(A',<')]$ and $[(A',<')] \ll [(A'',<'')]$. Then $(A,<)$ has the same order type as section of $(A',<')$ so that there is an order-preserving bijection f from A onto a section of A' Likewise there is an order-preserving bijection g from A' onto a section of A''. It is then trivial to show that $f \circ g$ is an order-preserving bijection from A onto a section of A''. It then clearly follows that $[(A,<)] \ll [(A'',<'')]$, which shows that $\ll$ is transitive.

Hence we have shown that $\ll$ satisfies all the requirements of a simple order. □

(b)

Proof. Following the hint, define the map $f : A \to E$ by setting

$$f(x) = [(S_x(A), \text{restriction } <)]$$

for any $x \in A$, noting that clearly $S_x(A)$ is well-ordered by the restricted $<$ so that the equivalence class is valid and in E.

Consider any x and y in A where $x < y$. Then clearly $x \in S_y(A)$ but $x \notin S_x(A)$ (since it is not true that $x < x$) so that $S_x(A)$ and $S_y(A)$ are distinct sets. We also clearly have that $S_x(A) = S_x(S_y(A))$ so that $S_x(A)$ has the same order type (the identity function is the required order-preserving map) as a section of $S_y(A)$. Hence

$$f(x) = [(S_x(A), \text{restriction } <)] \ll [(S_y(A), \text{restriction } <)] = f(y)$$

so that f preserves order since x and y were arbitrary.

Now we show that f is onto $S_\alpha(E)$. So consider any equivalence class $[(B,\prec)]$ in $S_\alpha(E)$ and hence

$$[(B,\prec)] \ll \alpha = [(A,<)]$$

so that by definition $(B,\prec)$ has the same order type as some section $S_x(A)$. Hence $[(B,\prec)]$ and $[(S_x(A), \text{restriction } <)]$ are the same equivalence class! Therefore

$$f(x) = [(S_x(A), \text{restriction } <)] = [(B,\prec)] \, ,$$

which of course shows the desired property since $[(B,\prec)]$ was arbitrary.

This shows that f is an order-preserving map from A onto $S_\alpha(E)$ so that they have the same order type. □

(c)

Proof. Consider any nonempty subset $D \subset E$. Thus there is an $\alpha = [(A,<)] \in D$. If α is the smallest element of D then we are done, so assume that this is not the case so that there is a $\beta \in D$ where $\beta \ll \alpha$. Now, it was shown in part (b) that $(A,<)$ has the same order type as the section $S_\alpha(E)$ so that this section must be well-ordered since A is. Also we have that $\beta \in S_\alpha(E)$ since $\beta \ll \alpha$. Thus $\beta \in D \cap S_\alpha(E)$ so that $D \cap S_\alpha(E)$ is a nonempty subset of $S_\alpha(E)$ so has a smallest element γ since $S_\alpha(E)$ is well-ordered. In particular, we of course have that $\gamma \lll \beta$, where we use $\lll$ to denote $\ll$ or equal to.

We claim that γ must be the smallest element of D. If not, then there is a $\delta \in D$ where $\delta \ll \gamma$. Of course we also then have that $\delta \ll \gamma \lll \beta \ll \alpha$ and hence $\delta \in S_\alpha(E)$. Therefore $\delta \in D \cap S_\alpha(E)$, but since $\delta \ll \gamma$ this contradicts the definition of γ as the smallest element of $D \cap S_\alpha(E)$. So it must be that in fact γ is the smallest element of D, which shows that E is well-ordered by $\ll$ since D was an arbitrary subset. □

(d)

Proof. Following the hint, suppose that E is countable so that there is a bijection $h : E \to \mathbb{Z}_+$. This of course gives rise to a well-ordering $<$ of $h(E) = \mathbb{Z}_+$ by simply ordering its elements according to its bijection with E, which was shown to be well-ordered in part (c). Then we have that $(\mathbb{Z}_+, <)$ is an element of $\mathcal{A}$ since $\mathbb{Z}_+$ is a subset of itself. Thus the equivalence class $\alpha = [(\mathbb{Z}_+, <)]$ is an element of E. But we know from part (b) that $(\mathbb{Z}_+, <)$ then has the same order type as the section $S_\alpha(E)$. Since we also know that $(\mathbb{Z}_+, <)$ has the same order type as E itself, it follows that E has the same order type as its section $S_\alpha(E)$. This was shown not to be possible in Exercise WO.2 part (b) so that a contradiction has been reached. So it must be that in fact E is uncountable as desired! □

Chapter 2 Topological Spaces and Continuous Functions

§13 Basis for a Topology

Exercise 13.1

Let X be a topological space; let A be a subset of X. Suppose that for each $x \in A$ there is an open set U containing x such that $U \subset A$. Show that A is open in X.

Solution:

Proof. For each $x \in A$ we can choose an open set U_x containing x such that $U_x \subset A$. We then claim that $\bigcup_{x \in A} U_x = A$. So first consider any $y \in \bigcup_{x \in A} U_x$ so that there is an $x \in A$ such that $y \in U_x$. Then clearly also $y \in A$ since $U_x \subset A$. Hence $\bigcup_{x \in A} U_x \subset A$ since y was arbitrary. Now consider $y \in A$ so that clearly $y \in U_y$. Then obviously $y \in \bigcup_{x \in A} U_x$ so that $A \subset \bigcup_{x \in A} U_x$ since y was arbitrary. Thus we have shown that $\bigcup_{x \in A} U_x = A$, and since each U_x is open, it follows from the definition of a topology that the union $\bigcup_{x \in A} U_x = A$ is open as well. □

Exercise 13.2

Consider the nine topologies on the set $X = \{a, b, c\}$ indicated in Example 1 of §12. Compare them; that is, for each pair of topologies, determine whether they are comparable, and if so, which is finer.

Solution:

We label each of the topologies in Figure 12.1 with an ordered pair (i, j) where $1 \leq i, j \leq 3$, i is the row, j is the column, and $(1, 1)$ is the upper left corner. The following matrix lists which of each pair is *finer*, or "Inc" if they are incomparable.

	$(1,1)$	$(1,2)$	$(1,3)$	$(2,1)$	$(2,2)$	$(2,3)$	$(3,1)$	$(3,2)$	$(3,3)$
$(1,1)$	$=$	$(1,2)$	$(1,3)$	$(2,1)$	$(2,2)$	$(2,3)$	$(3,1)$	$(3,2)$	$(3,3)$
$(1,2)$		$=$	Inc	Inc	Inc	Inc	$(1,2)$	$(3,2)$	$(3,3)$
$(1,3)$			$=$	$(1,3)$	Inc	$(2,3)$	$(1,3)$	Inc	$(3,3)$
$(2,1)$				$=$	Inc	$(2,3)$	Inc	$(3,2)$	$(3,3)$
$(2,2)$					$=$	Inc	Inc	Inc	$(3,3)$
$(2,3)$						$=$	$(2,3)$	Inc	$(3,3)$
$(3,1)$							$=$	$(3,2)$	$(3,3)$
$(3,2)$								$=$	$(3,3)$
$(3,3)$									$=$

We know that $\subsetneq$ forms a strict partial order on these topologies. So we can also list all the maximal simply ordered subsets, each in order:

$$(1,1) \subsetneq (2,2) \subsetneq (3,3)$$
$$(1,1) \subsetneq (3,1) \subsetneq (1,2) \subsetneq (3,2) \subsetneq (3,3)$$
$$(1,1) \subsetneq (3,1) \subsetneq (1,3) \subsetneq (2,3) \subsetneq (3,3)$$
$$(1,1) \subsetneq (2,1) \subsetneq (1,3) \subsetneq (2,3) \subsetneq (3,3)$$
$$(1,1) \subsetneq (2,1) \subsetneq (3,2) \subsetneq (3,3)$$

Exercise 13.3

Show that the collection $\mathcal{T}_\alpha$ given in Example 4 of §12 is a topology on X. Is the collection

$$\mathcal{T}_\infty = \{U \mid X - U \text{ is infinite or empty or all of } X\}$$

a topology on X?

Solution:

Recall that $\mathcal{T}_\alpha$ from Example 12.4 is the set of all subsets U of X such that $X-U$ either is countable or is all of X. First we show that $\mathcal{T}_\alpha$ is a topology on X.

Proof. First, clearly $\varnothing \in \mathcal{T}_\alpha$ since $X - \varnothing = X$ is all of X. Also $X \in \mathcal{T}_\alpha$ since $X - X = \varnothing$ is countable. Now suppose that $\mathcal{A}$ is a subcollection of $\mathcal{T}_\alpha$ so that $X-U$ is countable (or all of X) for every $U \in \mathcal{A}$. Then we have that

$$X - \bigcup \mathcal{A} = X - \bigcup_{A \in \mathcal{A}} A = \bigcap_{A \in \mathcal{A}} (X - A)$$

is countable (or all of X) since every $X - A$ is countable (or all of X). Therefore $\bigcup \mathcal{A} \in \mathcal{T}_\alpha$ by definition.

Now suppose that $U_1, \ldots, U_n$ are nonempty elements of $\mathcal{T}_\alpha$ so that $X - U_i$ is a countable subset of X or X itself for each $i \in \{1, \ldots, n\}$. Then we have

$$X - \bigcap_{i=1}^{n} U_i = \bigcup_{i=1}^{n} (X - U_i)$$

is a finite union of sets that are either countable subsets of X, or X itself. It then follows that the union is countable or X itself so that $\bigcap_{i=1}^{n} U_i \in \mathcal{T}_\alpha$ by definition. This completes the proof that $\mathcal{T}_\alpha$ is a topology on X. □

Now we claim that the collection $\mathcal{T}_\infty$ as defined above is not always a topology on X.

Proof. As a counterexample, let $X = \mathbb{Z}_+$ and suppose that $\mathcal{T}_\infty$ is a topology on X. Clearly if U is a finite subset of X, then $X - U$ is infinite since X is infinite so that U is open. Now consider the subcollection

$$\mathcal{A} = \{\{i\} \mid i \in \mathbb{Z}_+ \text{ and } i > 1\} = \{\{2\}, \{3\}, \ldots\} .$$

Then clearly we have that $\bigcup \mathcal{A} = \{2, 3, \ldots\}$ so that $X - \bigcup \mathcal{A} = \{1\}$ is neither infinite, empty, nor all of X. Therefore $\bigcup \mathcal{A}$ cannot be open, which violates property (2) of a topology. So it must be that $\mathcal{T}_\infty$ is not a topology, which of course contradicts our supposition that it is! □

Exercise 13.4

(a) If $\{\mathcal{T}_\alpha\}$ is a family of topologies on X, show that $\bigcap \mathcal{T}_\alpha$ is a topology on X. Is $\bigcup \mathcal{T}_\alpha$ a topology on X?

(b) Let $\{\mathcal{T}_\alpha\}$ be family of topologies on X. Show that there is a unique smallest topology on X containing all the collections $\mathcal{T}_\alpha$, and a unique largest topology contained in all $\mathcal{T}_\alpha$.

(c) If $X = \{a, b, c\}$, let

$$\mathcal{T}_1 = \{\varnothing, X, \{a\}, \{a, b\}\} \qquad \text{and} \qquad \mathcal{T}_2 = \{\varnothing, X, \{a\}, \{b, c\}\} \ .$$

Find the smallest topology containing $\mathcal{T}_1$ and $\mathcal{T}_2$, and the largest topology contained in $\mathcal{T}_1$ and $\mathcal{T}_2$.

Solution:

(a) First we show that $\bigcap \mathcal{T}_\alpha$ is a topology on X.

Proof. First, clearly since $\varnothing$ and X are in every $\mathcal{T}_\alpha$ since they are topologies, they are both in $\bigcap \mathcal{T}_\alpha$ so that property (1). Now suppose that $\mathcal{A}$ is a subcollection of $\bigcap \mathcal{T}_\alpha$. Consider any $\mathcal{T}_\beta$ and any $A \in \mathcal{A}$. Then A is also in $\bigcap \mathcal{T}_\alpha$ since $\mathcal{A} \subset \bigcap \mathcal{T}_\alpha$. It then follows that A is in our specific $\mathcal{T}_\beta$. Since A was arbitrary it follows that $\mathcal{A}$ is a subcollection of $\mathcal{T}_\beta$ so that $\bigcup \mathcal{A} \in \mathcal{T}_\beta$ also since $\mathcal{T}_\beta$ is a topology. Since $\mathcal{T}_\beta$ was also arbitrary it follows that $\bigcup \mathcal{A} \in \bigcap \mathcal{T}_\alpha$. Lastly, since the subcollection $\mathcal{A}$ was arbitrary, this shows property (2) for $\bigcap \mathcal{T}_\alpha$.

Finally, suppose that $U_1, \ldots, U_n$ are sets in $\bigcap \mathcal{T}_\alpha$. Consider any $\mathcal{T}_\beta$ so that clearly then $U_i \in \mathcal{T}_\beta$ for every $i \in \{1, \ldots, n\}$. It then follows that $\bigcap_{i=1}^{n} U_i \in \mathcal{T}_\beta$ since $\mathcal{T}_\beta$ is a topology. Since $\mathcal{T}_\beta$ was arbitrary, this shows that $\bigcap_{i=1}^{n} U_i \in \bigcap \mathcal{T}_\alpha$, which shows property (3) for $\bigcap \mathcal{T}_\alpha$. This completes the proof that $\bigcap \mathcal{T}_\alpha$ is a topology on X since all three properties have been shown. □

Now we claim that $\bigcup \mathcal{T}_\alpha$ is *not* generally a topology.

Proof. As a counterexample consider the set $X = \{a, b, c\}$, the topologies $\mathcal{T}_1 = \{\varnothing, X, \{a\}\}$ and $\mathcal{T}_2 = \{\varnothing, X, \{b\}\}$, and the collection of topologies $\mathcal{C} = \{\mathcal{T}_1, \mathcal{T}_2\}$. Then we clearly have that $\bigcup \mathcal{C} = \mathcal{T}_1 \cup \mathcal{T}_2 = \{\varnothing, X, \{a\}, \{b\}\}$, which is not a topology since $\mathcal{A} = \{\{a\}, \{b\}\}$ is a subcollection of $\bigcup \mathcal{C}$ but $\bigcup \mathcal{A} = \{a, b\}$ is not in $\bigcup \mathcal{C}$. □

(b) First we show that there is a unique smallest topology that contains each $\mathcal{T}_\alpha$.

Proof. It was proven in part (a) that $\bigcup \mathcal{T}_\alpha$ is not necessarily a topology. However, it is clearly always a subbasis for a topology since clearly $X \in \bigcup \mathcal{T}_\alpha$ since it is in each $\mathcal{T}_\alpha$ since they are topologies. Hence obviously then $\bigcup (\bigcup \mathcal{T}_\alpha) = X$ so that $\bigcup \mathcal{T}_\alpha$ is a subbasis by definition. Then let $\mathcal{T}_s$ be the topology generated by the subbasis $\bigcup \mathcal{T}_\alpha$. We claim that $\mathcal{T}_s$ is then the smallest topology that contains all the $\mathcal{T}_\alpha$ as subsets.

First, from the proof following the definition of a subbasis, we know that the set $\mathcal{B}$ of finite intersections of elements of $\bigcup \mathcal{T}_\alpha$ is a basis for the topology $\mathcal{T}_s$, and that $\mathcal{T}_s$ is the set of all unions of subcollections of $\mathcal{B}$.

We first show that every $\mathcal{T}_\alpha$ is indeed contained as a subset of $\mathcal{T}_s$. So consider any specific $\mathcal{T}_\beta$ and any $U \in \mathcal{T}_\beta$. Then clearly $U \in \bigcup \mathcal{T}_\alpha$ so that $U \in \mathcal{B}$ since $U = \bigcap \{U\}$ is a finite intersection of elements of $\bigcup \mathcal{T}_\alpha$. It then follows that $U \in \mathcal{T}_s$ since $U = \bigcup \{U\}$ is the union of a subcollection of $\mathcal{B}$. Since U was arbitrary, this shows that $\mathcal{T}_\beta \subset \mathcal{T}_s$, which shows the result since $\mathcal{T}_\beta$ was arbitrary.

Now we show that $\mathcal{T}_s$ is the smallest such topology as ordered by $\subsetneq$. So suppose that $\mathcal{T}$ is a topology that contains every $\mathcal{T}_\alpha$ as a subset. Consider any $U \in \mathcal{T}_s$ so that $U = \bigcup \mathcal{C}$ for some subcollection

$\mathcal{C} \subset \mathcal{B}$. Now consider any $Y \in \mathcal{C}$ so that also $Y \in \mathcal{B}$. Then $Y = \bigcap_{i=1}^{n} Y_i$ where each $Y_i \in \bigcup \mathcal{T}_\alpha$. Then each Y_i is in some $\mathcal{T}_\beta \subset \mathcal{T}$ so that also $Y_i \in \mathcal{T}$. Since $\mathcal{T}$ is a topology, it follows that the finite intersection $\bigcap_{i=1}^{n} Y_i = Y$ is also in $\mathcal{T}$. Since Y was arbitrary, this shows that $\mathcal{C} \subset \mathcal{T}$ so that $\mathcal{C}$ is a subcollection of $\mathcal{T}$. It then follows that $\bigcup \mathcal{C} = U$ is also in $\mathcal{T}$ since $\mathcal{T}$ is a topology. Since U was arbitrary, we have that $\mathcal{T}_s \subset \mathcal{T}$, which shows that $\mathcal{T}_s$ is the smallest topology since $\mathcal{T}$ was arbitrary.

It is easy to see that $\mathcal{T}_s$ is unique since, if both $\mathcal{T}_1$ and $\mathcal{T}_s$ are the smallest topologies that contain each $\mathcal{T}_\alpha$ as subsets, then we would have that both $\mathcal{T}_1 \subset \mathcal{T}_2$ and $\mathcal{T}_2 \subset \mathcal{T}_1$ so that $\mathcal{T}_1 = \mathcal{T}_2$. Really this follows from the more general fact that smallest elements in any order are always unique. □

Next we show that there is a unique largest topology that is contained in each $\mathcal{T}_\alpha$.

Proof. It was shown in part (a) that $\mathcal{T}_l = \bigcap \mathcal{T}_\alpha$ is a topology on X. We claim that in fact this is the unique largest topology contained in all $\mathcal{T}_\alpha$. First, clearly $\mathcal{T}_l = \bigcap \mathcal{T}_\alpha$ is contained in each $\mathcal{T}_\alpha$ since the intersection of a collection of sets is always a subset of every set in the collection. Now suppose that $\mathcal{T}$ is a topology that is contained in every $\mathcal{T}_\alpha$, i.e. $\mathcal{T} \subset \mathcal{T}_\alpha$ for every $\mathcal{T}_\alpha$. Then clearly for any $U \in \mathcal{T}$ we have that $U \in \mathcal{T}_\alpha$ for every $\mathcal{T}_\alpha$ so that $U \in \bigcap \mathcal{T}_\alpha = \mathcal{T}_l$. Thus $\mathcal{T} \subset \mathcal{T}_l$ since U was arbitrary. This shows that $\mathcal{T}_l$ is the largest such topology since $\mathcal{T}$ was arbitrary.

Clearly also $\mathcal{T}_l$ is unique since, if $\mathcal{T}_1$ and $\mathcal{T}_2$ are two such largest topologies that are contained in every $\mathcal{T}_\alpha$. Then we would have $\mathcal{T}_1 \subset \mathcal{T}_2$ and $\mathcal{T}_2 \subset \mathcal{T}_1$ so that $\mathcal{T}_1 = \mathcal{T}_2$. This also follows from the fact that the largest element in any ordered set (or collection of sets in this case) is unique. □

(c) Note that the proofs in part (b) are constructive so that we can construct these topologies as done in the proof. For the smallest topology containing $\mathcal{T}_1$ and $\mathcal{T}_2$ we have that

$$\bigcup \{\mathcal{T}_1, \mathcal{T}_2\} = \mathcal{T}_1 \cup \mathcal{T}_2 = \{\varnothing, X, \{a\}, \{a,b\}, \{b,c\}\}$$

is a subbasis for the smallest topology $\mathcal{T}_s$. Then the collection of all finite intersections of elements of this set is a basis for $\mathcal{T}_s$:

$$\mathcal{B} = \{\varnothing, X, \{a\}, \{b\}, \{a,b\}, \{b,c\}\} .$$

Then the topology $\mathcal{T}_s$ is the set of all unions of subcollections of $\mathcal{B}$:

$$\mathcal{T}_s = \{\varnothing, X, \{a\}, \{b\}, \{a,b\}, \{b,c\}\} = \mathcal{B}$$

so that evidently the basis and the topology are the same set here!

For the largest topology contained in $\mathcal{T}_1$ and $\mathcal{T}_2$ we have simply

$$\mathcal{T}_l = \bigcap \{\mathcal{T}_1, \mathcal{T}_2\} = \mathcal{T}_1 \cap \mathcal{T}_2 = \{\varnothing, X, \{a\}\} .$$

Exercise 13.5

Show that if $\mathcal{A}$ is a basis for a topology on X, then the topology generated by $\mathcal{A}$ equals the intersection of all topologies on X that contain $\mathcal{A}$. Prove the same if $\mathcal{A}$ is a subbasis.

Solution:

Suppose that $\mathcal{T}$ is the topology generated by basis $\mathcal{A}$, and $\mathcal{C}$ is the collection of topologies on X that contain $\mathcal{A}$ as a subset.

First we show that $\mathcal{T} = \bigcap \mathcal{C}$.

Proof. Consider $U \in \mathcal{T}$ and any $\mathcal{T}_c \in \mathcal{C}$ so that $\mathcal{A} \subset \mathcal{T}_c$. Then, since $\mathcal{A}$ generates $\mathcal{T}$, it follows from Lemma 13.1 that U is the union of elements of $\mathcal{A}$. Clearly then each of these elements of $\mathcal{A}$ is in $\mathcal{T}_c$ since $\mathcal{A} \subset \mathcal{T}_c$ so that their union is as well since $\mathcal{T}_c$ is a topology. Hence $U \in \mathcal{T}_c$ so that $\mathcal{T} \subset \mathcal{T}_c$ since U was arbitrary. Hence $\mathcal{T}$ is contained in all elements of $\mathcal{C}$ so that $\mathcal{T} \subset \bigcap \mathcal{C}$. Also, clearly $\mathcal{T}$ is a topology that contains $\mathcal{A}$ so that $\mathcal{T} \in \mathcal{C}$. Clearly then $\bigcap \mathcal{C} \subset \mathcal{T}$ so that $\mathcal{T} = \bigcap \mathcal{C}$ as desired. □

Next we show the same thing but when $\mathcal{A}$ is a subbasis.

Proof. Let $\mathcal{B}$ be the set of all finite intersections of elements of $\mathcal{A}$, which we know is a basis for $\mathcal{T}$ by the proof after the definition of a subbasis. We show that $\mathcal{B} \subset \mathcal{T}_c$ for all $\mathcal{T}_c \in \mathcal{C}$. So consider any set $B \in \mathcal{B}$ so that B is the finite intersection of elements of $\mathcal{A}$. Also consider any $\mathcal{T}_c \in \mathcal{C}$ so that each of these elements is in $\mathcal{T}_c$ since $\mathcal{A} \subset \mathcal{T}_c$. Since $\mathcal{T}_c$ is a topology, clearly the finite intersection of these elements, i.e. B, is in $\mathcal{T}_c$. Hence $\mathcal{B} \subset \mathcal{T}_c$ since B was arbitrary.

It then follows from what was shown before that $\mathcal{T} = \bigcap \mathcal{C}$ since $\mathcal{T}$ is the topology generated by the basis $\mathcal{B}$ and $\mathcal{B}$ is contained in each topology in $\mathcal{C}$. □

Exercise 13.6

Show that the topologies of $\mathbb{R}_l$ and $\mathbb{R}_K$ are not comparable.

Solution:

Proof. Let $\mathcal{T}_l$ and $\mathcal{T}_K$ be the topologies of $\mathbb{R}_l$ and $\mathbb{R}_K$, respectively. Also let $\mathcal{B}_l$ and $\mathcal{B}_K$ be the corresponding bases.

Consider $x = 0 \in \mathbb{R}$ and $B_l = [0, 1)$, which clearly contains 0 and is a basis element of $\mathcal{B}_l$. Let B_K be any basis element of $\mathcal{B}_K$ that contains 0. Then B_K is either (a, b) or $(a, b) - K$ for some $a < b$. In either case it must be that $a < 0 < b$ so that clearly $a < a/2 < 0 < b$. Also $a/2 \notin K$ since $a/2 < 0$ so that we have $a/2 \in (a, b)$ and $a/2 \in (a, b) - K$. Clearly also $a/2 \notin [0, 1)$ so that it cannot be that $B_K \subset B_l$. We have therefore shown that

$$\begin{gathered}
\exists x \in \mathbb{R} \exists B_l \in \mathcal{B}_l \left[x \in B_l \wedge \forall B_K \in \mathcal{B}_K \left(x \in B_K \Rightarrow B_K \not\subset B_l\right)\right] \\
\exists x \in \mathbb{R} \exists B_l \in \mathcal{B}_l \left[x \in B_l \wedge \forall B_K \in \mathcal{B}_K \left(x \notin B_K \vee B_K \not\subset B_l\right)\right] \\
\exists x \in \mathbb{R} \exists B_l \in \mathcal{B}_l \left[x \in B_l \wedge \neg \exists B_K \in \mathcal{B}_K \left(x \in B_K \wedge B_K \subset B_l\right)\right] \\
\exists x \in \mathbb{R} \exists B_l \in \mathcal{B}_l \neg \left[x \notin B_l \vee \exists B_K \in \mathcal{B}_K \left(x \in B_K \wedge B_K \subset B_l\right)\right] \\
\exists x \in \mathbb{R} \exists B_l \in \mathcal{B}_l \neg \left[x \in B_l \Rightarrow \exists B_K \in \mathcal{B}_K \left(x \in B_K \wedge B_K \subset B_l\right)\right] \\
\neg \forall x \in \mathbb{R} \forall B_l \in \mathcal{B}_l \left[x \in B_l \Rightarrow \exists B_K \in \mathcal{B}_K \left(x \in B_K \wedge B_K \subset B_l\right)\right] \\
\neg \forall x \in \mathbb{R} \forall B_l \in \mathcal{B}_l \left[x \in B_l \Rightarrow \exists B_K \in \mathcal{B}_K \left(x \in B_K \subset B_l\right)\right]
\end{gathered}$$

This shows by the negation of Lemma 13.3 that $\mathcal{T}_K$ is not finer than $\mathcal{T}_l$.

Now consider again $x = 0 \in \mathbb{R}$ and $B_K = (-1, 1) - K$, which clearly contains 0 and is a basis element of $\mathcal{B}_K$. Let B_l be any basis element of $\mathcal{B}_l$ that contains 0 so that $B_l = [a, b)$ where $a \leq 0 < b$. Clearly we have that $1/b > 0$ and there is an $n \in \mathbb{Z}_+$ where $n > 1/b$ since the positive integers have no upper bound. We then have

$$\begin{aligned}
&0 < 1/b < n \\
&0 < 1 < bn && \text{(since } b > 0) \\
&0 < 1/n < b && \text{(since } n > 1/b > 0)
\end{aligned}$$

so that $1/n \in [0,b) = B_l$. However, clearly $1/n \in K$ so that $1/n \notin (-1,1) - K = B_K$. Hence it must be that $B_l \not\subset B_K$. This shows that $\mathcal{T}_l$ is not finer than $\mathcal{T}_K$ by the negation of Lemma 13.3 as before.

This completes the proof that $\mathcal{T}_K$ and $\mathcal{T}_l$ are not comparable. □

Exercise 13.7

Consider the following topologies on $\mathbb{R}$:

$$\begin{aligned}
\mathcal{T}_1 &= \text{the standard topology,} \\
\mathcal{T}_2 &= \text{the topology of } \mathbb{R}_K, \\
\mathcal{T}_3 &= \text{the finite compliment topology,} \\
\mathcal{T}_4 &= \text{the upper limit topology, having all sets } (a,b] \text{ as basis,} \\
\mathcal{T}_5 &= \text{the topology having all sets } (-\infty,a) = \{x \mid x < a\} \text{ as basis.}
\end{aligned}$$

Determine, for each of these topologies, which of the others it contains.

Solution:

We claim that $\mathcal{T}_3 \subsetneq \mathcal{T}_1 \subsetneq \mathcal{T}_2 \subsetneq \mathcal{T}_4$ and $\mathcal{T}_5 \subsetneq \mathcal{T}_1 \subsetneq \mathcal{T}_2 \subsetneq \mathcal{T}_4$ but that $\mathcal{T}_3$ and $\mathcal{T}_5$ are incomparable.

Let $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}_4$, and $\mathcal{B}_5$ be the given bases corresponding to the above topologies, noting that $\mathcal{T}_3$ is defined directly rather than generated from a basis.

First we show that $\mathcal{T}_3 \subsetneq \mathcal{T}_1$.

Proof. Consider any $U \in \mathcal{T}_3$ so that $\mathbb{R} - U$ is finite or $U = \mathbb{R}$. Clearly in the latter case $U \in \mathcal{T}_1$ since it is a topology. In the former case $\mathbb{R} - U$ is a finite set of real numbers so that its elements can be enumerated as $\{x_1, x_2, \ldots, x_n\}$ for some $n \in \mathbb{Z}_+$ where $x_1 < x_2 < \cdots < x_n$. Then clearly we have that

$$U = (-\infty, x_1) \cup \left[\bigcup_{k=1}^{n-1} (x_k, x_{k+1})\right] \cup (x_n, \infty) .$$

Each of these sets is an interval (a,b) or the union of such intervals. For example, the set $(-\infty, x_1)$ can be covered by the countable union of intervals

$$\bigcup_{k=1}^{\infty} (x_1 - k - 1, x_1 - k + 1)$$

and similarly for the interval (x_n, ∞). Hence the union U is an element of $\mathcal{T}_1$ by Lemma 13.1. Since U was arbitrary, this shows that $\mathcal{T}_3 \subset \mathcal{T}_1$.

Now, clearly the interval $(-1,1)$ is in $\mathcal{T}_1$ since it is a basis element. However, we also have that $\mathbb{R} - (-1,1) = (-\infty,-1] \cup [1,\infty)$ is neither finite nor all of $\mathbb{R}$. Hence $(-1,1) \notin \mathcal{T}_3$. This shows that $\mathcal{T}_1$ cannot be a subset of $\mathcal{T}_3$ so that $\mathcal{T}_3 \subsetneq \mathcal{T}_1$ as desired. □

Next we show that $\mathcal{T}_5 \subsetneq \mathcal{T}_1$ also.

Proof. Consider any $x \in \mathbb{R}$ and any basis element $B_5 \in \mathcal{B}_5$ containing x. Then $B_5 = (-\infty, a)$ where $x < a$. Let $B_1 = (x-1, a)$, which is a basis element in $\mathcal{B}_1$. Also clearly B_1 contains x and is a subset of B_5. This proves that $\mathcal{T}_5 \subset \mathcal{T}_1$ by Lemma 13.3.

Now consider $x = -1$ and basis element $B_1 = (-2, 0)$ in $\mathcal{B}_1$, noting that obviously $x \in B_1$, and hence $-2 < x < 0$. Let B_5 be any element of $\mathcal{B}_5$ containing x so that $B_5 = (-\infty, a)$ where $x < a$. Clearly then $-3 < -2 < x < a$ so that $-3 \in B_5$. However, since $-3 \notin (-2, 0) = B_1$, this shows that $B_5 \not\subset B_1$. This suffices to show that $\mathcal{T}_1 \not\subset \mathcal{T}_5$ by the negation of Lemma 13.3. Therefore $\mathcal{T}_5 \subsetneq \mathcal{T}_1$ as desired. □

Now we show that $\mathcal{T}_3$ and $\mathcal{T}_5$ are not comparable.

Proof. First consider the set $U = \mathbb{R} - \{0\}$ so that $U \in \mathcal{T}_3$ since $\mathbb{R} - U = \{0\}$ is obviously finite. Now suppose that $U \in \mathcal{T}_5$ as well. Then, since clearly $1 \in U$, there must be a basis element $B_5 \in \mathcal{B}_5$ where $1 \in B_5$ and $B_5 \subset U$ by the definition of a topological basis. Then $B_5 = (-\infty, a)$ where $1 < a$. However, since $0 < 1 < a$ as well, it must be that $0 \in B_5$, and hence $0 \in U$ since $B_5 \subset U$. As this clearly contradicts the definition of U, it has to be that U is not in fact in $\mathcal{T}_5$ so that $\mathcal{T}_3 \not\subset \mathcal{T}_5$.

Now consider the set $U = (-\infty, 0)$, which is clearly in $\mathcal{T}_5$ since it is a basis element. However, since $\mathbb{R} - U = [0, \infty)$ is clearly neither all of $\mathbb{R}$ nor finite, it follows that $U \notin \mathcal{T}_3$. This shows that $\mathcal{T}_5 \not\subset \mathcal{T}_3$, which completes the proof that the two are incomparable. □

Now, the fact that $\mathcal{T}_1 \subsetneq \mathcal{T}_2$ was shown in Lemma 13.4. All that remains to be shown is that $\mathcal{T}_2 \subsetneq \mathcal{T}_4$ since the rest of the relations follow from the transitivity of proper inclusion.

Proof. First consider any basis element $B_2 \in \mathcal{B}_2$ and any $x \in B_2$. Either B_2 is (a, b) or $(a, b) - K$ for $a < b$ so that $a < x < b$ with $x \notin K$. In the former case clearly the set $B_4 = (a, x]$ is in $\mathcal{B}_4$, $x \in B_4$, and $B_4 \subset B_2$. In the latter case we have the following:

Case: $x \leq 0$. Then here again $B_4 = (a, x]$ is in $\mathcal{B}_4$, $x \in B_4$, and $B_4 \subset B_2$ since $y \notin K$ for any $y \in B_4$ since then $a < y \leq x \leq 0$.

Case: $x > 0$. Then let n be the smallest positive integer where $n > 1/x$, which exists since $\mathbb{Z}_+$ has no upper bound and is well-ordered. It then follows that $0 < 1/n < x$ and there are no integers m such that $1/n < 1/m \leq x$. So let $a' = \max(a, 1/n)$ and set $B_4 = (a', x]$ so that, for any $y \in B_4$, both $a \leq a' < y \leq x < b$ and $1/n \leq a' < y < x$, and hence $y \in (a, b)$ and $y \notin K$. Therefore $y \in (a, b) - K = B_2$. Since y was arbitrary, this shows that $B_4 \subset B_2$, noting that also clearly $x \in B_4$ and $B_4 \in \mathcal{B}_4$.

Hence in any case it follows that $\mathcal{T}_2 \subset \mathcal{T}_4$ from Lemma 3.13.

Now let $x = -1$ and $B_4 = (-2, -1]$ so that clearly $x \in B_4$ and $B_4 \in \mathcal{B}_4$. Then let B_2 be any basis element in $\mathcal{B}_2$ that contains x. Then we have that B_2 is either (a, b) or $(a, b) - K$ where $a < x < b$ and $x \notin K$.

Case: $0 < b$. Then $a < x = -1 < 0 \leq b$ so that 0 is in both (a, b) and $(a, b) - K$ since clearly $0 \notin K$, and thus $0 \in B_2$. However, clearly $0 \notin (-2, -1] = B_4$.

Case: $0 \geq b$. Then $a < x < (x + b)/2 < b \leq 0$ so that $(x + b)/2 \in B_2$ since $(x + b)/2$ is not in K. Clearly also though $(x + b)/2 \notin (-2, x] = B_4$ since $x < (x + b)/2$.

Thus in either case we have that $B_2 \not\subset B_4$. This shows the negation of Lemma 13.3 so that $\mathcal{T}_4 \not\subset \mathcal{T}_2$. Hence $\mathcal{T}_2 \subsetneq \mathcal{T}_4$ as desired. □

It is perhaps a rather surprising fact that, though it has been shown that the K and lower limit topology are incomparable (Exercise 13.6), the K topology and the upper limit topology *are* comparable as was just shown.

Exercise 13.8

(a) Apply Lemma 13.2 to show that the countable collection

$$\mathcal{B} = \{(a, b) \mid a < b, \; a \text{ and } b \text{ rational}\}$$

is a basis that generates the standard topology on $\mathbb{R}$.

(b) Show that the collection

$$\mathcal{C} = \{[a, b) \mid a < b, \; a \text{ and } b \text{ rational}\}$$

is a basis that generates a topology different from the lower limit topology on $\mathbb{R}$.

Solution:

(a)

Proof. Let $\mathcal{T}$ be the standard topology on $\mathbb{R}$. First, clearly $\mathcal{B}$ is a collection of open sets of $\mathcal{T}$ since each element is a basis element in the standard basis (i.e. an open interval). Now consider any $U \in \mathcal{T}$ and any $x \in U$. Then there is a standard basis element $B' = (a', b')$ such that $x \in B'$ and $B' \subset U$ since $\mathcal{T}$ is generated by the standard basis. Then $a' < x < b'$ so that, since the rationals are order-dense in the reals (shown in Exercise 4.9 part (d)), there are rational a and b such that $a' < a < x < b < b'$. Let $B = (a, b)$ so that clearly $x \in B$, $B \subset B' \subset U$, and $B \in \mathcal{B}$. This shows that $\mathcal{B}$ is a basis for $\mathcal{T}$ by Lemma 13.2 since U and $x \in U$ were arbitrary. □

(b)

Proof. First we must show that $\mathcal{C}$ is a basis at all. Clearly, for any $x \in \mathbb{R}$ we have that there is an element in $\mathcal{C}$ containing x, for example $[x, x+1)$. Now suppose that $C_1 = [a_1, b_1)$ and $C_2 = [a_2, b_2)$ are two elements of $\mathcal{C}$ and that $x \in C_1 \cap C_2$. Then obviously $a_1 \le x < b_1$ and $a_2 \le x < b_2$. Let $a = \max(a_1, a_2)$ and $b = \min(b_1, b_2)$ and $C = [a, b)$ so that clearly $C \in \mathcal{C}$. Also clearly $a \le x < b$ since both $a_1 \le x < b_1$ and $a_2 \le x < b_2$, a is a_1 or a_2, and b is b_1 or b_2. Therefore C contains x. Now consider any $y \in C$ so that $a_1 \le a \le y < b \le b_1$ and $a_2 \le a \le y < b \le b_2$ and hence $y \in C_1$ and $y \in C_2$. This shows that $C \subset C_1 \cap C_2$ since y was arbitrary. By definition this suffices to show that $\mathcal{C}$ is a basis for a topology.

So let $\mathcal{T}$ be the topology generated by $\mathcal{C}$ and $\mathcal{T}_l$ be the lower limit topology. Now consider $U = [x, x+1)$ where x is any irrational number, for example $x = \pi$. Let C be any basis element in $\mathcal{C}$ containing x so that $C = [a, b)$ where a and b are rational. It must be that $a \neq x$ since a is rational but x is not. Also, since C contains x it has to be that $a \le x$. So it has to be that $a < x$, but then $a \in C$ but $a \notin [x, x+1) = U$. This shows that C is not a subset of U. Hence we have shown

$$\exists x \in U \forall C \in \mathcal{C} \, (x \in C \Rightarrow C \not\subset U)$$
$$\exists x \in U \forall C \in \mathcal{C} \, (x \notin C \vee C \not\subset U)$$
$$\neg \forall x \in U \exists C \in \mathcal{C} \, (x \in C \wedge C \subset U) \; .$$

This shows that $U \notin \mathcal{T}$ by the definition of a generated topology. However, clearly we have that $U \in \mathcal{T}_l$ since it is a lower limit basis element. This suffices to show that $\mathcal{T}$ and $\mathcal{T}_l$ are different topologies. □

§16 The Subspace Topology

Exercise 16.1

Show that if Y is a subspace of X and A is a subspace of Y, then the topology A inherits as a subspace of Y is the same as the topology it inherits as a subspace of X.

Solution:

Proof. Let $\mathcal{T}$ be the topology on X and $\mathcal{T}_Y$ be the subspace topology that Y inherits from X. Also let $\mathcal{T}_A$ and $\mathcal{T}_A'$ be the topologies that A inherits as a subspace of Y and X, respectively. Therefore we must show that $\mathcal{T}_A = \mathcal{T}_A'$. Now, by definition of subspace topologies we have that,

$$\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}\} \qquad \mathcal{T}_A = \{A \cap U \mid U \in \mathcal{T}_Y\} \qquad \mathcal{T}_A' = \{A \cap U \mid U \in \mathcal{T}\} .$$

Now suppose that $W \in \mathcal{T}_A$ so that $W = A \cap V$ for some $V \in \mathcal{T}_Y$. Then we have that $V = Y \cap U$ for some $U \in \mathcal{T}$, and hence

$$W = A \cap V = A \cap (Y \cap U) = (A \cap Y) \cap U = A \cap U$$

since we have that $A \cap Y = A$ since $A \subset Y$. Since $U \in \mathcal{T}$ this clearly shows that $W \in \mathcal{T}_A'$ so that $\mathcal{T}_A \subset \mathcal{T}_A'$ since W was arbitrary.

Then, for any $W \in \mathcal{T}_A'$, we have that $W = A \cap U$ for some $U \in \mathcal{T}$. Let $V = Y \cap U$ so that clearly $V \in \mathcal{T}_Y$. Then as before we have that $A = A \cap Y$ since $A \subset Y$ so that

$$W = A \cap U = (A \cap Y) \cap U = A \cap (Y \cap U) = A \cap V ,$$

and thus $W \in \mathcal{T}_A$ since $V \in \mathcal{T}_Y$. Since W was arbitrary this shows that $\mathcal{T}_A' \subset \mathcal{T}_A$, which completes the proof that $\mathcal{T}_A = \mathcal{T}_A'$. ☐

Exercise 16.2

If $\mathcal{T}$ and $\mathcal{T}'$ are topologies on X and $\mathcal{T}'$ is strictly finer than $\mathcal{T}$, what can you say about the corresponding subspace topologies on the subset Y of X?

Solution:

Let $\mathcal{T}_Y$ and $\mathcal{T}_Y'$ be the subspace topologies on Y corresponding to $\mathcal{T}$ and $\mathcal{T}'$, respectively. We claim that $\mathcal{T}_Y$ is finer than $\mathcal{T}_Y$ but not necessarily strictly finer.

Proof. First, we have that

$$\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}\} \qquad \mathcal{T}_Y' = \{Y \cap U \mid U \in \mathcal{T}'\}$$

by the definition of subspace topologies. So for any $V \in \mathcal{T}_Y$ we have that $V = Y \cap U$ where $U \in \mathcal{T}$. Then also $U \in \mathcal{T}'$ since $\mathcal{T}'$ is finer than $\mathcal{T}$. This shows that $V \in \mathcal{T}_Y'$ since $V = Y \cap U$ where $U \in \mathcal{T}'$. Hence $\mathcal{T}_Y'$ is finer than $\mathcal{T}_Y$ since V was arbitrary.

To show that it is not necessarily strictly finer, consider the sets $X = \{a, b, c\}$ and $Y = \{a, b\}$ so that clearly $Y \subset X$. Consider also the topologies

$$\mathcal{T} = \{\varnothing, X, \{a, b\}\} \qquad \mathcal{T}' = \{\varnothing, X, \{a, b\}, \{c\}\}$$

on X so that clearly $\mathcal{T}'$ is strictly finer than $\mathcal{T}$. This results in the subspace topologies

$$\mathcal{T}_Y = \{\varnothing, Y\} \qquad \mathcal{T}_Y' = \{\varnothing, Y\} ,$$

which are clearly the same so that $\mathcal{T}_Y'$ is not strictly finer than $\mathcal{T}_Y'$, noting that it is technically still finer. However, if we instead have the topologies

$$\mathcal{T} = \{\varnothing, X, \{a, b\}\} \qquad \mathcal{T}' = \{\varnothing, X, \{a, b\}, \{b\}\}$$

then

$$\mathcal{T}_Y = \{\varnothing, Y\} \qquad\qquad \mathcal{T}'_Y = \{\varnothing, Y, \{b\}\}$$

so that $\mathcal{T}'_Y$ *is* strictly finer than $\mathcal{T}_Y$. Thus we can say nothing about the strictness of relation of the subspace topologies. □

Exercise 16.3

Consider the set $Y = [-1, 1]$ as a subspace of $\mathbb{R}$. Which of the following sets are open in Y? Which are open in $\mathbb{R}$?

$$\begin{aligned}
A &= \left\{x \mid \tfrac{1}{2} < |x| < 1\right\}, \\
B &= \left\{x \mid \tfrac{1}{2} < |x| \leq 1\right\}, \\
C &= \left\{x \mid \tfrac{1}{2} \leq |x| < 1\right\}, \\
D &= \left\{x \mid \tfrac{1}{2} \leq |x| \leq 1\right\}, \\
E &= \{x \mid 0 < |x| < 1 \text{ and } 1/x \notin \mathbb{Z}_+\}.
\end{aligned}$$

Solution:

Lemma 16.3.1. *If $a, b \in \mathbb{R}$ such that $0 \leq a < b$ then the following are true:*

$$\{x \in \mathbb{R} \mid a < |x| < b\} = (-b, -a) \cup (a, b) \qquad \{x \in \mathbb{R} \mid a \leq |x| \leq b\} = [-b, -a] \cup [a, b]$$
$$\{x \in \mathbb{R} \mid a \leq |x| < b\} = (-b, -a] \cup [a, b) \qquad \{x \in \mathbb{R} \mid a < |x| \leq b\} = [-b, -a) \cup (a, b].$$

Proof. We prove only the first of these as the rest follow from nearly identical arguments. Let $A = \{x \in \mathbb{R} \mid a < |x| < b\}$ and $B = (-b, -a) \cup (a, b)$ so that we must show that $A = B$.

So consider $x \in A$ so that $a < |x| < b$. If $x \geq 0$ then $|x| = x$ so that $a < x < b$ and hence $x \in (a, b)$. If $x < 0$ then $|x| = -x$ so that $a < -x < b$, and thus $-a > x > -b$ so that $x \in (-b, -a)$. Thus in either case $x \in B$ so that $A \subset B$.

Now let $x \in B$ so that either $x \in (-b, -a)$ or $x \in (a, b)$. In the former case we have that $x < -a \leq 0$ since $a \geq 0$ so that $|x| = -x$ and therefore

$$x \in (-b, -a) \Rightarrow -b < x < -a \Rightarrow b > -x = |x| > a \Rightarrow x \in A.$$

In the latter case we have that $x > a \geq 0$ so that $|x| = x$ and therefore

$$x \in (a, b) \Rightarrow a < x = |x| < b \Rightarrow x \in A.$$

This shows that $B \subset A$ since x was arbitrary, and thus $A = B$ as desired. □

Lemma 16.3.2. *Suppose that X is a topological space and $Y \subset X$ with the subspace topology. Then, if a set $U \subset Y$ is open in X, then it is also open in Y.*

Proof. So suppose that $U \subset Y$ is open in X. Then we have that $Y \cap U = U$ is also open in Y by the definition of the subspace topology. □

Main Problem.

First we claim that A is open in both $\mathbb{R}$ and Y.

Proof. We have from Lemma 16.3.1 that $A = (-1, -\frac{1}{2}) \cup (\frac{1}{2}, 1)$ which is clearly the union of basis elements so that A is open in $\mathbb{R}$. We also have that $A \subset Y$ so that A is open in Y by Lemma 16.3.2 since it is open in $\mathbb{R}$. □

Next we claim that B is open in Y but not in $\mathbb{R}$.

Proof. By Lemmma 16.3.1 we have that $B = [-1, -\frac{1}{2}) \cup (\frac{1}{2}, 1]$. First, consider the sets $(-2, -\frac{1}{2})$ and $(\frac{1}{2}, 2)$, which are clearly both basis elements and therefore open in $\mathbb{R}$. We then have that $(-2, -\frac{1}{2}) \cap Y = [-1, -\frac{1}{2})$ and $(\frac{1}{2}, 2) \cap Y = (\frac{1}{2}, 1]$ so that these sets are open in Y by the definition of the subspace topology. Clearly then their union $B = [-1, -\frac{1}{2}) \cup (\frac{1}{2}, 1]$ is then also open in Y.

It is also easy to see that B is not open in $\mathbb{R}$. For example, $-1 \in B$ but for any basis element $B' = (a, b)$ containing -1 we have that $a < -1 < b$ so that $a < (a-1)/2 < -1 < b$ and hence $(a-1)/2 \in B'$. Clearly though $(a-1)/2 \notin B$ so that B' cannot be a subset of B. Thus suffices to show that B is not open by the definition of the topology of $\mathbb{R}$ generated by its basis. □

We claim that C is open neither in $\mathbb{R}$ nor Y.

Proof. By Lemmma 16.3.1 we have that $C = (-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1)$. If $\mathcal{B}$ is the standard basis on $\mathbb{R}$, then, by Lemma 16.1, the set $\mathcal{B}_Y = \{B \cap Y \mid B \in \mathcal{B}\}$ is a basis for the subspace Y. So consider the point $x = \frac{1}{2}$ and any basis element $B_Y \in \mathcal{B}_Y$ containing x. Then we have that $B_Y = Y \cap B_X$ for some basis element $B_X = (a, b)$ in $\mathcal{B}$, and thus $a < x < b$ since $x \in B_X$. Let $a' = \max(a, -\frac{1}{2})$ and set $y = (a' + x)/2$ so that

$$a \leq a' < (a' + x)/2 = y < x < b\,,$$

and hence $y \in (a, b) = B_X$. Also we have

$$-1 < -\tfrac{1}{2} \leq a' < (a' + x)/2 = y < x = \tfrac{1}{2} < 1$$

so that $y \in [-1, 1] = Y$. Therefore $y \in Y \cap B_X = B_Y$. However, since $-\frac{1}{2} < y < \frac{1}{2}$, clearly $y \notin C$ so that B_Y cannot be a subset of C. Since the basis element $B_Y \in \mathcal{B}_Y$ was arbitrary, this suffices to show that C cannot be open in Y since $\mathcal{B}_Y$ is a basis. Since also clearly $C \subset Y$, it follows from the contrapositive of Lemma 16.3.2 that C is not open in $\mathbb{R}$ either. □

Next we claim that D is also not open in $\mathbb{R}$ or Y.

Proof. This follows from basically the same argument as the previous proof, again using the point $x = \frac{1}{2}$ to show that any basis element of Y that contains x cannot be a subset of D. □

Lastly, we claim that E is open in both $\mathbb{R}$ and Y.

Proof. First, it is trivial to show that

$$E = \{x \in \mathbb{R} \mid 0 < |x| < 1\} - K = [(-1, 0) \cup (0, 1)] - K\,,$$

where we have used Lemma 16.3.1. Now consider any $x \in E$ so that $x \in (-1, 0) \cup (0, 1)$ and $x \notin K$. If $x \in (-1, 0)$ then clearly the basis element $(-1, 0)$ contains x and is a subset of E since $(-1, 0) \cap K = \varnothing$.

On the other hand, if $x \in (0, 1)$ then $x \notin K$ so that $1/x \notin \mathbb{Z}_+$. From this it follows from Exercise 4.9 part (b) that there is exactly one positive integer n such that $n < 1/x < n + 1$. We then have that $1/(n+1) < x < 1/n$. So let $B = (1/(n+1), 1/n)$ so that clearly $x \in B$, $B \cap K = \varnothing$, and B is a basis element of the standard topology on $\mathbb{R}$. Since $B \cap K = \varnothing$ and clearly $0 < 1/(n+1) < 1/n \leq 1$, it also follows that $B \subset E$.

Hence in either case there is a basis element of $\mathbb{R}$ that contains x and is a subset of E. This suffices to show that E is open in $\mathbb{R}$. Since clearly $E \subset Y$, we also clearly have that E is open in Y by Lemma 16.3.2. □

Exercise 16.4

A map $f : X \to Y$ is said to be an ***open map*** if for every open set U of X, the set $f(U)$ is open in Y. Show that $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are open maps.

Solution:

Proof. Suppose that U is an open subset of $X \times Y$. Consider any $x \in \pi_1(U)$ so that there is a $y \in Y$ such that $(x, y) \in U$. Then there is a basis element $A \times B$ of the product topology on $X \times Y$ where $(x, y) \in A \times B \subset U$. Then A and B are open sets of X and Y, respectively, since $A \times B$ is a basis element of the product topology. Clearly we have that $x \in A$ since $(x, y) \in A \times B$. Now, for any $x' \in A$, we have that $(x', y) \in A \times B$ so that $(x', y) \in U$. Hence $x' = \pi_1(x', y) \in \pi_1(U)$, which shows that $A \subset \pi_1(U)$ since x' was arbitrary. Then, since A is an open subset of X, there is a basis element A' where $x \in A' \subset A \subset \pi_1(U)$. This suffices to show that $\pi_1(U)$ is an open subset of X since x was arbitrary. An analogous argument shows that π_2 is also an open map. $\square$

Exercise 16.5

Let X and X' denote a single set in the topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively; let Y and Y' denote a single set in the topologies $\mathcal{U}$ and $\mathcal{U}'$, respectively. Assume these sets are nonempty.

(a) Show that if $\mathcal{T}' \supset \mathcal{T}$ and $\mathcal{U}' \supset \mathcal{U}$, then the product topology on $X' \times Y'$ is finer than the product topology on $X \times Y$.

(b) Does the converse of (a) hold? Justify your answer.

Solution:

In what follows let $\mathcal{T}_p'$ and $\mathcal{T}_p$ denote the product topologies on $X' \times Y'$ and $X \times Y$, respectively.

(a)

Proof. Consider any $W \in \mathcal{T}_p$ and any $(x, y) \in W$, noting that obviously $W \subset X \times Y$. Then there is a basis element $U \times V$ of $\mathcal{T}_p$ such that $(x, y) \in U \times V$ and $U \times V \subset W$. By the definition of the product topology, we have that U and V are open sets in $\mathcal{T}$ and $\mathcal{U}$, respectively. Then we also have that $U \in \mathcal{T}'$ and $V \in \mathcal{U}'$ since $\mathcal{T} \subset \mathcal{T}'$ and $\mathcal{U} \subset \mathcal{U}'$. Hence $U \times V$ is also a basis element of $\mathcal{T}_p'$. Since we know that $(x, y) \in U \times V$, $U \times V \subset W$, and $(x, y) \in W$ was arbitrary, this suffices to show that W is an open subset of $X' \times Y'$ and hence $W \in \mathcal{T}_p'$. This in turn shows that $\mathcal{T}_p \subset \mathcal{T}_p'$ since W was arbitrary. $\square$

(b) We claim that the converse does not always hold.

Proof. As a counterexample consider $A = \{a, b, c, d\}$ so that clearly

$$\begin{aligned} \mathcal{T}' &= \{\varnothing, A, \{a, b\}, \{c, d\}\} \\ \mathcal{T} &= \{\varnothing, A, \{a, b\}, \{c, d\}, \{c\}, \{d\}, \{a, b, c\}, \{a, b, d\}\} \end{aligned}$$

are topologies on A. Clearly also $\mathcal{T}'$ is not finer than $\mathcal{T}$. Similarly let $B = \{1, 2, 3, 4\}$ so that

$$\begin{aligned} \mathcal{U}' &= \{\varnothing, B, \{1, 2\}, \{3, 4\}\} \\ \mathcal{U} &= \{\varnothing, B, \{1, 2\}, \{3, 4\}, \{3\}, \{4\}, \{1, 2, 3\}, \{1, 2, 4\}\} \end{aligned}$$

are topologies on B, also noting that clearly $\mathcal{U}'$ is not finer that $\mathcal{U}$. Now let $X = X' = \{a, b\}$ and $Y = Y' = \{1, 2\}$ so that clearly X and X' are in topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively, and Y and Y' are in $\mathcal{U}$ and $\mathcal{U}'$, respectively.

Then the bases for the product topologies $\mathcal{T}_p$ on $X \times Y$ and $\mathcal{T}'_p$ on $X' \times Y'$ are then

$$\mathcal{B} = \{\varnothing, X \times Y\} \qquad \mathcal{B}' = \{\varnothing, X' \times Y'\} = \{\varnothing, X \times Y\} = \mathcal{B},$$

respectively, since there are no subsets of X in $\mathcal{T}$ or $\mathcal{T}'$ other than $\varnothing$ and X itself, and similarly no subsets of Y in $\mathcal{U}$ or $\mathcal{U}'$ other than $\varnothing$ and Y. Since their bases are the same, clearly $\mathcal{T}_p = \mathcal{T}'_p$ so that it is true that $\mathcal{T}'_p$ is finer than $\mathcal{T}_p$ (though not strictly so). □

Exercise 16.6

Show that the countable collection

$$\{(a, b) \times (c, d) \mid a < b \text{ and } c < d \text{ and } a, b, c, d \text{ are rational}\}$$

is a basis for $\mathbb{R}^2$.

Solution:

Proof. It was proven in Exercise 13.8 part (a) that the set

$$\mathcal{B} = \{(a, b) \mid a < b,\ a \text{ and } b \text{ rational}\}$$

is a basis for the standard topology on $\mathbb{R}$. It then follows that

$$\mathcal{D} = \{B \times C \mid B, C \in \mathcal{B}\}$$

is a basis for the standard topology on $\mathbb{R}^2$ by Theorem 15.1. Clearly we have

$$\mathcal{D} = \{(a, b) \times (c, d) \mid a < b \text{ and } c < d \text{ and } a, b, c, d \text{ are rational}\},$$

which shows the desired result. □

Exercise 16.7

Let X be an ordered set. If Y is a proper subset of X that is convex in X, does it follow that Y is an interval or a ray in X?

Solution:

We claim that Y is not always an interval or a ray in X.

Proof. As a counterexample consider $X = \mathbb{Q}$ and the proper subset $Y = \{x \in \mathbb{Q} \mid x^2 < 2\}$. We claim that Y is convex but not an interval or a ray.

First, consider $a, b \in Y$ where $a < b$, thus $a^2, b^2 < 2$. Also consider $x \in (a, b)$ so that $a < x < b$. If $x \geq 0$ then $0 \leq x < b$ so that $x^2 < b^2 < 2$. If $x < 0$ then $a < x < 0$ so that $2 > a^2 > x^2$. Thus in either case $x^2 < 2$ so that $x \in Y$. Since x was arbitrary, this shows that $(a, b) \subset Y$ so that Y is convex since a and b were arbitrary.

Now, clearly Y cannot be a ray with no lower bound since then there would be an x in the ray where $x < -2$ so that $x^2 > 4 > 2$ and hence $x \notin Y$. Similarly Y cannot be a ray with no upper bound since then the ray would contain an $x > 2$ so that $x^2 > 4 > 2$ and thus $x \notin Y$. So suppose that

$Y = [a, b]$ for some $a, b \in X = \mathbb{Q}$ where $a \leq b$. Now, it cannot be that $b^2 = 2$ since then $b = \sqrt{2}$, which is not rational. Similarly it cannot be that $a^2 = 2$ for the same reason.

Case: $b^2 < 2$. Then there is a rational p where $b < p < \sqrt{2}$ since the rationals are order-dense in the reals. Let $x = \max(0, p)$ so that $b < p \leq x$ and hence $x \notin [a, b]$. However, if $0 < p$ then $x = p$ so that $x^2 = p^2 < 2$, and if $0 \geq p$ then $x = 0$ so that $x^2 = 0 < 2$. Thus either way $x \in Y$ and $x \notin [a, b]$, which shows that Y cannot be $[a, b]$.

Case: $b^2 > 2$. Then $\sqrt{2} < b$ since $0 < 2 < b^2$. If $\sqrt{2} < a$ then clearly for any $x \in [a, b]$ we have that $0 < \sqrt{2} < a \leq x$ so that $2 < x^2$ and hence $x \notin Y$. If $a < \sqrt{2}$ then there is a rational p such that $a < \sqrt{2} < p < b$ since the rationals are order-dense in the reals. Hence $2 < p^2$ so that $p \notin Y$. Either way there is an $x \in [a, b]$ where $x \notin Y$ so that Y cannot be $[a, b]$.

Similar arguments show that neither $Y = (a, b)$, $Y = [a, b)$, nor $Y = (a, b]$ for $a, b \in X = \mathbb{Q}$ and $a < b$. Hence Y cannot be an interval. Thus Y is convex but neither an interval nor a ray in X. This shows the desired result. □

Exercise 16.8

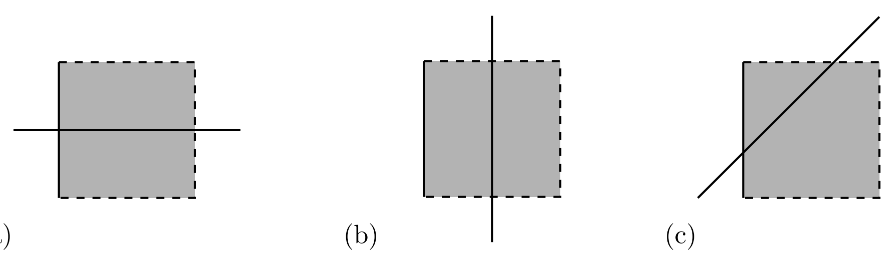
If L is a straight line in the plane, describe the topology L inherits as a subspace of $\mathbb{R}_\ell \times \mathbb{R}$ and as a subspace of $\mathbb{R}_\ell \times \mathbb{R}_\ell$. In each case it is a familiar topology.

Solution:

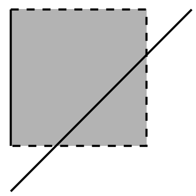
First, let $\mathbb{R}_u$denote the reals with the upper limit topology, with a basis containing all intervals $(a, b]$ for $a < b$. Also let $\mathbb{R}_d$denote the reals with the discrete topology, which can clearly be generated by a basis containing intervals $[a, b]$ for $a \leq b$. This is easy to see as $[a, a] = \{a\}$ is a basis element so that any subset of $\mathbb{R}$ can be considered a union of such basis elements. It is then easy to show that $\mathbb{R}_\ell$and $\mathbb{R}_u$are both strictly finer than the standard topology on $\mathbb{R}$ (this was shown in Lemma 13.4 for $\mathbb{R}_\ell$), but that $\mathbb{R}_\ell$and $\mathbb{R}_u$are incomparable. Clearly $\mathbb{R}_d$is strictly finer than both of these since it is the finest possible topology on $\mathbb{R}$.

Now, regarding the main problem, we do not yet have the tools show formally show how topologies on a line L compare to topologies on $\mathbb{R}$, so we will have to discuss this informally. We can see that, in some sense, a line L in the plane is like a copy of the real line so that we can discuss topologies on L is being in some sense the same as topologies on $\mathbb{R}$.

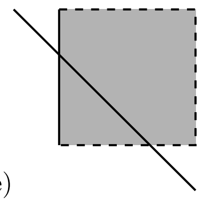
The product topology $\mathbb{R}_\ell \times \mathbb{R}$ is the topology is generated by the basis containing sets of the form $[a, b) \times (c, d)$ where $a < b$ and $c < d$ by Theorem 15.1. Then, for a line L in the plane, it can intersect such a basis element in a variety of ways, which are illustrated below:



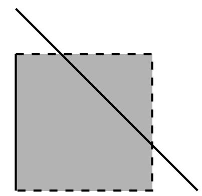
(a) (b) (c)



(d)



(e)



(f)

Clearly the intersection of L and these basis elements results in some kind of interval on L. Such intervals then form the basis for the subspace topology on L by Lemma 16.1 since they are the intersection of L and a basis element in the superspace. Another point is that the orientation of the line L with regard to the way in which it is a copy of $\mathbb{R}$ is important. For example, in Figure (a) above, if L is oriented in the natural way with the negative reals on the left and the positive reals on the right, then the resulting intervals are of the form $[a, b)$, which would result in a topology like $\mathbb{R}_\ell$. The opposite orientation results in intervals of the form $(a, b]$ as basis elements, generating a topology like $\mathbb{R}_u$.

Now, for a line L such as that illustrated in Figure (a), every possible basis element of $\mathbb{R}_\ell \times \mathbb{R}$ that intersects L results the half-open intervals as described above depending on the orientation of L. This is not the case for all lines, however, and is dependent on its slope in the plane. For example, lines with positive slope can intersect basis elements as in Figure (c), which result in half open intervals $[a, b)$ (or $(a, b]$ depending on orientation), or they can intersect them as in Figure (d), which result in open intervals (a, b). However, since the topologies $\mathbb{R}_\ell$and $\mathbb{R}_u$are strictly finer than the standard topology, the subspace topology formed on L would be like these (which depends on orientation) rather than like the standard topology. Lastly, we note that, for any appropriate interval on the line L, we can clearly always find a basis element B in $\mathbb{R}_\ell \times \mathbb{R}$ such that the intersection of B with L is the interval. For this reason, these intervals form the basis elements of the topology on L.

With all these considerations in mind, we list the topologies on $\mathbb{R}$ that the subspace topologies on L are like based on line directions and orientations for product topologies $\mathbb{R}_\ell \times \mathbb{R}$ and $\mathbb{R}_\ell \times \mathbb{R}_\ell$:

L	$\mathbb{R}_\ell \times \mathbb{R}$	$\mathbb{R}_\ell \times \mathbb{R}_\ell$
$\rightarrow$	$\mathbb{R}_\ell$	$\mathbb{R}_\ell$
$\nearrow$	$\mathbb{R}_\ell$	$\mathbb{R}_\ell$
$\uparrow$	$\mathbb{R}$	$\mathbb{R}_\ell$
$\nwarrow$	$\mathbb{R}_u$	$\mathbb{R}_d$
$\leftarrow$	$\mathbb{R}_u$	$\mathbb{R}_u$
$\swarrow$	$\mathbb{R}_u$	$\mathbb{R}_u$
$\downarrow$	$\mathbb{R}$	$\mathbb{R}_u$
$\searrow$	$\mathbb{R}_\ell$	$\mathbb{R}_d$

We note that $\mathbb{R}$ simply denotes the standard topology.

Exercise 16.9

Show that the dictionary order topology on the set $\mathbb{R} \times \mathbb{R}$ is the same as the product topology on $\mathbb{R}_d \times \mathbb{R}$, where $\mathbb{R}_d$ denotes $\mathbb{R}$ in the discrete topology. Compare this topology with the standard topology on $\mathbb{R}^2$.

Solution:

In what follows let $\mathcal{T}_d$ denote the dictionary order topology on $\mathbb{R} \times \mathbb{R}$, and let $\mathcal{T}_p$ denote the product topology on $\mathbb{R}_d \times \mathbb{R}$. Also let $\prec$ denote the dictionary ordering of $\mathbb{R} \times \mathbb{R}$. First we show that $\mathcal{T}_d = \mathcal{T}_p$.

Proof. First we note that clearly the dictionary order on $\mathbb{R} \times \mathbb{R}$ has no largest or smallest elements so that, by definition, $\mathcal{T}_d$ has as basis elements intervals $((x, y), (x', y'))$, that is the set of all points $z \in \mathbb{R} \times \mathbb{R}$ where $(x, y) \prec z \prec (x', y')$. Clearly the set $\{\{x\} \mid x \in \mathbb{R}\}$ is a basis for $\mathbb{R}_d$. Hence, by Theorem 15.1, the set $\mathcal{B}_p = \{\{x\} \times (a, b) \mid x \in \mathbb{R} \text{ and } a < b\}$ is a basis for the product topology $\mathcal{T}_p$.

So consider any $(x, y) \in \mathbb{R} \times \mathbb{R}$ and any basis element $B_d = ((a, b), (a', b'))$ of $\mathcal{T}_d$ that contains (x, y). Hence $(a, b) \prec (x, y) \prec (a', b')$.

Case: $a = x$: Then since $(a, b) \prec (x, y)$, it has to be that $b < y$.

Case: $a = x = a'$. Then it also has to be that $y < b'$ since $(x, y) \prec (a', b')$. Then the set $B_p = \{x\} \times (b, b')$ is a basis element of $\mathcal{T}_p$ that contains (x, y) and is a subset of B_d.

Case: $a = x < a'$. Then it is easy to show that the set $B_p = \{x\} \times (b, y+1)$ is a basis element of $\mathcal{T}_p$ that contains (x, y) and is a subset of B_d.

Case: $a < x$:

Case: $x = a'$. Then it has to be that $y < b'$ since $(x, y) \prec (a', b')$. Then it is easy to show that the set $B_p = \{x\} \times (y - 1, b')$ is a basis element of $\mathcal{T}_p$ that contains (x, y) and is a subset of B_d.

Case: $a = x < a'$. Then it is easy to show that the set $B_p = \{x\} \times (y - 1, y + 1)$ is a basis element of $\mathcal{T}_p$ that contains (x, y) and is a subset of B_d.

In every case and sub-case it follows from Lemma 13.3 that $\mathcal{T}_d \subset \mathcal{T}_p$.

Now suppose $(x, y) \in \mathbb{R} \times \mathbb{R}$ and $B_p = \{x\} \times (a, b)$ is a basis element of $\mathcal{T}_p$ containing (x, y). Also let B_d be the interval in the dictionary order $((x, a), (x, b))$, which is clearly a basis element of $\mathcal{T}_d$. It is then trivial to show that $B_p = B_d$ so that $x \in B_d \subset B_p$, which shows that $\mathcal{T}_p \subset \mathcal{T}_d$ by Lemma 13.3. This suffices to show that $\mathcal{T}_d = \mathcal{T}_p$ as desired. □

We now claim that this topology $\mathcal{T}_d = \mathcal{T}_p$ is strictly finer than the standard topology on $\mathbb{R} \times \mathbb{R}$. We denote the latter by simply $\mathcal{T}$.

Proof. Since it was just shown that $\mathcal{T}_d = \mathcal{T}_p$, it suffices to show that either one is strictly finer than the standard topology. It shall be most convenient to use the product topology $\mathcal{T}_p$. So first consider any $(x, y) \in \mathbb{R}^2$ and any basis element $B = (a, b) \times (c, d)$ of $\mathcal{T}$ containing (x, y). Hence $a < x < b$ and $c < y < d$. It is then trivial to show that the set $\{x\} \times (c, d)$, which is clearly a basis element of $\mathcal{T}_p$, contains (x, y) and is a subset of B. This shows that $\mathcal{T}_p$ is finer than $\mathcal{T}$ by Lemma 13.3.

To show that it is strictly finer, consider the point $(0, 0)$ and the set $B_p = \{0\} \times (-1, 1)$, which clearly contains $(0, 0)$ and is a basis element of $\mathcal{T}_p$. Now consider any basis element $B = (a, b) \times (c, d)$ of $\mathcal{T}$ that also contains $(0, 0)$. It then follows that $a < 0 < b$ and $c < 0 < d$. Consider then the point $x = (a + 0)/2 = a/2$ so that clearly $a < x < 0 < b$ and hence $x \in (a, b)$. Thus the point $(x, 0) \in B$, but also $(x, 0) \notin B_p$ since $x < 0$ so that $x \neq 0$. This shows that B cannot be a subset of B_p. Since B was an arbitrary basis element of $\mathcal{T}$, this shows that $\mathcal{T}$ is *not* finer than $\mathcal{T}_p$ by the negation of Lemma 13.3.

This suffices to show that $\mathcal{T}_p$ is strictly finer than $\mathcal{T}$ as desired. □

Exercise 16.10

Let $I = [0, 1]$. Compare the product topology on $I \times I$, the dictionary order topology on $I \times I$, and the topology $I \times I$ inherits as a subspace of $\mathbb{R} \times \mathbb{R}$ in the dictionary order topology.

Solution:

First, we assume that the product topology on $I \times I$ is the product of I with the order topology as this seems to be the standard when no topology is explicitly specified. Denote this product topology by $\mathcal{T}_p$. Let $\mathcal{T}_d$ denote the dictionary order topology on $I \times I$, and let $\mathcal{T}_s$ denote the subspace topology on $I \times I$ inherited as a subspace of $\mathbb{R} \times \mathbb{R}$ in the dictionary order topology. Lastly, let $\prec$ denote the dictionary order on $I \times I$ and $\mathbb{R} \times \mathbb{R}$. To avoid ambiguity we also use the notation $x \times y$ to denote the ordered pair (x, y) and reserve parentheses for open intervals.

First we claim that $\mathcal{T}_p$ and $\mathcal{T}_d$ are incomparable.

Proof. First, consider the point $0 \times 1 \in I \times I$ and $B_p = [0, 1/2) \times (1/2, 1]$, which is a basis element of $\mathcal{T}_p$ that clearly contains 0×1. Note that B_p is a basis element because $[0, 1/2)$ and $(1/2, 1]$ are both basis elements in the order topology on I since 0 and 1 are the smallest and largest elements of I, respectively. Now consider any interval $B_d = (a \times b, a' \times b')$ in the dictionary order on $I \times I$ that contains 0×1, which is of course a basis element of $\mathcal{T}_d$. Then we have that $a \times b$ and $a' \times b'$ are in $I \times I$ with $a \times b \prec 0 \times 1 \prec a' \times b'$. Hence $0 < a'$ or $0 = a'$ and $1 < b'$. As 1 is the largest element of I, the latter case is not possible so that it must be that $0 < a'$. Let $x = (0 + a')/2 = a'/2$ so that clearly $0 < x < a'$. Then we have that $a \times b' \prec x \times 0 \prec a' \times b'$ so that the point $x \times 0$ is in B_d. However, clearly $0 \notin (1/2, 1]$ so that $x \times 0 \notin B_p$. This shows that B_d cannot be a subset of B_p.

Here we note that, in the dictionary order on $I \times I$, the smallest element is 0×0 while the largest is 1×1. With this in mind, the above argument for an open interval also applies to the half-open intervals $[0 \times 0, a \times b)$ and $(a \times b, 1 \times 1]$, which are of course also basis elements of $\mathcal{T}_d$. This then shows that $\mathcal{T}_d$ is not finer than $\mathcal{T}_p$ by the negation of Lemma 13.3.

Now consider the point $0 \times 1/2$ and the interval $B_d = (0 \times 0, 0 \times 1)$ in the dictionary ordering, which is therefore a basis element of $\mathcal{T}_d$, and clearly also contains $0 \times 1/2$. Consider also any basis element $B_p = A \times B$ of $\mathcal{T}_p$ that contains $0 \times 1/2$. Since $0 \in A$ and A must be a basis element of the order topology on I, it has to be that $A = [0, a)$ for some $0 < a \leq 1$. Then let $x = (0 + a)/2 = a/2$ so that $0 < x < a$, and thus $x \in A$. Then, since $1/2 \in B$ (since $0 \times 1/2 \in A \times B$), we have that $x \times 1/2 \in A \times B = B_p$ as well. However, we also clearly have that $0 \times 1 \prec x \times 1/2$ since $0 < x$ so that $x \times 1/2 \notin (0 \times 0, 0 \times 1) = B_d$. This shows that B_p cannot be a subset of B_d. As B_p was an arbitrary basis element of $\mathcal{T}_p$, this shows by the negation of Lemma 13.3 that $\mathcal{T}_p$ is not finer than $\mathcal{T}_d$.

This suffices to show that $\mathcal{T}_d$ and $\mathcal{T}_p$ are incomparable. $\square$

Next we claim that $\mathcal{T}_s$ is strictly finer than $\mathcal{T}_p$.

Proof. Consider any $x \times y \in I \times I$ and suppose that $B_p = A \times B$ is a basis element of $\mathcal{T}_p$ that contains $x \times y$. First suppose that $A = (a, b)$ and $B = (c, d)$ so that of course $a, b, c, d \in I$, $a < x < b$, and $c < y < d$. It is then trivial to show that the interval $B_s = (x \times c, x \times d)$ in the dictionary order also contains $x \times y$, is a basis element of $\mathcal{T}_s$ (since $B_s \subset I \times I$ so that $B_s \cap (I \times I) = B_s$), and is a subset of B_p. A similar argument can be made if A is an interval of the form $[0, a)$ or $(a, 1]$. If $B = (c, 1]$ and A is still (a, b), then let X be the interval $(x \times c, x \times 2)$ in the dictionary order so that we have $B_s = X \cap (I \times I) = \{x\} \times (c, 1]$ is a basis element of $\mathcal{T}_s$ that contains $x \times y$ and is a subset of B_p. A similar argument applies if $B = [0, d)$ and/or when the interval A is half-open. This shows that $\mathcal{T}_s$ is finer than $\mathcal{T}_p$ by Lemma 13.3.

The argument above that shows that $\mathcal{T}_p$ is not finer than $\mathcal{T}_d$ using the negation of Lemma 13.3 applies equally well to show that $\mathcal{T}_p$ is not finer than $\mathcal{T}_s$. This of course suffices to show the desired result that $\mathcal{T}_s$ is strictly finer than $\mathcal{T}_p$. $\square$

Lastly, we claim that $\mathcal{T}_s$ is also strictly finer than $\mathcal{T}_d$.

Proof. First consider any point $x \times y$ in $I \times I$ and let B_d be a basis element of $\mathcal{T}_d$ that contains $x \times y$ so that it is some kind of interval with endpoints $a \times b$ and $a' \times b'$ in $I \times I$. We note here that, since $\mathbb{R} \times \mathbb{R}$ has no smallest or largest elements, basis elements of the dictionary order topology there can only be open intervals. Now, if B_d is an open interval in $I \times I$ then clearly then the same interval $B = (a \times b, a' \times b')$ is a basis element in the dictionary order topology of $\mathbb{R} \times \mathbb{R}$, though though the two intervals can in general be different sets. For example the interval $(0 \times 0, 1 \times 1)$ in $\mathbb{R} \times \mathbb{R}$ contains the point 0×100 whereas the same interval in $I \times I$ does not since $0 \times 100 \notin I \times I$. It is, however, trivial to show that $B \cap (I \times I) = B_d$ so that B_d is basis element of $\mathcal{T}_s$.

If we have that B_d is the half-open interval $[0 \times 0, a' \times b')$ then let $B = (0 \times -1, a' \times b')$, which is clearly a basis element of the dictionary order topology on $\mathbb{R} \times \mathbb{R}$. It is then easy to see that $B \cap (I \times I) = B_d$ again so that it is a basis element of $\mathcal{T}_s$. If B_d is the half-open interval $(a \times b, 1 \times 1]$, then the open interval $(a \times b, 1 \times 2)$ is a basis element of the dictionary order topology on $\mathbb{R} \times \mathbb{R}$ and has the same result. Hence in all cases B_d is also a basis element of $\mathcal{T}_s$, and that it trivially is a subset of itself, and it contains $x \times y$. This shows that $\mathcal{T}_s$ is finer than $\mathcal{T}_d$ by Lemma 13.3.

To show that it is strictly finer, consider the point 0×1 and the open interval $B = (0 \times 0, 0 \times 2)$, which is clearly a basis element of the dictionary order topology in $\mathbb{R} \times \mathbb{R}$. It is then easy to prove that $B_s = B \cap (I \times I) = \{0\} \times (0, 1] = (0 \times 0, 0 \times 1]$ so that B_s is a basis element of $\mathcal{T}_s$. Now consider any basis element B_d of $\mathcal{T}_d$ that contains 0×1 so that B_d is some type of dictionary-order interval with endpoints $a \times b$ and $a' \times b'$, both in $I \times I$. The only way the interval can be closed above is if $a' \times b' = 1 \times 1$, in which case clearly $1 \times 1 \in B_d$ but $1 \times 1 \notin B_s$. So assume that it is open above so that $0 \times 1 \prec a' \times b'$, and hence either $0 < a'$ or $0 = a'$ and $1 < b'$. The latter case cannot be since 1 is the largest element of I and $b' \in I$. Therefore it has to be that $0 < a'$. So let $x = (0 + a')/2 = a'/2$ so that $0 < x < a'$ and thus $0 \times 1 \prec x \times 0 \prec a' \times b'$. From this it follows that $x \times 0$ is in B_d. However, clearly $x \times 0 \notin B_s$ since $0 \times 1 \prec x \times 0$.

Hence in any case we have shown that, while they both contain 0×1, B_d cannot be a subset of B_s. Since B_d was an arbitrary basis element, this shows that $\mathcal{T}_d$ is not finer than $\mathcal{T}_s$ by the negation of Lemma 13.3. This shows the desired result that $\mathcal{T}_s$ is strictly finer than $\mathcal{T}_d$. □

§17 Closed Sets and Limit Points

Exercise 17.1

Let $\mathcal{C}$ be a collection of subsets of the set X. Suppose that $\varnothing$ and X are in C, and that finite unions and arbitrary intersections of elements of $\mathcal{C}$ are in $\mathcal{C}$. Show that the collection

$$\mathcal{T} = \{X - C \mid C \in \mathcal{C}\}$$

is a topology on X.

Solution:

Proof. First, clearly $\varnothing$ and X are in $\mathcal{T}$ since $\varnothing = X - X$ and $X = X - \varnothing$ and both X and $\varnothing$ are in $\mathcal{C}$. This shows the first defining property of a topology.

Now consider an arbitrary sub-collection $\mathcal{A}$ of $\mathcal{T}$. Then, for each $A \in \mathcal{A}$, we have that $A = X - B$ for some $B \in \mathcal{C}$ since also $A \in \mathcal{T}$. So let $\mathcal{B} = \{B \in \mathcal{C} \mid X - B \in \mathcal{A}\}$. Then we have that

$$\bigcup \mathcal{A} = \bigcup_{A \in \mathcal{A}} A = \bigcup_{B \in \mathcal{B}} (X - B) = X - \bigcap_{B \in \mathcal{B}} B = X - \bigcap \mathcal{B}$$

by DeMorgan's law. By the definition of $\mathcal{C}$ we have that $\bigcap \mathcal{B} \in \mathcal{C}$ since it is an arbitrary intersection of elements of $\mathcal{C}$. It then follows that $\bigcup \mathcal{A} = X - \bigcap \mathcal{B}$ is in $\mathcal{T}$ by definition. This shows the second defining property of a topology.

Lastly, suppose that $\mathcal{A}$ is a nonempty finite sub-collection of $\mathcal{T}$, which of course can be expressed as $\mathcal{A} = \{A_k \mid k \in \{1, \ldots, n\}\}$ for some positive integer n. Then, again we have that that $A_k = X - B_k$ for some $B_k \in \mathcal{C}$ for all $k \in \{1, \ldots, n\}$ since $A_k \in \mathcal{T}$. Then we have

$$\bigcap \mathcal{A} = \bigcap_{k=1}^{n} A_k = \bigcap_{k=1}^{n} (X - B_k) = X - \bigcup_{k=1}^{n} B_k$$

by DeMorgan's law. Then clearly $\bigcup_{k=1}^{n} B_k$ is in $\mathcal{C}$ by definition since it is a finite union of elements of $\mathcal{C}$. It then follows that $\bigcap \mathcal{A} = X - \bigcup_{k=1}^{n} B_k$ is in $\mathcal{T}$ by definition. Since $\mathcal{A}$ was an arbitrary finite sub-collection, this shows the third defining property of a topology. Hence $\mathcal{T}$ is a topology by definition. □

Exercise 17.2

Show that if A is closed in Y and Y is closed in X, then A is closed in X.

Solution:

Proof. Since A is closed in Y, it follows from Theorem 17.2 that $A = B \cap Y$ where B is some closed set in X. Hence by definition $X - B$ is open in X. Also, since Y is closed in X, we have that $X - Y$ is open in X by definition. We then have

$$X - A = X - (B \cap Y) = (X - B) \cup (X - Y)$$

by DeMorgan's law. Since both $X - B$ and $X - Y$ are open in X, clearly their union must also be open since we are in a topological space. Hence $X - A$ is open in X so that A is closed in X by definition. □

Exercise 17.3

Show that if A is closed in X and B is closed in Y, then $A \times B$ is closed in $X \times Y$.

Solution:

Lemma 17.3.1. *If X, Y, A, and B are sets then $X \times Y - A \times B = (X - A) \times Y \cup X \times (Y - B)$.*

Proof. We show this via logical equivalences:

$$\begin{aligned}
(x, y) \in X \times Y - A \times B &\Leftrightarrow (x, y) \in X \times Y \wedge (x, y) \notin A \times B \\
&\Leftrightarrow (x \in X \wedge y \in Y) \wedge \neg(x \in A \wedge y \in B) \\
&\Leftrightarrow (x \in X \wedge y \in Y) \wedge (x \notin A \vee y \notin B) \\
&\Leftrightarrow (x \in X \wedge y \in Y \wedge x \notin A) \vee (x \in X \wedge y \in Y \wedge y \notin B) \\
&\Leftrightarrow (x \in X - A \wedge y \in Y) \vee (x \in X \wedge y \in Y - B) \\
&\Leftrightarrow (x, y) \in (X - A) \times Y \vee (x, y) \in X \times (Y - B) \\
&\Leftrightarrow (x, y) \in (X - A) \times Y \cup X \times (Y - B)
\end{aligned}$$

as desired. □

Main Problem.

Proof. Since A is closed we have that $X - A$ is open in X. Since also Y itself is open in Y, we have that $(X - A) \times Y$ is a basis element in the product topology by definition, and is therefore obviously open. An analogous argument shows that $X \times (Y - B)$ is also open in the product topology since B is closed in Y. Hence their union is also open in the product topology, but by Lemma 17.3.1 we have

$$(X - A) \times Y \cup X \times (Y - B) = X \times Y - A \times B$$

so that $X \times Y - A \times B$ is also open in the product topology. It then follows by definition that $A \times B$ is closed as desired. □

Exercise 17.4

Show that if U is open in X and A is closed in X, then $U - A$ is open in X, and $A - U$ is closed in X.

Solution:

Lemma 17.4.1. *If A, B, and C are sets then $A - (B - C) = (A - B) \cup (A \cap C)$.*

Proof. We show this by a sequence of logical equivalences:

$$\begin{aligned} x \in A - (B - C) &\Leftrightarrow x \in A \wedge x \notin B - C \\ &\Leftrightarrow x \in A \wedge \neg(x \in B \wedge x \notin C) \\ &\Leftrightarrow x \in A \wedge (x \notin B \vee x \in C) \\ &\Leftrightarrow (x \in A \wedge x \notin B) \vee (x \in A \wedge x \in C) \\ &\Leftrightarrow x \in A - B \vee x \in A \cap C \\ &\Leftrightarrow x \in (A - B) \cup (A \cap C) \end{aligned}$$

as desired. □

Corollary 17.4.2. *If $A \subset X$ and $B = X - A$, then $A = X - B$.*

Proof. By Lemma 17.4.1, we have that

$$X - B = X - (X - A) = (X - X) \cup (X \cap A) = \varnothing \cup (X \cap A) = X \cap A = A$$

since $A \subset X$. □

Main Problem.

Proof. First, since A is closed in X, we have that $B = X - A$ is open in X, and it follows from Corollary 17.4.2 that $A = X - B$. Then we have that

$$U - A = U - (X - B) = (U - X) \cup (U \cap B)$$

by Lemma 17.4.1. Since $U \subset X$, it follows that $U - X = \varnothing$, and hence

$$U - A = \varnothing \cup (U \cap B) = U \cap B\,.$$

Then, since both U and B are open, their intersection is as well and therefore $U - A$ is open.

Next, we have by Lemma 17.4.1

$$X - (A - U) = (X - A) \cup (X \cap U) = B \cup (X \cap U) = B \cup U.$$

since $U \subset X$ so that $X \cap U = U$. Since both B and U are open, clearly their union is as well and hence $X - (A - U)$ is open. This of course means that $A - U$ is closed by definition. □

Exercise 17.5

Let X be an ordered set in the order topology. Show that $\overline{(a, b)} \subset [a, b]$. Under what conditions does equality hold?

Solution:

Proof. First, the closed interval $[a, b]$ is closed (hence why it is called such!) because clearly its compliment is

$$X - [a, b] = (-\infty, a) \cup (b, \infty)$$

and we know that open rays are always open so that their union is as well. Clearly also $[a, b]$ contains (a, b). Hence $[a, b]$ is a closed set containing (a, b). Since $\overline{(a, b)}$ is defined as the intersection of closed sets that contain (a, b) clearly we have that $\overline{(a, b)} \subset [a, b]$ as desired. □

The conditions required for equality are such that $[a, b]$ is also a subset of $\overline{(a, b)}$ and, in particular both a and b must be in $\overline{(a, b)}$. Since clearly $a, b \notin (a, b)$, it has to be that they are both limit points of (a, b). This is equivalent to the condition that a has no immediate successor and b no immediate predecessor. We show only the first of these since the second is analogous.

Proof. ($\Rightarrow$) We show the contrapositive of this. So suppose that a *does* have an immediate successor c. Then the open ray $(-\infty, c)$ is an open set that contains a but does not intersect (a, b). This is easy to see, because if they did intersect, there would be an $x \in (a, b)$ where also $x \in (-\infty, c)$. From these it follows that $a < x < c$, which contradicts the fact that c is the immediate successor of a. Hence by definition a is not a limit point of (a, b).

($\Leftarrow$) Suppose that a is not a limit point of (a, b). Then there is an open set U containing a that does not intersect (a, b). From this it follows that there is a basis element B containing a such that $B \subset U$, and thus B also cannot intersect (a, b) (as, if it did, then so would U). Suppose that B is the open interval (c, d) so that $c < a < d$. It also must be that $d < b$ for otherwise, for any element of x of (a, b), we would have $c < a < x < b \leq d$ so that $x \in (c, d) = B$ and B and (a, b) would not be disjoint. We claim that d is the immediate successor of a. If this is not the case then there would be an x such that $c < a < x < d$ and hence $x \in (c, d) = B$. Also $a < x < d < b$ so that also $x \in (a, b)$. Therefore B and (a, b) would not be disjoint. Similar arguments can be made if B are other types of basis element in the order topology. (Actually B cannot be of the form $(e, f]$ for largest element f of X since then any element of (a, b) would also be in B and they would not be disjoint.) □

It is also worth noting that the Hausdorff axiom (and therefore also the T_1 axiom since it is implied by the Hausdorff axiom) is not sufficient for general equivalence of $[a, b]$ and $\overline{(a, b)}$. For example the order topology on $\mathbb{Z}$ results in the discrete topology so that every subset is both open and closed. Thus for any pair x_1, x_2 in $\mathbb{Z}$, the sets $\{x_1\}$ and $\{x_2\}$ are neighborhoods of x_1 and x_2, respectively, that are disjoint. This shows that this topology is a Hausdorff space. However, the fact that a has an immediate successor in $\mathbb{Z}_+$ is sufficient to show that $[a, b] \neq \overline{(a, b)}$ per what was just shown above.

Exercise 17.6

Let A, B, and A_α denote subsets of a space X. Prove the following:

(a) If $A \subset B$, then $\overline{A} \subset \overline{B}$.

(b) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

(c) $\overline{\bigcup A_a} \supset \bigcup \overline{A}_\alpha$; give an example where equality fails.

Solution:

(a)

Proof. Suppose that $A \subset B$ and consider any $x \in \overline{A}$. Consider any neighborhood U of x so that U intersects A by Theorem 17.5 part (a). Hence there is a point $y \in U \cap A$ so that $y \in U$ and $y \in A$. But then clearly $y \in B$ also since $A \subset B$. Therefore $y \in U \cap B$ so that U intersects B. Since U was an arbitrary neighborhood of x, this shows that $x \in \overline{B}$, again by Theorem 17.5 part (a). This of course shows that $\overline{A} \subset \overline{B}$ as desired since x was arbitrary. □

(b)

Proof. ($\subset$) We show this by contrapositive. So suppose that $x \notin \overline{A} \cup \overline{B}$. Then clearly $x \notin \overline{A}$ and $x \notin \overline{B}$. Thus, by Theorem 17.5 part (a), there is an open set U_A such that U_A does not intersect A, and likewise an open U_B that does not intersect B. Let $U = U_A \cap U_B$, which is clearly open since U_A and U_B are. We also note that U contains x since both U_A and U_B do. Then it must be that U does not intersect A since, if it did, then U_A would also intersect A since $U \subset U_A$. Similarly, U cannot intersect B. Thus, for all $y \in U$, $y \notin A$ and $y \notin B$. This is logically equivalent to saying that there is no $y \in U$ where $y \in A$ or $y \in B$, therefore there is no $y \in U$ where $y \in A \cup B$. Hence U and $A \cup B$ do not intersect. Since U is open and contains x, this shows that $x \notin \overline{A \cup B}$, again by Theorem 17.5 part (a). Therefore, by contrapositive, $x \in \overline{A \cup B}$ implies that $x \in \overline{A} \cup \overline{B}$ so that $\overline{A \cup B} \subset \overline{A} \cup \overline{B}$.

($\supset$) Consider any $x \in \overline{A} \cup \overline{B}$ and any neighborhood U of x. If $x \in \overline{A}$ then U intersects A by Theorem 17.5 part (a). Hence there is a $y \in U \cap A$ so that $y \in U$ and $y \in A$. Then clearly $y \in A \cup B$ so that y is also in $U \cap (A \cup B)$. Hence U intersects $A \cup B$. An analogous argument shows that this is also true if $x \in \overline{B}$ instead. Since U was an arbitrary neighborhood, this shows that $x \in \overline{A \cup B}$ by Theorem 17.5 part (a). Hence $\overline{A} \cup \overline{B} \subset \overline{A \cup B}$ since x was arbitrary. □

(c)

Proof. Consider any $x \in \bigcup \overline{A}_\alpha$ so that there is a particular β where $x \in \overline{A}_\beta$. Suppose that U is any open set containing x so that U intersects A_β by Theorem 17.5 part (a) since $x \in \overline{A}_\beta$. Then clearly U also intersects $\bigcup A_\alpha$ since $A_\beta \subset \bigcup A_\alpha$. Since U was an arbitrary open set containing x, this shows that $x \in \overline{\bigcup A_\alpha}$ by Theorem 17.5 part (a). This shows that $\bigcup \overline{A}_\alpha \subset \overline{\bigcup A_\alpha}$ since x was arbitrary, which is of course the desired result. □

As an example where equality fails, consider the standard topology on $\mathbb{R}$ and the sets $A_n = (1/n, 2]$ for $n \in \mathbb{Z}_+$. It is then trivial to show that $\bigcup A_n = (0, 2]$ so that clearly 0 is a limit point of $\bigcup A_n$, and hence $0 \in \overline{\bigcup A_n}$. However, for any $n \in \mathbb{Z}_+$, the open interval $(-1, 1/n)$ is clearly an open set containing 0 that is disjoint from $(1/n, 2] = A_n$. This shows that $0 \notin \overline{A}_n$ for every $n \in \mathbb{Z}_+$ by Theorem 17.5 part (a), from which it follows that $0 \notin \bigcup \overline{A}_n$. Hence $\overline{\bigcup A_n}$ is not a subset of $\bigcup \overline{A}_n$ and thus $\overline{\bigcup A_n} \neq \bigcup \overline{A}_n$.

Exercise 17.7

Criticize the following "proof" that $\overline{\bigcup A_\alpha} \subset \bigcup \overline{A}_\alpha$: if $\{A_\alpha\}$ is a collection of sets in X and if $x \in \overline{\bigcup A_\alpha}$, then every neighborhood U of x intersects $\bigcup A_\alpha$. Thus U must intersect some A_α, so that x must belong to the closure of some A_α. Therefore, $x \in \bigcup \overline{A}_\alpha$.

Solution:

The problem with this "proof" is that, just because every neighborhood U intersects *some* A_α, it does not mean that *every* U intersects a single A_α, which is what is required for x to be in $\overline{A}_\alpha$. This is illustrated in the counterexample above at the end of Exercise 17.6 part (c). There, every neighborhood of 0 clearly intersects *some* set $A_n = (1/n, 2]$, but, for any given $n \in \mathbb{Z}_+$, not every neighborhood of 0 intersects A_n, for example the neighborhood $(-1, 1/n)$ does not.

Exercise 17.8

Let A, B, and A_α denote subsets of a space X. Determine whether the following equations hold; if an equality fails, determine whether one of the inclusions $\supset$ or $\subset$ holds.

(a) $\overline{A \cap B} = \overline{A} \cap \overline{B}$.
(b) $\overline{\bigcap A_\alpha} = \bigcap \overline{A}_\alpha$.
(c) $\overline{A - B} = \overline{A} - \overline{B}$.

Solution:

(a) We claim that $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$ but equality is not always true.

Proof. Consider any $x \in \overline{A \cap B}$ and any open set U containing x. Then by, Theorem 17.5 part (a), U intersects $A \cap B$, from which it immediately follows that U intersects both A and B. However, since U was an arbitrary neighborhood of x, it follows from Theorem 17.5 part (a) again that x is in both $\overline{A}$ and $\overline{B}$. Hence $x \in \overline{A} \cap \overline{B}$, which shows that $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$ since x was arbitrary.

Now consider the standard topology on $\mathbb{R}$ with $A = [-1, 0)$ and $B = (0, 1]$. As these are clearly disjoint, we have that $A \cap B = \varnothing$ so that $\overline{A \cap B} = \varnothing$ also. However, since we also clearly have that $\overline{A} = [-1, 0]$ and $\overline{B} = [0, 1]$, it follows that $\overline{A} \cap \overline{B} = \{0\}$. Thus clearly $\overline{A \cap B} = \varnothing \neq \{0\} = \overline{A} - \overline{B}$ as desired. □

(b) We again claim that $\overline{\bigcap A_\alpha} \subset \bigcap \overline{A}_\alpha$ but that equality is not generally true.

Proof. Consider any $x \in \overline{\bigcap A_\alpha}$ and any open set U of x. Then, by Theorem 17.5 part (a), U intersects $\bigcap A_\alpha$ so that, for any particular A_β, U intersects A_β. This shows that $x \in \overline{A}_\beta$ by Theorem 17.5 part (a) so that $x \in \overline{A}_\alpha$ for every α since β was arbitrary. Hence $x \in \bigcap \overline{A}_\alpha$, which shows that $\overline{\bigcap A_\alpha} \subset \bigcap \overline{A}_\alpha$ since x was arbitrary.

As in part (a), equality fails if we have $A_1 = [-1, 0)$ and $A_2 = (0, 1]$ in the standard topology on $\mathbb{R}$. By the same argument as in part (a) it follows that $\overline{\bigcap_{n=1}^{2} A_n} = \varnothing \neq \{0\} = \bigcap_{n=1}^{2} \overline{A}_n$. □

(c) Here we claim that $\overline{A - B} \supset \overline{A} - \overline{B}$ but that the converse does not always hold.

Proof. Consider any $x \in \overline{A} - \overline{B}$ and any open set U containing x. Then $x \in \overline{A}$ so that every open set containing x intersects A by Theorem 17.5 part (a). Also $x \notin \overline{B}$ so that there is an open set V containing x that does not intersect B, also by Theorem 17.5 part (a). Let $W = U \cap V$ so that W contains x since both $x \in U$ and $x \in V$. Now, since W is also an open set containing x, W intersects A so that there is a $y \in W$ where also $y \in A$. It also cannot be that $y \in B$ since we have $y \in W \subset V$

so that then V would intersect B. Therefore $y \in A - B$. Also we have $y \in W \subset U$ so that also $y \in U$. Hence U intersects $A - B$, which shows that $x \in \overline{A - B}$ by Theorem 17.5 part (a) since U was an arbitrary neighborhood of x. Therefore $\overline{A - B} \supset \overline{A} - \overline{B}$ as desired since x was arbitrary.

As a counterexample to equality, consider the standard topology on $\mathbb{R}$ with $A = [0, 2]$ and $B = (1, 3]$. Then clearly $\overline{A} = A = [0, 2]$ and $\overline{B} = [1, 3]$, from which it is easily shown that $\overline{A} - \overline{B} = [0, 1)$. But we also have $A - B = [0, 1]$ so that obviously $\overline{A - B} = [0, 1]$ as well. Therefore $\overline{A - B} = [0, 1] \neq [0, 1) = \overline{A} - \overline{B}$ as desired. □

Exercise 17.9

Let $A \subset X$ and $B \subset Y$. Show that in the space $X \times Y$,

$$\overline{A \times B} = \overline{A} \times \overline{B}.$$

Solution:

Proof. ($\subset$) Consider $(x, y) \in \overline{A \times B}$. Also suppose that U and V are any open sets in X and Y, respectively, that contain x and y, respectively. Then $U \times V$ is a basis element of the product topology on $X \times Y$, by definition, that contains (x, y). It then follows from Theorem 17.5 part (b) that $U \times V$ intersects $A \times B$ and hence there is a point $(w, z) \in U \times V$ where also $(w, z) \in A \times B$. Then $w \in U$ and $w \in A$ so that U intersects A, and hence $x \in \overline{A}$ by Theorem 17.5 part (a) since U was an arbitrary neighborhood of x. An analogous argument shows that $y \in \overline{B}$. Therefore $(x, y) \in \overline{A} \times \overline{B}$ so that $\overline{A \times B} \subset \overline{A} \times \overline{B}$ since x was arbitrary.

($\supset$) Now suppose that (x, y) is any point in $\overline{A} \times \overline{B}$ so that $x \in \overline{A}$ and $y \in \overline{B}$. Suppose also that $U \times V$ is any basis element of $X \times Y$ that contains (x, y) so that by definition U and V are open in X and Y, respectively. Since $x \in \overline{A}$ and U is an open set where $x \in U$, it follows from Theorem 17.5 part (a) that U intersects A. Thus there is $w \in U$ where $w \in A$ as well. An analogous argument shows that V intersects B so that there is a $z \in V$ where also $z \in B$. We therefore have that $(w, z) \in U \times V$ and $(w, z) \in A \times B$ so that $U \times V$ intersects $A \times B$. Since $U \times V$ was an arbitrary basis element containing (x, y), it follows from Theorem 17.5 part (b) that $(x, y) \in \overline{A \times B}$. This shows that $\overline{A} \times \overline{B} \subset \overline{A \times B}$ since the point (x, y) was arbitrary. □

Exercise 17.10

Show that every order topology is Hausdorff.

Solution:

Proof. Suppose that X is an ordered set with the order topology. Consider a pair of distinct points x_1 and x_2 in X. Since X is an order, x_1 and x_2 must be comparable since they are distinct, so we can assume that $x_1 < x_2$ without loss of generality.

Case: x_2 is the immediate successor of x_1. Then, if X has a smallest element a then clearly the set $U_1 = [a, x_2)$ is a neighborhood (because it is a basis element) of x_1. If X has no smallest element then there is an $a < x_1$ so that $U_1 = (a, x_2)$ is a neighborhood of x_1. Similarly $U_2 = (x_1, b]$ or $U_2 = (x_1, b)$ is a neighborhood of x_2, where b is either the largest element of X or $x_2 < b$, respectively. Either way, for any $y \in U_1$ we have that $y < x_2$ so that $y \leq x_1$ since x_2 is the immediate successor of x_1. Hence it is not true that $y > x_1$ so that $y \notin U_2$. This shows that U_1 and U_2 are disjoint.

Case: x_2 is not the immediate successor of x_1. Then there is an $x \in X$ where $x_1 < x < x_2$. So let $U_1 = [a, x)$ (or $U_1 = (a, x)$) for the smallest element a of X (or some $a < x_1$). Similarly let $U_2 = (x, b]$ (or $U_2 = (x, b)$) for the largest element b of X (or some $x_2 < b$). Either way U_1 and U_2 are neighborhoods of x_1 and x_2, respectively. If $y \in U_1$ then $y < x$ so that clearly it is not true that $y > x$ so that $x \notin U_2$. Hence again U_1 and U_2 are disjoint.

Thus in either case we have shown that X is a Hausdorff space as desired since x_1 and x_2 were an arbitrary pair. □

Exercise 17.11

Show that the product of two Hausdorff spaces is Hausdorff.

Solution:

Proof. Suppose that X and Y are Hausdorff spaces and consider two distinct points (x_1, y_1) and (x_2, y_2) in $X \times Y$. Since these points are distinct, it has to be that $x_1 \neq x_2$ or $y_1 \neq y_2$. In the first case x_1 and x_2 are distinct points of X so that there are disjoint neighborhoods U_1 and U_2 of x_1 and x_2, respectively. This of course follows from the fact that X is a Hausdorff space. Then we have that $U_1 \times Y$ and $U_2 \times Y$ are both basis elements, and therefore open sets, in the product space $X \times Y$ since Y itself is obviously an open set of Y. Clearly also $(x_1, y_1) \in U_1 \times Y$ and $(x_2, y_2) \in U_2 \times Y$ so that $U_1 \times Y$ is a neighborhood of (x_1, y_1) and $U_2 \times Y$ is a neighborhood of (x_2, y_2).

Then, for any $(x, y) \in U_1 \times Y$ we have that $x \in U_1$ so that $x \notin U_2$ since they are disjoint. Then it has to be that $(x, y) \notin U_2 \times Y$. This suffices to show that $U_1 \times Y$ and $U_2 \times Y$ are disjoint since (x, y) was arbitrary. Thus $X \times Y$ is a Hausdorff space since the points (x_1, y_1) and (x_2, y_2) were arbitrary. An analogous argument in the case in which $y_1 \neq y_2$ shows the same result. □

Exercise 17.12

Show that a subspace of a Hausdorff space is Hausdorff.

Solution:

Proof. Suppose that X is a Hausdorff space and that Y is a subset of X. Consider any two distinct points y_1 and y_2 in Y so that of course also $y_1, y_2 \in X$. Then there are neighborhoods U_1 and U_2 of y_1 and y_2, respectively, that are disjoint since X is Hausdorff. Since U_1 is open in X, we have that $V_1 = U_1 \cap Y$ is open in Y by the definition of a subspace topology. Clearly also V_1 contains y_1 since $y_1 \in U_1$ and $y_1 \in Y$. Similarly $V_2 = U_2 \cap Y$ is an open set of Y that contains y_2. Then, for any $x \in V_1$ clearly $x \in U_1$ so that $x \notin U_2$ since U_1 and U_2 are disjoint. Then $x \notin U_2 \cap Y = V_2$. Since x was arbitrary, this shows that V_1 and V_2 are disjoint, which then shows that Y is a Hausdorff space as desired. □

Exercise 17.13

Show that X is Hausdorff if and only if the ***diagonal*** $\Delta = \{x \times x \mid x \in X\}$ is closed in $X \times X$.

Solution:

Proof. (⇒) Suppose that X is Hausdorff and consider any point $x \times y \in X \times X$ where $x \times y \notin \Delta$. Then it must be that $x \neq y$ so that there are disjoint neighborhood U of x and V of y since X is Hausdorff. Then $U \times V$ is a basis element of $X \times X$, by the definition of a product topology, and is therefore open. Now consider any point $w \times z \in U \times V$ so that $w \in U$ and $z \in V$. Then it has to be that $w \neq z$ since U and V are disjoint, which shows that $w \times z \notin \Delta$. Since $w \times z$ was an arbitrary point of $U \times V$, this shows that $U \times V$ does not intersect Δ. Since also $U \times V$ is open and contains $x \times y$, this shows that $x \times y$ is not a limit point of Δ. Moreover, since $x \times y$ was an arbitrary element of $X \times X$ that is not in Δ, it follows that Δ must contain all of its limit points and is therefore closed by Corollary 17.7.

(⇐) Now suppose that Δ is closed and suppose that x and y are distinct points in X. Then $x \times y \notin \Delta$ so that $x \times y$ cannot be a limit point of Δ (since it contains all its limit points by Corollary 17.7). Hence there is an open set T in $X \times X$ that contains $x \times y$ and does not intersect Δ. It then follows that there is a basis element $U \times V$ of $X \times X$ containing $x \times y$ where $U \times V \subset T$. Then U and V are both open in X by the definition of the product topology, and clearly $x \in U$ and $y \in V$. It also follows that $U \times V$ does not intersect Δ since, if it did, then T would as well.

Suppose that U and V are not disjoint so that there is a $z \in U$ where also $z \in V$. Then clearly $z \times z \in U \times V$ but we also have that $z \times z \in \Delta$ so that $U \times V$ intersects Δ. As we know that this cannot be the case, it has to be that U and V are disjoint. This shows that X is Hausdorff as desired since U is a neighborhood of x, V is a neighborhood of y, and x and y were arbitrary distinct points of X. □

Exercise 17.14

In the finite compliment topology on $\mathbb{R}$, to what point or points does the sequence $x_n = 1/n$ converge?

Solution:

We claim that this sequence converges to every point in $\mathbb{R}$.

Proof. Suppose that this is not the case so that there is point $a \in \mathbb{R}$ where the sequence does not converge to A. Then there is an open set U containing a such that, for every $N \in \mathbb{Z}_+$, there is an $n \geq N$ where $x_n \notin U$. It is easy to see that $x_n \notin U$ for an infinite number of $n \in \mathbb{Z}_+$. For, if this were not the case, then there would be an $N \in \mathbb{Z}_+$ where $x_n \in U$ for every $n \geq N$. We know, though, that there must be an $n \geq N$ where $x_n \notin U$.

Moreover, clearly every x_n in the sequence is distinct so that there are an infinite number of points not in U. Since each of these points are still in X, we have that $X - U$ is infinite. As this is the finite compliment topology and U is open, this can only be the case if $X - U = X$ itself, in which case it would have be that $U = \varnothing$ since $U \subset X$. This is not possible since U contains a. So it seems that a contradiction has been reached, which shows the desired result. □

In fact, this is true for any sequence for which the image of the sequence $\{x_n \mid n \in \mathbb{Z}_+\}$ is infinite. This is to say that any such sequence converges to every point of $\mathbb{R}$. Note also that this shows that the finite compliment topology on $\mathbb{R}$ is not a Hausdorff space by the contrapositive of Theorem 17.10.

Exercise 17.15

Show the T_1 axiom is equivalent to the condition that for each pair of points of X, each has a neighborhood not containing the other.

Solution:

Note that, though it does not say so above, the points must be distinct since any neighborhood containing x obviously has to contain x.

Proof. ($\Rightarrow$) Suppose that a space X satisfies the T_1 axiom and consider any two distinct points x and y of X. Then the point $\{x\}$ is closed since it is finite, and hence it also contains all of its limit points by Corollary 17.7. Since the point y is not in $\{x\}$ (since $y \neq x$), it cannot be a limit point of $\{x\}$. Hence there is a neighborhood U of y that does not intersect $\{x\}$. Hence $x \notin U$. An analogous argument involving $\{y\}$ shows that there is a neighborhood V of x that does not contain y. Since x and y were arbitrary points, this shows the desired property.

($\Leftarrow$) Now suppose that, for each pair of distinct points in X, each point has a neighborhood that does not contain the other point. As in the proof of Theorem 17.8, it suffices to show that every one-point set is closed, since any finite set can be expressed as the finite union of such sets, which is also then closed by Theorem 17.1. So let $\{x\}$ be such a one-point set and consider any $y \notin \{x\}$ so that clearly $y \neq x$. Then, since x and y are distinct, there is a neighborhood U of y such that U does not contain x. Therefore U and $\{x\}$ are disjoint. This shows that y is not a limit point of $\{x\}$, which shows that $\{x\}$ contains all its limit points since $y \notin \{x\}$ was arbitrary. Hence $\{x\}$ is closed as desired by Corollary 17.7. □

Exercise 17.16

Consider the five topologies on $\mathbb{R}$ given in Exercise 7 of §13.

(a) Determine the closure of the set $K = \{1/n \mid n \in \mathbb{Z}_+\}$ under each of these topologies.

(b) Which of these topologies satisfy the Hausdorff axiom? the T_1 axiom?

Solution:

Lemma 17.16.1. *Suppose that $\mathcal{T}$ and $\mathcal{T}'$ are topologies on X and $\mathcal{T}'$ is finer than $\mathcal{T}$. If $\mathcal{T}$ satisfies the T_1 axiom, then so does $\mathcal{T}'$. Similarly, if $\mathcal{T}$ is Hausdorff, then so is $\mathcal{T}'$.*

Proof. First, suppose that $\mathcal{T}$ satisfies the T_1 axiom and consider any finite subset A of X. Then A is closed in $\mathcal{T}$ by the T_1 axiom so that by definition $X - A$ is open in $\mathcal{T}$ and hence $X - A \in \mathcal{T}$. Then $X - A \in \mathcal{T}'$ as well since $\mathcal{T} \subset \mathcal{T}'$ so that $X - A$ is open in $\mathcal{T}'$. Hence A is closed in $\mathcal{T}'$ by definition. Since A was an arbitrary finite set, this shows that $\mathcal{T}'$ also satisfies the T_1 axiom.

Now suppose that $\mathcal{T}$ is Hausdorff, and consider any two distinct points x and y in X. Then there are neighborhoods U of x and V of y, both in $\mathcal{T}$, that do not intersect since $\mathcal{T}$ is Hausdorff. Then clearly $U, V \in \mathcal{T}'$ as well since $\mathcal{T} \subset \mathcal{T}'$. Hence U and V are neighborhoods of x and y, respectively, in $\mathcal{T}'$ that do not intersect. This shows that $\mathcal{T}'$ is Hausdorff as desired since x and y were arbitrary points of X. □

Main Problem.

First we summarize what we claim about these topologies on $\mathbb{R}$ for both parts:

Topology	Definition	T_1	Hausdorff	$\overline{K}$
$\mathcal{T}_1$	Standard	Yes	Yes	$K \cup \{0\}$
$\mathcal{T}_2$	$\mathbb{R}_K$	Yes	Yes	K
$\mathcal{T}_3$	Finite complement	Yes	No	$\mathbb{R}$
$\mathcal{T}_4$	Upper limit	Yes	Yes	K
$\mathcal{T}_5$	Basis of $(-\infty, a)$ sets	No	No	$\{x \in \mathbb{R} \mid 0 \leq x\}$

Next we justify these claims for each part.

(a) First we show that $\overline{K} = K \cup \{0\}$ in $\mathcal{T}_1$.

Proof. ($\subset$) Consider any real number x and suppose that $x \notin K \cup \{0\}$, hence $x \notin K$ and $x \neq 0$. Since $x \neq 0$, we have

Case: $x < 0$. Then clearly the open set $(x-1, 0)$ contains x but does not intersect K since $0 < y$ for every $y \in K$, but $y < 0$ for every $y \in (x-1, 0)$.

Case: $x > 0$. If $1 < x$, then $(1, x+1)$ contains x but does not intersect K since $y \leq 1$ for every $y \in K$, but $1 < y$ for every $y \in (1, x+1)$. If $1 \geq x$ it follows from the fact that $x \notin K$ that there is a positive integer n where $n < 1/x < n+1$, and hence $1/(n+1) < x < 1/n$. Then clearly the open set $(1/(n+1), 1/n)$ contains x, but we also have that it does not intersect K. If it did, then there would be an integer m where $1/(n+1) < 1/m < 1/n$ so that $n < m < n+1$, which we know is not possible since $n+1$ is the immediate successor of n in $\mathbb{Z}_+$.

Thus in all cases there is a neighborhood of x that does not intersect K. This of course shows that $x \notin \overline{K}$ by Theorem 17.5 part (a). We have therefore shown that $x \notin K \cup \{0\}$ implies that $x \notin \overline{K}$. By contrapositive, this shows that $\overline{K} \subset K \cup \{0\}$.

($\supset$) Now consider any neighborhood U of 0 so that there is a basis element (a, b) containing 0 that is a subset of U. Then $a < 0 < b$. Clearly there is an $n \in \mathbb{Z}_+$ large enough where $a < 0 < 1/n < b$ and hence $1/n \in (a, b) \subset U$. Since also $1/n \in K$, we have that U intersects K. Since U was an arbitrary neighborhood, this shows that 0 is in $\overline{K}$ by Theorem 17.5 part (a). Since also clearly any $x \in K$ is also in $\overline{K}$, it follows that $\overline{K} \supset K \cup \{0\}$. □

Next we show that $\overline{K} = K$ in $\mathcal{T}_2$, which is to say that K is already closed.

Proof. First, clearly $K \subset \overline{K}$ basically by definition. Now consider any $x \notin K$. Then clearly the set $B = (x-1, x+1) - K$ is a basis element of $\mathcal{T}_2$. Also it clearly contains x since $x \notin K$ and also does not intersect K since $y \in B$ means that $y \notin K$. This shows that x is not in $\overline{K}$ by Theorem 17.5 part (b). Since x was arbitrary this shows that $x \notin K$ implies that $x \notin \overline{K}$. Thus $\overline{K} \subset K$ by contrapositive. This suffices to show that $\overline{K} = K$ as desired. □

Now we show that $\overline{K} = \mathbb{R}$ in $\mathcal{T}_3$.

Proof. Consider any real x and any neighborhood U of x. Then U is open in $\mathcal{T}_3$ so that $\mathbb{R} - U$ must be finite, noting that $\mathbb{R} - U$ cannot be all of $\mathbb{R}$ since U would then have to be empty since $U \subset \mathbb{R}$, whereas we know that $x \in U$. It then follows that there are a finite number of real numbers not in U. However, clearly K is an infinite set so that there must be an element of K that *is* in U. This shows that K intersects U so that x is in $\overline{K}$ by Theorem 17.5 part (a) since U was an arbitrary neighborhood. Hence $\mathbb{R} \subset \overline{K}$ since x was arbitrary. Clearly also $\overline{K} \subset \mathbb{R}$ so that $\overline{K} = \mathbb{R}$. □

Next we show that $\overline{K} = K$ in $\mathcal{T}_4$ so that K is closed.

Proof. Clearly $K \subset \overline{K}$. So consider any real x where $x \notin K$.

Case: $x \leq 0$. Then the set $B = (x-1, x]$ is clearly a basis element of $\mathcal{T}_4$ that contains x. For any $y \in K$ we have that $x \leq 0 < y$ so that $y \notin B$. Hence B does not intersect K.

Case: $x > 0$. If $1 \leq x$ then it has to be that $1 < x$ since $1 = 1/1 \in K$ but $x \notin K$, and hence $x \neq 1$. Then $B = (1, x]$ is clearly a basis element of $\mathcal{T}_4$ and contains x. This also clearly does not intersect K since $y \leq 1$ for any $y \in K$ so that $y \notin B$. On the other hand, if $1 > x$ then there is an integer n where $n < 1/x < n+1$ so that $1/(n+1) < x < 1/n$ since $x \notin K$. It then follows that the set $B = (1/(n+1), x]$ is a basis element of $\mathcal{T}_4$ that contains x and does not intersect K.

Hence in all cases there is a basis element B containing x that does not intersect K. This shows that $x \notin \overline{K}$ by Theorem 17.5 part (b). Hence we have shown that $x \notin K$ implies that $x \notin \overline{K}$, which shows by contrapositive that $\overline{K} \subset K$. Therefore $\overline{K} = K$ as desired. □

Lastly we show that $\overline{K} = \{x \in \mathbb{R} \mid 0 \le x\}$ in $\mathcal{T}_5$.

Proof. First, let $A = \{x \in \mathbb{R} \mid 0 \le x\}$ and consider any $x \in A$ and any basis element $B = (-\infty, a)$ containing x. Hence clearly $0 \le x$ since $x \in A$ and $x < a$ since $x \in B$. Thus $0 \le x < a$ so that there is an integer n large enough that $0 < 1/n < a$. Then $1/n \in B$ and also clearly $1/n \in K$. Thus B intersects K. Since B was any neighborhood of x it follows from Theorem 17.5 part (b) that $x \in \overline{K}$. Hence $A \subset \overline{K}$ since x was arbitrary.

Now suppose that $x \notin A$ so that $x < 0$. Then the set $B = (-\infty, 0)$ is clearly a basis element of $\mathcal{T}_5$ that contains x. Since $0 < y$ for any $y \in K$, it follows that $y \notin B$, and thus B cannot intersect K. Hence by Theorem 17.5 part (b) we have that $x \notin \overline{K}$. This shows that $\overline{K} \subset A$ by contrapositive, which completes the proof that $\overline{K} = A$. □

(b) First we show that $\mathcal{T}_1$, $\mathcal{T}_2$, and $\mathcal{T}_4$ are Hausdorff spaces and satisfy the T_1 axiom.

Proof. First consider any two distinct points $x, y \in \mathbb{R}$. Without loss of generality, we can assume that $x < y$. Let $z = (x+y)/2$ so that clearly $x < z < y$. Then obviously the open intervals $(x-1, z)$ and $(z, y+1)$ are disjoint open sets in $\mathcal{T}_1$ that contain x and y, respectively. This shows that $\mathcal{T}_1$ is a Hausdorff space and therefore also satisfies the T_1 axiom by Theorem 17.8.

It then follows that $\mathcal{T}_2$ and $\mathcal{T}_4$ are also both Hausdorff and satisfy the T_1 axiom. This follows from Lemma 17.16.1 since it was shown in Exercise 13.7 that $\mathcal{T}_1 \subset \mathcal{T}_2 \subset \mathcal{T}_4$. □

Next we show that $\mathcal{T}_3$ satisfies the T_1 axiom but is not a Hausdorff space.

Proof. So first consider any finite subset A of $\mathbb{R}$. Let $U = X - A$ so that clearly $A = X - (X - A) = X - U$. Then, since $X - U = A$ is finite, it follows that U is open in $\mathcal{T}_3$ by the definition of the finite complement topology. Hence by definition A is closed in $\mathcal{T}_3$ since $X - A = U$ is open. This shows that $\mathcal{T}_3$ satisfies the T_1 axiom since A was an arbitrary finite subset of $\mathbb{R}$.

To show that $\mathcal{T}_3$ is not Hausdorff, consider any open set U containing 0 and any open set V containing 1. It then has to be that $\mathbb{R} - U$ is finite since it cannot be that $\mathbb{R} - U = \mathbb{R}$ itself since then U would have to be empty (which we know is not the case since $0 \in U$) since $U \subset \mathbb{R}$. Likewise $\mathbb{R} - V$ is also finite. Thus there are a finite number of real numbers that are not in U and a finite number that are not in V. From this it clearly follows that there are a finite number of real numbers x where $x \notin U$ or $x \notin V$. Since we have

$$x \notin U \vee x \notin V \Leftrightarrow \neg(x \in U \wedge x \in V) \Leftrightarrow \neg(x \in U \cap V) \Leftrightarrow x \notin U \cap V\,,$$

it has to be that there are a finite number of reals numbers that are not in $U \cap V$. But since $\mathbb{R}$ is infinite, this means that there are an infinite number of real numbers that *are* in $U \cap V$. Hence $U \cap V \neq \varnothing$, i.e. they intersect. Since U and V were arbitrary neighborhoods, this shows that $\mathcal{T}_3$ is not Hausdorff by the negation of the definition. □

Lastly we prove that $\mathcal{T}_5$ is neither a Hausdorff space nor satisfies the T_1 axiom.

Proof. First consider the distinct real numbers 0 and 1. Consider then any open set V containing 1 so that there is a basis element $B = (-\infty, a)$ that contains 1 and is a subset of U. Clearly we have that $0 \in B$ since $0 < 1 < a$ and hence $0 \in U$ since $B \subset U$. Since U was an arbitrary neighborhood of 1, it follows there is no neighborhood of 1 that does not contain 0. Hence $\mathcal{T}_5$ does not satisfy the T_1 axiom by the negation of Exercise 17.15. It also then follows that $\mathcal{T}_5$ is not a Hausdorff space by the contrapositive of Theorem 17.8. □

Exercise 17.17

Consider the lower limit topology on $\mathbb{R}$ and the topology given by the basis $\mathcal{C}$ of Exercise 8 of §13. Determine the closures of the intervals $A = (0, \sqrt{2})$ and $B = (\sqrt{2}, 3)$ in these two topologies.

Solution:

Recall that $\mathcal{C} = \{[a,b) \mid a < b,\ a \text{ and } b \text{ rational}\}$ from Exercise 13.8, noting that it was shown there that this basis generates a topology different from the lower limit topology. Denote the lower limit topology by $\mathcal{T}_l$, and denote the topology generated by $\mathcal{C}$ by $\mathcal{T}_c$.

Lemma 17.17.1. *The closure of an open interval (a,b) is $[a,b)$ in the lower limit topology on $\mathbb{R}$.*

Proof. First let $A = (a,b)$ and $C = [a,b)$ so that we must show that $\overline{A} = C$.

($\supset$) Consider any $x \in C$.

Case: $x = a$. Consider any basis element $B = [c,d)$ that contains $x = a$ so that $c \leq x = a < d$. Let $e = \min(b,d)$ so that $a < e$ since both $a < d$ and $a < b$. Then of course there is a real y between a and e so that $a < y < e$. Thus we have $c \leq a < y < e \leq d$ so that $y \in B$. Also $a < y < e \leq b$ so that $y \in A$. Hence B intersects A so that $x = a \in \overline{A}$ by Theorem 17.5 part (b) since B was an arbitrary basis element.

Case: $x \neq a$. Then it has to be that $x \in (a,b) = A$ so that $x \in \overline{A}$ since obviously $A \subset \overline{A}$.

This shows that $C \subset \overline{A}$ since x was arbitrary.

($\subset$) Now consider any real x where $x \notin C$ so that either $x < a$ or $x \geq b$. If $x < a$ then the basis element $B = [x,a)$ clearly contains x but does not intersect A. If $x \geq b$ then the basis element $B = [b, x+1)$ contains x and does not intersect A. Either way it follows from Theorem 17.5 part (b) that $x \notin \overline{A}$. Since x was arbitrary, the contrapositive shows that $\overline{A} \subset C$. $\square$

Lemma 17.17.2. *The closure of an open interval (a,b) in $\mathcal{T}_c$ is $[a,b)$ if b is rational and $[a,b]$ if b is irrational.*

Proof. Let $A = (a,b)$. Consider any real x, and we shall consider an exhaustive list of cases that will show whether $x \in \overline{A}$ or $x \notin \overline{A}$.

Case: $x < a$. Obviously there is a rational p where $p < x$ since the rationals are unbounded below. Similarly, there is a rational q where $x < q < a$ since the rationals are order-dense in the reals. The set $B = [p,q)$ is then clearly a basis element of $\mathcal{T}_c$ that contains x. It is also trivial to show that B does not intersect A since $q < a$, which shows that $x \notin \overline{A}$ by Theorem 17.5 part (b) whether b is rational or not.

Case: $x = a$. Consider any basis element $B = [p,q)$ (where p and q are rational) that contains $x = a$ so that $p \leq x = a < q$. Let $d = \min(b,q)$ so that $a < d$ since both $a < q$ and $a < b$. Then of course there is a real y between a and d so that $a < y < d$. Thus we have $p \leq a < y < d \leq q$ so that $y \in B$. Also $a < y < d \leq b$ so that $y \in A$. Hence B intersects A so that $x = a \in \overline{A}$ by Theorem 17.5 part (b) since B was an arbitrary basis element. Note that this is true whether or not b is rational.

Case: $a < x < b$. Then clearly $x \in (a,b) = A$ so that $x \in \overline{A}$ since obviously $A \subset \overline{A}$.

Case: $x = b$.

> Case: b is rational. Then there is another rational q where $q > b$ since the rationals are unbounded above. Then clearly the set $B = [b,q)$ is a basis element of $\mathcal{T}_c$ that contains b. Also clearly B does not intersect A since $y \in A$ implies that $y < b$ and hence $y \notin B$. This shows that $x = b \notin \overline{A}$ by Theorem 17.5 part (b).
>
> Case: b is irrational. Then consider any basis element $B = [p,q)$ containing b so that p and q are rational. Thus $p \leq b < q$, but since p is rational but b is not, it has to be that $p < b < q$.

Let $c = \max(p, a)$ so that $c < b$ since both $a < b$ and $p < b$. There is then a real y where $c < y < b$ so that $a \le c < y < b$ and hence $y \in A$. Also $p \le c < y < b < q$ so that also $y \in B$. Therefore B and A intersect, which shows that $x = b \in \overline{A}$ by Theorem 17.5 part (b) since B was arbitrary.

Case: $x > b$. Then there are clearly rationals p and q where $b < p < x$ and $x < q$. Then clearly the set $B = [p, q)$ is a basis element that contains x and does not intersect A. This of course shows that $x \notin \overline{A}$ by Theorem 17.5 part (b) again, noting that this is true regardless of the rationality of b.

These cases taken together show the desired results. □

Main Problem.

First, it follows directly from Lemma 17.17.1 that that $\overline{A} = [0, \sqrt{2})$ and $\overline{B} = [\sqrt{2}, 3)$ in $\mathcal{T}_l$. It is worth noting that $\overline{A}$ and $\overline{B}$ are both basis elements of $\mathcal{T}_l$, which is interesting since they are closures and therefore closed. This of course implies that basis elements in $\mathcal{T}_l$ are both open and closed, which is indeed the case and is easy to see after a little thought.

It also follows directly from Lemma 17.17.2 that $\overline{A} = [0, \sqrt{2}]$ and $\overline{B} = [\sqrt{2}, 3)$ in $\mathcal{T}_c$ since $\sqrt{2}$ is irrational and 3 is rational.

Exercise 17.18

Determine the closures of the following subsets of the ordered square:

$$\begin{aligned}
A &= \{(1/n) \times 0 \mid n \in \mathbb{Z}_+\}\ , \\
B &= \left\{(1 - 1/n) \times \tfrac{1}{2} \mid n \in \mathbb{Z}_+\right\}\ , \\
C &= \{x \times 0 \mid 0 < x < 1\}\ , \\
D &= \left\{x \times \tfrac{1}{2} \mid 0 < x < 1\right\}\ , \\
E &= \left\{\tfrac{1}{2} \times y \mid 0 < y < 1\right\}\ .
\end{aligned}$$

Solution:

We assume that the ordered square refers to the set $X = [0, 1]^2$ with the dictionary order topology. Denote the dictionary order on X by $\prec$.

Definition 17.18.1. *For a topology on $\mathbb{R}$ and some subset $A \subset \mathbb{R}$, consider a point $x \in \mathbb{R}$. We say that x is a limit point of A from above if every neighborhood containing x also contains a point y where $y \in A$ and $x < y$. Similarly, a point x is a limit point of A from below if every neighborhood containing x also contains a point y where $y \in A$ and $y < x$.*

Note that a point can be a limit point from both below and above.

Lemma 17.18.2. *Suppose that A is a subset of the real interval $[0, 1]$ and that $B = \{x \times b \mid x \in A\}$ for some $b \in [0, 1]$ so that $B \subset X = [0, 1]^2$. Then the point $x \times y$ is a limit point of B in the dictionary order topology on the unit square if and only if either $y = 1$ and x is a limit point of A from above* or *$y = 0$ and x is a limit point of A from below in the order topology on $[0, 1]$.*

Proof. ($\Rightarrow$) We show this by contrapositive. So suppose that $y \neq 1$ or x is not a limit point of A from above *and* that $y \neq 0$ or x is not a limit point from below.

Case: $y \neq 0$ and $y \neq 1$. Clearly then $0 < y < 1$. If $y = b$ then the dictionary order interval $(x \times 0, x \times 1)$ is a basis element that contains $x \times y$ and that does not contain any other points of B, if indeed $x \in A$ so that $x \times y = x \times b$ is in B. If $y < b$ then the dictionary order interval $(x \times 0, x \times b)$ is a basis element with the same properties. Lastly, if $y > b$ then the dictionary order interval $(x \times b, x \times 1)$ is a basis element that contains $x \times y$ but no points of B.

Case: $y = 0$ or $y = 1$. If $y = 0$ then we have

Case: $x = 0$. Then, if $b = y = 0$, we have that the dictionary order interval $[0 \times 0, 0 \times 1)$ is a basis element containing $x \times y = 0 \times 0$ but no other points of B, if indeed $x = 0 \in A$ so that $x \times y \in B$. If $b \neq 0$ then $0 < b$ so that the interval $[0 \times 0, 0 \times b)$ is a basis element with the same properties.

Case: $x \neq 0$. Then $0 < x$ and it has to be that x is a not a limit point of A from below. Thus there is an interval (c, d) or $(c, 1]$ (or $[0, d)$ in which case let $c = 0$ in what follows) that contains x but no other points $y \in A$ where $y < x$. If $b = y = 0$ then it is easy to show that $(c \times 1, x \times 1)$ (or $(c \times 1, x \times 1]$ if $x = 1$) is a basis element that contains $x \times y$ but no other points of B, if indeed $x \in A$ so that $x \times y \in B$. If $b \neq y = 0$ then $0 < b$ so that $(c \times 1, x \times b)$ is a basis element with the same property.

If $y = 1$, then an analogous argument shows analogous results.

Thus in all cases and sub-cases it follows that $x \times y$ is not a limit point of B, which shows the desired result by contrapositive.

($\Leftarrow$) Now suppose that either $y = 1$ and x is a limit point of A from above or $y = 0$ and x is a limit point of A from below. In the first case consider any dictionary order interval $C = (a \times c, d \times e)$ that contains $x \times y$. Then it has to be that $x < d$ since otherwise it would have to be that $y = 1 < e$ since $x \times y \prec d \times e$, which is of course impossible. Then, since x is a limit point of A from above, it follows that the open set $[0, d)$ contains a point $z \in A$ where $x < z$ so that $x < z < d$. It then follows that the point $z \times b$ is in both C and B, and is of course distinct from $x \times y$ since $x < z$. The same argument can be made if C is a basis element in the form of $[0 \times 0, d \times e)$ or $(a \times c, 1 \times 1]$. This suffices to show that $x \times y$ is a limit point of B since C was an arbitrary basis element.

An analogous argument can be made in the case when $y = 0$ and x is a limit point of A from below, which shows the desired result. □

Main Problem.

First we claim that $\overline{A} = A \cup \{0 \times 1\}$.

Proof. First, let $K = \{1/n \mid n \in \mathbb{Z}_+\} \subset [0, 1]$ so that clearly $A = \{x \times 0 \mid x \in K\}$. It is easy to show that 0 is the only limit point of K and it is a limit point from above only. It then follows from Lemma 17.18.2 that 0×1 is the only limit point of A so that $\overline{A} = A \cup \{0 \times 1\}$ since the closure is the union of the set and the set of its limit points. □

Next we claim that $\overline{B} = B \cup \{1 \times 0\}$.

Proof. This time let $L = \{1 - 1/n \mid n \in \mathbb{Z}_+\}$ so that clearly $B = \left\{x \times \frac{1}{2} \mid x \in L\right\}$. It is trivial to show that 1 is the only limit point of L and that it is a limit point from below only. Hence 1×0 is the only limit point of B by Lemma 17.18.2 so that the result follows. □

Now we claim that $\overline{C} = C \cup \{1 \times 0\} \cup \{x \times 1 \mid 0 \leq x < 1\}$.

Proof. First, we clearly have that $C = \{x \times 0 \mid x \in (0, 1)\}$. It is easy to show that every point of $(0, 1)$ is a limit point both from above and below, that 0 is a limit point from above only, and that 1 is a limit point from below only. Thus it follows that the set of limit points of C are then $\{x \times 0 \mid 0 < x \leq 1\} \cup \{x \times 1 \mid 0 \leq x < 1\}$ by Lemma 17.18.2. As many of these points are contained in C itself, the result follows. □

We claim that $\overline{D} = D \cup \{x \times 0 \mid 0 < x \leq 1\} \cup \{x \times 1 \mid 0 \leq x < 1\}$.

Proof. The limit points of D are the same as for C above for the same reasons, i.e. $\{x \times 0 \mid 0 < x \leq 1\} \cup \{x \times 1 \mid 0 \leq x < 1\}$. The result then follows. $\square$

Lastly, we claim that $\overline{E} = \left\{\frac{1}{2} \times y \mid 0 \leq y \leq 1\right\} = \left\{\frac{1}{2}\right\} \times [0, 1]$, noting that clearly $E = \left\{\frac{1}{2}\right\} \times (0, 1)$.

Proof. Let $F = \left\{\frac{1}{2}\right\} \times [0, 1]$ so that we must show that $\overline{E} = F$.

($\subset$) Consider any $x \times y$ where $x \times y \notin F$ so that simply $x \neq \frac{1}{2}$ since it has to be that $y \in [0, 1]$. If $x < \frac{1}{2}$ then the basis element $[0 \times 0, \frac{1}{2} \times 0)$ clearly contains $x \times y$ but no elements of E. If $x > \frac{1}{2}$ then the basis element $(\frac{1}{2} \times 1, 1 \times 1]$ clearly contains $x \times y$ but no elements of E either. This shows that $x \times y$ is a not in $\overline{E}$ by Theorem 17.5 part (b). Hence $\overline{E} \subset F$ by contrapositive.

($\supset$) Consider any $x \times y \in F$ so that $x = \frac{1}{2}$ and $y \in [0, 1]$. If $y \in (0, 1)$ then $x \times y \in E$ so that $x \times y \in \overline{E}$ since obviously $E \subset \overline{E}$. If $y = 0$ then consider any dictionary order interval $F = (a \times c, b \times d)$ containing $x \times y = \frac{1}{2} \times 0$. In particular we have that $\frac{1}{2} \times 0 \prec b \times d$ so that either $\frac{1}{2} < b$, or $b = \frac{1}{2}$ and $0 < d$. In the first case we have that $\frac{1}{2} \times \frac{1}{2}$ is in both F and E. In the second case let $z = d/2$ so that we have $0 < z < d \leq 1$. Then clearly the point $\frac{1}{2} \times z$ is in F, but we also have that $\frac{1}{2} \times z$ is in E since $0 < z < 1$. The same argument applies if the basis element F is of the form $[0 \times 0, b \times d)$ or $(a \times c, 1 \times 1]$. A similar argument shows an analogous result in the case when $y = 1$. This shows by Theorem 17.5 part (b) that $x \times y \in \overline{E}$ since F was an arbitrary basis element, which of course shows that $\overline{E} \supset F$ since $x \times y$ was arbitrary. $\square$

Exercise 17.19

If $A \subset X$, we define the ***boundary*** of A by the equation

$$\operatorname{Bd} A = \overline{A} \cap \overline{(X - A)}\,.$$

(a) Show that $\operatorname{Int} A$ and $\operatorname{Bd} A$ are disjoint, and $\overline{A} = \operatorname{Int} A \cup \operatorname{Bd} A$.

(b) Show that $\operatorname{Bd} A = \varnothing \Leftrightarrow A$ is both open and closed.

(c) Show that U is open $\Leftrightarrow \operatorname{Bd} U = \overline{U} - U$.

(d) If U is open, is it true that $U = \operatorname{Int}(\overline{U})$? Justify your answer.

Solution:

(a)

Proof. Consider any $x \in \operatorname{Int} A$ so that there is a neighborhood of x that is entirely contained in A. Then, for any $y \in U$, we have that $y \in A$ and hence $y \notin X - A$. This shows that U does not intersect $X - A$, which suffices to show that x is not in the closure of $X - A$ by Theorem 17.5 part (a). Thus x is not in the boundary of A since $\operatorname{Bd} A = \overline{A} \cap \overline{(X - A)}$. This of course shows that $\operatorname{Int} A$ and $\operatorname{Bd} A$ are disjoint since x was arbitrary.

To show that $\overline{A} = \operatorname{Int} A \cup \operatorname{Bd} A$, first consider any $x \in \overline{A}$. If $x \in \operatorname{Int} A$ then clearly $x \in \operatorname{Int} A \cup \operatorname{Bd} A$, so assume that $x \notin \operatorname{Int} A$. Consider any neighborhood U of X. Then it has to be that U is not a subset of A since otherwise x would be in the union of open subsets of A and hence in the interior. It then follows that there is a point $y \in U$ where $y \notin A$ and therefore $y \in X - A$. This shows that U intersects $X - A$ so that x is in the closure of $X - A$ since U was an arbitrary neighborhood. Since also $x \in \overline{A}$, we have that $x \in \overline{A} \cap \overline{(X - A)} = \operatorname{Bd} A$. Hence clearly $x \in \operatorname{Int} A \cup \operatorname{Bd} A$ so that $\overline{A} \subset \operatorname{Int} A \cup \operatorname{Bd} A$ since x was arbitrary.

Now consider any $x \in \operatorname{Int} A \cup \operatorname{Bd} A$. If $x \in \operatorname{Int} A$ then also $x \in \overline{A}$ since we have that $\operatorname{Int} A \subset A \subset \overline{A}$. On the other hand, if $x \in \operatorname{Bd} A = \overline{A} \cap \overline{(X - A)}$ then of course $x \in \overline{A}$. This shows that $\operatorname{Int} A \cup \operatorname{Bd} A \subset \overline{A}$ in either case since x was arbitrary. Since both directions have been shown, it follows that $\overline{A} = \operatorname{Int} A \cup \operatorname{Bd} A$ as desired. □

(b)

Proof. ($\Rightarrow$) First suppose that $\operatorname{Bd} A = \varnothing$. Then by part (a) we have that $\overline{A} = \operatorname{Int} A \cup \operatorname{Bd} A = \operatorname{Int} A \cup \varnothing = \operatorname{Int} A$. Hence $A \subset \overline{A} = \operatorname{Int} A$ so that $A = \operatorname{Int} A$ since it is also always the case that $\operatorname{Int} A \subset A$. This shows that A is open since $\operatorname{Int} A$ is always open. We also have $\overline{A} = \operatorname{Int} A \subset A$ so that $A = \overline{A}$ since it is always also the case that $A \subset \overline{A}$. This of course shows that A is also closed since $\overline{A}$ is always closed.

($\Leftarrow$) Now suppose that A is both open and closed. It then follows that $\overline{A} = A = \operatorname{Int} A$. So consider any $x \in \overline{A}$ so that also $x \in \operatorname{Int} A$. Then there is a neighborhood U of x contained entirely in A. Thus, for any point $y \in U$, we have that $y \in A$ so that $y \notin X - A$, which shows that U does not intersect $X - A$. Since U is a neighborhood of x, this shows that $x \notin \overline{X - A}$ by Theorem 17.5 part (a). Then, since x was an arbitrary element of $\overline{A}$, it follows that $\overline{A}$ and $\overline{X - A}$ are disjoint so that $\operatorname{Bd} A = \overline{A} \cap \overline{(X - A)} = \varnothing$ as desired. □

(c)

Proof. ($\Rightarrow$) First suppose that U is open and consider any $x \in \operatorname{Bd} U$. Then we have that $x \in \overline{U}$ and $x \in \overline{X - U}$ since $\operatorname{Bd} U = \overline{U} \cap \overline{(X - U)}$ by definition. Suppose for the moment that $x \in U$ so that U itself is a neighborhood of x since it is open. For any $y \in U$ we have that $y \notin X - U$, and hence U does not intersect $X - U$. This shows that x is not in $\overline{X - U}$ by Theorem 17.5 part (a), which is a contradiction since we know it is. Thus it must be that $x \notin U$ so that $x \in \overline{U} - U$. This of course shows that $\operatorname{Bd} U \subset \overline{U} - U$ since x was arbitrary.

Now consider any $x \in \overline{U} - U$ so that clearly $x \in \overline{U}$. Since also $x \notin U$, it follows that $x \in X - U$ so that of course $x \in \overline{X - U}$ as well. Hence $x \in \overline{U} \cap \overline{(X - U)} = \operatorname{Bd} U$, which shows that $\overline{U} - U \subset \operatorname{Bd} U$ since x was arbitrary. This suffices to show that $\operatorname{Bd} U = \overline{U} - U$ as desired.

($\Leftarrow$) Now suppose that $\operatorname{Bd} U = \overline{U} - U$ and consider any $x \in U$. Then we have that $x \notin \overline{U} - U = \operatorname{Bd} U = \overline{U} \cap \overline{(X - U)}$. Since we know that $x \in \overline{U}$ (since $U \subset \overline{U}$), it must be that $x \notin \overline{X - U}$. Thus, by Theorem 17.5 part (a), there is a neighborhood of V of x that does not intersect $X - U$. This means that, for any point $y \in V$, we have that $y \notin X - U$. Since of course $y \in X$, it follows that y must be in U. This shows that $V \subset U$ since y was arbitrary. Hence V is a neighborhood of x that is entirely contained in U so that x is in the union of open sets contained in U, hence $x \in \operatorname{Int} U$. Since x was an arbitrary element of U, this shows that $U \subset \operatorname{Int} U$. As it is always the case that $\operatorname{Int} U \subset U$ as well, we have that $U = \operatorname{Int} U$ so that U is open since $\operatorname{Int} U$ is always open. □

(d) We claim that this is not generally true.

Proof. As a counterexample consider the set $U = \mathbb{R} - \{0\}$ in the finite complement topology on $\mathbb{R}$. Clearly U is open as its complement $\mathbb{R} - U = \{0\}$ is finite. It is also obvious that U is an infinite set.

Now consider any real number x and any neighborhood V of x. It cannot be that $\mathbb{R} - V$ is all of $\mathbb{R}$ since then V would be empty, and we know that $x \in V$. So it must be that $\mathbb{R} - V$ is finite since V is open, which means that there are only a finite number of real numbers that are *not* in V. However, since U is infinite, there must be an element of U that *is* in V (in fact there are an infinite number of such elements). Hence V intersects U so that $x \in \overline{U}$ by Theorem 17.5 part (a). Since $x \in \mathbb{R}$ was arbitrary, it must be that $\overline{U}$ is all of $\mathbb{R}$.

Clearly $\mathbb{R}$ is open (since the a set is always open in any topology on that set) so that $\text{Int}\,(\overline{U}) = \text{Int}\,\mathbb{R} = \mathbb{R}$. Then, since $0 \in \mathbb{R} = \text{Int}\,(\overline{U})$ but $0 \notin U$, we have that $U \neq \text{Int}\,(\overline{U})$. □

Exercise 17.20

Find the boundary and the interior of each of the following subsets of $\mathbb{R}^2$:

(a) $A = \{x \times y \mid y = 0\}$
(b) $B = \{x \times y \mid x > 0 \text{ and } y \neq 0\}$
(c) $C = A \cup B$
(d) $D = \{x \times y \mid x \text{ is rational}\}$
(e) $E = \{x \times y \mid 0 < x^2 - y^2 \leq 1\}$
(f) $F = \{x \times y \mid x \neq 0 \text{ and } y \leq 1/x\}$

Solution:

(a) It is easy to show that A is closed so that $\overline{A} = A$, and that also $\overline{\mathbb{R} - A} = A$ so that $\text{Bd}\,A = A$. It is also easy to see that no basis element and therefore no neighborhood of any point in A is contained entirely within A. From this it follows that $\text{Int}\,A = \varnothing$.

(b) It is easy to show that B is open so that $\text{Int}\,B = B$. It is likewise not difficult to prove that $\overline{B} = \{x \times y \mid x \geq 0\}$. We then have from Exercise 17.19 part (c) that $\text{Bd}\,B = \overline{B} - B = \{x \times y \mid x = 0\} \cup \{x \times y \mid x > 0 \text{ and } y = 0\}$.

(c) Here we have that $C = A \cup B = \{x \times y \mid y = 0\} \cup \{x \times y \mid x > 0\}$. It is then easy to show that the closure is $\overline{C} = \{x \times y \mid y = 0\} \cup \{x \times y \mid x \geq 0\}$. We also have that $\mathbb{R} - C = \{x \times y \mid x \leq 0 \text{ and } y \neq 0\}$ so that $\overline{\mathbb{R} - C} = \{x \times y \mid x \leq 0\}$. From these we clearly then have

$$\text{Bd}\,C = \overline{C} \cap \overline{(\mathbb{R} - C)} = \{x \times y \mid x < 0 \text{ and } y = 0\} \cup \{x \times y \mid x = 0\} .$$

It is also not difficult to show that $\text{Int}\,C = \{x \times y \mid x > 0\}$.

(d) Clearly we have that $\overline{D}$ is all of $\mathbb{R}^2$ as a consequence of the fact that the rationals are order-dense in the reals. Also, since any neighborhood of any point in D will intersect a point $x \times y$ with irrational x, it follows that no point of D is in its interior. Thus $\text{Int}\,D = \varnothing$ so that $\overline{D} = \text{Int}\,D \cup \text{Bd}\,D = \varnothing \cup \text{Bd}\,D = \text{Bd}\,D$ by Exercise 17.19 part (a), and hence $\text{Bd}\,D = \overline{D} = \mathbb{R}^2$.

(e) It should be fairly obvious by this point that

$$\text{Bd}\,E = \{x \times y \mid |y| = |x|\} \cup \{x \times y \mid x^2 - y^2 = 1\}$$

and $\text{Int}\,E = \{x \times y \mid 0 < x^2 - y^2 < 1\}$. This would be easy but tedious to prove rigorously.

(f) First we clearly have that $\text{Int}\,F = \{x \times y \mid x \neq 0 \text{ and } y < 1/x\}$. We also have that $\overline{F} = \{x \times y \mid x = 0\} \cup \{x \times y \mid x \neq 0 \text{ and } y \leq 1/x\}$. By Exercise 17.19 part (a) we have that $\overline{F} = \text{Int}\,F \cup \text{Bd}\,F$ and that $\text{Int}\,F \cap \text{Bd}\,F = \varnothing$ so that $\text{Bd}\,F = \overline{F} - \text{Int}\,F$. Thus we have that $\text{Bd}\,F = \{x \times y \mid x = 0\} \cup \{x \times y \mid x \neq 0 \text{ and } y = 1/x\}$. Again these facts are not difficult to show rigorously but would be tedious.

Exercise 17.21

(Kuratowski) Consider the collection of all subsets A of the topological space X. The operations of closure $A \to \overline{A}$ and complementation $A \to X - A$ are functions from the collection to itself.

(a) Show that starting with a given set A, one can form no more than 14 distinct sets by applying these two operations successively.

(b) Find a subset A of $\mathbb{R}$ (in its usual topology) for which the maximum of 14 is obtained.

Solution:

For the following we introduce the following notation to make things simpler. If A is a subset of a topological space X then denote

$$cA = \overline{A} \qquad xA = X - C$$
$$iA = \operatorname{Int} A \qquad bA = \operatorname{Bd} A\,.$$

We can consider these (c, x, i, and b) as operators on sets that can be chained together in the obvious way so that, for example, $cxiA = \overline{X - \operatorname{Int} A}$.

Lemma 17.21.1. *For a subset A of topological space X, $X = cA \cup ixA$, and cA and ixA are disjoint*

Proof. First, it is obvious that $cA \cup ixA \subset X$ since each of the sets in the union is a subset of X. Now consider any $x \in X$ and suppose that $x \notin cA = \overline{A}$. Then by Lemma 17.5 part (a) there is an open set U containing x where U does not intersect A. For any $y \in U$ we thus have that $y \notin A$ and hence $y \in X - A = xA$. This shows that $U \subset xA$ since y was arbitrary, which suffices to show that $x \in \operatorname{Int}(xA) = ixA$ since U is a neighborhood of x. This of course shows that $x \in cA \cup ixA$ so that $X \subset cA \cup ixA$ since x was arbitrary. This completes the proof that $X = cA \cup ixA$.

To show that cA and ixA are disjoint, consider any $x \in cA$. Consider any neighborhood U of x so that U intersects A by Lemma 17.5 part (a). Hence there is a point $y \in U$ where also $y \in A$, from which it follows that $y \notin X - A = xA$. This suffices to show that U is not a subset of xA. Since U is an arbitrary neighborhood, this shows that $x \notin \operatorname{Int}(xA) = ixA$. This of course shows that cA and ixA are disjoint as desired. □

Lemma 17.21.2. *For a subsets A and B of topological space X where $A \subset B$, we have the following:*

(a) $cA \subset cB$

(b) $iA \subset iB$

(c) $ccA = cA$

(d) $iiA = iA$

(e) $xxA = A$

(f) $xcA = ixA$

(g) $xiA = cxA$

(h) $icicA = icA$

(i) $ciciA = ciA$.

Proof. (a) This was shown in Exercise 17.6 part (a).

(b) Consider any $x \in iA$ so that there is a neighborhood U of x that is totally contained in A. Then clearly U is also totally contained in B as well since, for any $x \in U$, we have that $x \in A$ and hence $x \in B$ since $A \subset B$. This shows that $x \in iB$ since U is a neighborhood of x. Hence $iA \subset iB$ since x was arbitrary.

(c) Since $cA = \overline{A}$ is closed, we clearly have $ccA = cA$.

(d) Since $iA = \operatorname{Int} A$ is open, its interior is itself, i.e. $iiA = iA$.

(e) Obviously $xxA = X - (X - A) = A$ since $A \subset X$.

(f) We have by Lemma 17.21.1 that $X = cA \cup ixA$ where cA, and ixA are mutually disjoint. From this it follows that $ixA = X - cA = xcA$.

(g) We have

$$\begin{aligned} cxA &= xxcxA && \text{(by (e))} \\ &= xixxA && \text{(by (f))} \end{aligned}$$

$$= xiA \qquad \text{(by (e) again)}$$

as desired.

(h) First we have that $icA = iicA$ by (d). Also clearly $icA = c(icA) = cicA$ since a set is always a subset of its closure. Hence by (b) we have that $icA = iicA = i(icA) \subset i(cicA) = icicA$. Now, we also have that $icA = i(cA) \subset cA$ since the interior of a set is always a subset of the set. Hence by (a) and (c) we have $cicA = c(icA) \subset c(cA) = ccA = cA$. It then follows from (b) that $icicA = i(cicA) \subset i(cA) = icA$ as well. This of course shows that $icicA = icA$ as desired.

(i) Lastly, we have

$$\begin{aligned}
ciciA &= cicixxA && \text{(by (e))} \\
&= cicxcxA && \text{(by (f))} \\
&= cixicxA && \text{(by (g))} \\
&= cxcicxA && \text{(by (f))} \\
&= xicicxA && \text{(by (g))} \\
&= xicxA && \text{(by (h))} \\
&= cxcxA && \text{(by (g))} \\
&= cixxA && \text{(by (f))} \\
&= ciA && \text{(by (e))}
\end{aligned}$$

as desired. □

Main Problem.

(a)

Proof. We are interested in sequences applying the operators c and x to a subset A. By Lemma 17.21.2 (c) and (e) we have that $ccA = cA$ and $xxA = A$. Thus there is no point in ever applying c or x twice in a row since that would clearly result in a set that we have seen before. We are then interested only in sequences that apply alternating c and x. If we apply the closure c first, we obtain the following sequence:

$$\begin{aligned}
A &= A \\
cA &= cA \\
xcA &= ixA && \text{(by Lemma 17.21.2f)} \\
cxcA &= cixA && \text{(previous result)} \\
xcxcA &= xcixA && \text{(previous result)} \\
&= ixixA && \text{(by Lemma 17.21.2f)} \\
&= icxxA && \text{(by Lemma 17.21.2g)} \\
&= icA && \text{(by Lemma 17.21.2e)} \\
cxcxcA &= cicA && \text{(previous result)} \\
xcxcxcA &= xcicA && \text{(previous result)} \\
&= ixicA && \text{(by Lemma 17.21.2f)} \\
&= icxcA && \text{(by Lemma 17.21.2g)} \\
&= icixA && \text{(by Lemma 17.21.2f)}
\end{aligned}$$

If we apply the next operation we obtain

$$cxcxcxcA = ciciixA \qquad \text{(previous result)}$$

$$= cixA\,, \qquad \text{(by Lemma 17.21.2i)}$$

which is the same as the fourth set above. Therefore we can get at most 7 distinct sets by applying c first, including A itself. If we instead apply x first then we get the following sequence:

$$\begin{aligned}
xA &= xA \\
cxA &= cxA \\
xcxA &= ixxA && \text{(corresponding result above)} \\
&= iA && \text{(by Lemma 17.21.2e)} \\
cxcxA &= ciA && \text{(previous result)} \\
xcxcxA &= xciA && \text{(previous result)} \\
&= ixiA && \text{(by Lemma 17.21.2f)} \\
&= icxA && \text{(by Lemma 17.21.2g)} \\
cxcxcxA &= cicxA && \text{(previous result)} \\
xcxcxcxA &= xcicxA && \text{(previous result)} \\
&= ixicxA && \text{(by Lemma 17.21.2f)} \\
&= icxcxA && \text{(by Lemma 17.21.2g)} \\
&= icixxA && \text{(by Lemma 17.21.2f)} \\
&= iciA && \text{(by Lemma 17.21.2e)}
\end{aligned}$$

Again if we try to apply the next operation we get

$$\begin{aligned}
cxcxcxcxA &= ciciA && \text{(previous result)} \\
&= ciA && \text{(by Lemma 17.21.2i)}
\end{aligned}$$

which as before is the same as the fourth set in the sequence. Hence we have at most 7 distinct sets in this sequence for a total of 14 potentially distinct sets as desired. □

Note that this only shows that there can be *no more than* 14 distinct sets. It could be that there are always less than 14 in general. While there are certainly sets that generate less than 14 distinct sets, the next part shows the existence of a topology and a set that does result in 14 distinct sets. This of course shows that 14 is the lowest possible bound in general.

(b) We claim that $A = (-3,-2) \cup (-2,-1) \cup ([0,1] \cap \mathbb{Q}) \cup \{2\}$ in the standard topology on $\mathbb{R}$ is a set that results in 14 distinct sets when the operational sequences from part (a) are applied. We do not prove each sequential operation as this is easy but would be prohibitively tedious. First we enumerate the first sequence, starting with A.

Operations	Set
A	$(-3,-2) \cup (-2,-1) \cup ([0,1] \cap \mathbb{Q}) \cup \{2\}$
cA	$[-3,-1] \cup [0,1] \cup \{2\}$
$xcA = ixA$	$(-\infty,-3) \cup (-1,0) \cup (1,2) \cup (2,\infty)$
$cxcA = cixA$	$(-\infty,-3] \cup [-1,0] \cup [1,\infty)$
$xcxcA = icA$	$(-3,-1) \cup (0,1)$
$cxcxcA = cicA$	$[-3,-1] \cup [0,1]$
$xcxcxcA = icixA$	$(-\infty,-3) \cup (-1,0) \cup (1,\infty)$

Next we enumerate the next sequence of 7 sets, starting with xA:

Operations	Set
xA	$(-\infty,-3]\cup\{-2\}\cup[-1,0)\cup((0,1)-\mathbb{Q})\cup(1,2)\cup(2,\infty)$
cxA	$(-\infty-3]\cup\{-2\}\cup[-1,\infty)$
$xcxA=iA$	$(-3,-2)\cup(-2,-1)$
$cxcxA=ciA$	$[-3,-1]$
$xcxcxA=icxA$	$(-\infty,-3)\cup(-1,\infty)$
$cxcxcxA=cicxA$	$(-\infty,-3]\cup[-1,\infty)$
$xcxcxcxA=iciA$	$(-3,1)$

It is easy to see that these are 14 distinct sets.

We do note that, in an interval containing only rationals (or only irrationals), such as $[0,1]\cap\mathbb{Q}$ used as part of A, clearly every point in the interval is a limit point, including any irrational (or rational) points. This is because any open interval containing any real always contains both rationals and irrationals on account of $\mathbb{Q}$ being order-dense in $\mathbb{R}$. For the same reason no point of such an interval of rationals (or irrationals) is in its interior. If, for example $C=[0,1]\cap\mathbb{Q}$, this clearly then results in $cC=[0,1]$ and $iC=\varnothing$. Indeed this property of this part of A is crucial in its success in generating 14 distinct sets.

§18 Continuous Functions

Exercise 18.1

Prove that for functions $f:\mathbb{R}\to\mathbb{R}$, the ϵ-δ definition of continuity implies the open set definition.

Solution:

Recall that that $f:\mathbb{R}\to\mathbb{R}$ is continuous at a point $x\in\mathbb{R}$ if, for every real $\epsilon>0$, there is a real $\delta>0$ such that $|f(y)-f(x)|<\epsilon$ for every real y where $|y-x|<\delta$. We say that f itself is continuous if it is continuous at every $p\in\mathbb{R}$.

Proof. Suppose that $f:\mathbb{R}\to\mathbb{R}$ is continuous by the ϵ-δ definition above. We show that this implies the open set definition by showing that f satisfies (4) in Theorem 18.1. So consider any $x\in\mathbb{R}$ and any neighborhood V of $f(x)$. Then of course there is a basis element (c,d) containing $f(x)$ such that $(c,d)\subset V$. Let $\epsilon=\min(f(x)-c,d-f(x))$, noting that $\epsilon>0$ since $c<f(x)<d$. It is then trivial to show that $(f(x)-\epsilon,f(x)+\epsilon)\subset(c,d)\subset V$ and contains x.

Then, since f is continuous at x, there is $\delta>0$ such that $|y-x|<\delta$ implies that $|f(y)-f(x)|<\epsilon$ for any real y. Let $U=(x-\delta,x+\delta)$, which is clearly a neighborhood of x. Now consider any $z\in f(U)$ so that $z=f(y)$ for some $y\in U$. Then we have that $x-\delta<y<x+\delta$ so that clearly $-\delta<y-x<\delta$, from which it follows that $|y-x|<\delta$. We then know that $|z-f(x)|=|f(y)-f(x)|<\epsilon$ since f is continuous. Hence $-\epsilon<z-f(x)<\epsilon$ so that $f(x)-\epsilon<z<f(x)+\epsilon$, and thus $z\in V$ since $(f(x)-\epsilon,f(x)+\epsilon)\subset V$. Since $z\in f(U)$ was arbitrary, this shows that $f(U)\subset V$, which shows that (4) holds for f since x was also arbitrary. □

Exercise 18.2

Suppose that $f:X\to Y$ is continuous. If x is a limit point of the subset A of X, is it necessarily true the $f(x)$ is a limit point of $f(A)$?

Solution:

This is not necessarily true.

Proof. As a counterexample consider a constant function $f : X \to Y$ defined by $f(x) = y_0$ for any $x \in X$ and some $y_0 \in Y$. It was shown in Theorem 18.2 part (a) that this is continuous. However, clearly $f(A) = \{y_0\}$ for any subset A of X. So even if x is a limit point of A, no neighborhood of $f(x)$ can intersect $f(A)$ in a point other than $f(x) = y_0$ since y_0 is the only point in $f(A)$! Therefore $f(x)$ is not a limit point of $f(A)$. □

Exercise 18.3

Let X and X' denote a single set in two topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively. Let $i : X' \to X$ be the identity function.

(a) Show that i is continuous $\Leftrightarrow \mathcal{T}'$ is finer than $\mathcal{T}$.

(b) Show that i is a homeomorphism $\Leftrightarrow \mathcal{T}' = \mathcal{T}$.

Solution:

(a)

Proof. First note that clearly the inverse of the identity function is itself with the domain and image reversed, and that for any subset $A \subset X = X'$ we have $i(A) = i^{-1}(A) = A$.

($\Rightarrow$) Suppose that i is continuous and consider any open set $U \in \mathcal{T}$. Then we have that $i^{-1}(U) = U$ is open in $\mathcal{T}'$ since i is continuous. Since U was arbitrary, this shows that $\mathcal{T} \subset \mathcal{T}'$ so that $\mathcal{T}'$ is finer.

($\Leftarrow$) Now suppose that $\mathcal{T}'$ is finer so that $\mathcal{T} \subset \mathcal{T}'$. Consider any open set $U \in \mathcal{T}$ so that also clearly $U \in \mathcal{T}'$, i.e. U is also open in $\mathcal{T}'$. Since $i^{-1}(U) = U$, this shows that i is continuous by the definition of continuity. □

(b)

Proof. Clearly i is a bijection since its domain and image are the same set, and $i^{-1} = i$. We then have that

$$\begin{aligned} i \text{ is a homeomorphism } &\Leftrightarrow i \text{ and } i^{-1} \text{ are both continuous} \\ &\Leftrightarrow \mathcal{T}' \text{ is finer than } \mathcal{T} \text{ and } \mathcal{T} \text{ is finer than } \mathcal{T}' \quad \text{(by part (a) applied twice)} \\ &\Leftrightarrow \mathcal{T} \subset \mathcal{T}' \text{ and } \mathcal{T}' \subset \mathcal{T} \\ &\Leftrightarrow \mathcal{T}' = \mathcal{T} \end{aligned}$$

as desired. □

Exercise 18.4

Given $x_0 \in X$ and $y_0 \in Y$, show that the maps $f : X \to X \times Y$ and $g : Y \to X \times Y$ defined by

$$f(x) = x \times y_0 \qquad \text{and} \qquad g(y) = x_0 \times y$$

are imbeddings.

Solution:

We only show that f is an imbedding of X in $X \times Y$ as the argument for g is entirely analogous.

Proof. First, it is easy to see and trivial to formally show that f is injective. The function f can be of course be defined as $f(x) = f_1(x) \times f_2(x)$ where $f_1 : X \to X$ is the identity function and $f_2 : X \to Y$ is the constant function that maps every element of X to y_0. Since these have both been proven to be continuous in the text, it follows that f is continuous by Theorem 18.4.

Now let f' be the function obtained by restricting the range of f to $f(X) = \{x \times y_0 \mid x \in X\}$. Since f is injective, it follows that f' is a bijection. It follows from Theorem 18.2 part (e) that f' is continuous. Clearly the inverse function f'^{-1} is equal to the projection function π_1 so that $f'^{-1}(x, y) = x$. This was shown to be continuous in the proof of Theorem 18.4. This suffices to show that f' is a homeomorphism, which shows the f is an imbedding of X in $X \times Y$. ☐

Exercise 18.5

Show that the subspace (a, b) of $\mathbb{R}$ is homeomorphic with $(0, 1)$ and the subspace $[a, b]$ of $\mathbb{R}$ is homeomorphic with $[0, 1]$.

Solution:

First we show that (a, b) is homeomorphic to $(0, 1)$.

Proof. First let $X = (a, b)$ and $Y = (0, 1)$, and define the map $f : X \to Y$ by

$$f(x) = \frac{x - a}{b - a}$$

for any $x \in X$, noting that this is defined since $a < b$ so that $b - a > 0$. It is trivial to show that f is a bijection.

Now, f is a linear function that could just as well be defined as a map from $\mathbb{R}$ to $\mathbb{R}$, and clearly this would be continuous by basic calculus. It then follows from Theorem 18.2 part (d) that restricting its domain to X means that it is still continuous. We also clearly have from basic algebra that its inverse is the function $f^{-1} : Y \to X$ defined by

$$f^{-1}(y) = a + y(b - a)$$

for $y \in Y$. As this is also linear, it too is continuous by the same argument. This suffices to show that f is a homeomorphism. ☐

The exact same argument shows that $[a, b]$ is homeomorphic to $[0, 1]$ by simply setting $X = [a, b]$ and $Y = [0, 1]$ in the above proof. It is assumed that here again $a < b$ even though the interval $[a, b]$ is valid if $a = b$ and simply becomes $[a, b] = [a, a] = \{a\}$. However, clearly this set cannot be homeomorphic to $[0, 1]$ since it is finite whereas $[0, 1]$ is uncountable.

Exercise 18.6

Find a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous at precisely one point.

Solution:

For any real x define

$$f(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ x & x \notin \mathbb{Q}\,. \end{cases}$$

We claim that this is continuous only at $x = 0$.

Proof. As it is easier to do so, we show this using the ϵ-δ definition of continuity, which we know implies the open set definition by Exercise 18.1. First we note that $f(0) = 0$ since 0 is rational. Now consider any $\epsilon > 0$ and let $\delta = \epsilon$. Suppose real y where $|y - 0| = |y| < \delta$. If y is rational then $y = 0$ so that $|f(y) - f(0)| = |0 - 0| = |0| = 0 < \epsilon$. If y is irrational then $|f(y) - f(0)| = |y - 0| = |y| < \delta = \epsilon$ again. Since ϵ was arbitrary this shows that f is continuous at $x = 0$.

Now consider any $x \neq 0$. Let $\epsilon = |x|/2$, noting that $\epsilon > 0$ since $x \neq 0$ so that $|x| > 0$. Now consider any $\delta > 0$.

Case: $x \in \mathbb{Q}$. Then $f(x) = 0$ but there is clearly an irrational y close enough to x so that $|y - x| < \min(\epsilon, \delta)$, and hence both $|y - x| < \epsilon$ and $|y - x| < \delta$. We also have that $f(y) = y$. We then have that

$$2\epsilon = |x| = |x - 0| \leq |x - y| + |y - 0| < \epsilon + |y|$$

so that

$$\epsilon < |y| = |f(y)| = |f(y) - 0| = |f(y) - f(x)|\,.$$

Case: $x \notin \mathbb{Q}$. Then $f(x) = x$, and there is clearly a rational y close enough to x that $|y - x| < \delta$. We then also have $f(y) = 0$ so that

$$|f(y) - f(x)| = |0 - x| = |x| = 2\epsilon > \epsilon$$

since $\epsilon > 0$.

Hence in either case there is a y such that $|y - x| < \delta$ but $|f(y) - f(x)| \geq \epsilon$. This suffices to show that f is not continuous at x. □

Exercise 18.7

(a) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is "continuous from the right," that is

$$\lim_{x \to a^+} f(x) = f(a)\,,$$

for each $a \in \mathbb{R}$. Show that f is continuous when considered as a function from $\mathbb{R}_\ell$ to $\mathbb{R}$.

(b) Can you conjecture what functions $f : \mathbb{R} \to \mathbb{R}$ are continuous when considered as maps from $\mathbb{R}$ to $\mathbb{R}_\ell$? We shall return to this question in Chapter 3.

Solution:

Lemma 18.7.1. *In the topology $\mathbb{R}_\ell$, every basis element is both open and closed.*

Proof. Consider any basis element $B = [a, b)$, which is clearly open since basis elements are always open. We then have that the complement of this set is $C = \mathbb{R} - B = (-\infty, a) \cup [b, \infty)$. We claim that this complement is also open so that B is closed by definition. To see this, define the sets $C_n = [a-n-1, a-n+1) \cup [b+n-1, b+n+1)$ for $n \in \mathbb{Z}_+$. Clearly each C_n is open since it is the union of two basis elements. It is also trivial to show that $C = \bigcup_{n\in\mathbb{Z}_+} C_n$, which is then also open since it is a union of open sets. □

Lemma 18.7.2. *The only open sets in the standard topology on $\mathbb{R}$ that are both open and closed are $\varnothing$ and $\mathbb{R}$ itself.*

Proof. First, clearly both $\varnothing$ and $\mathbb{R}$ are both open and closed since they are compliments of each other and are both open by the definition of a topology. Now suppose that U is a nonempty subset of $\mathbb{R}$ that is both open and closed. Suppose also that $U \neq \mathbb{R}$ so that $U \subsetneq \mathbb{R}$ and hence there is a $y \in \mathbb{R}$ where $y \notin U$. We show that the existence of such a U results in a contradiction, which of course shows the desired result since it implies that $U = \mathbb{R}$ if $U \neq \varnothing$. Since U is nonempty we have that there is an $x \in U$ and it must be that $x \neq y$ since $x \in U$ but $y \notin U$.

If $x < y$ then define the set $A = \{z > x \mid z \notin U\}$. Clearly we have that A is nonempty since $y \in A$, and that x is a lower bound of A. It then follows that A has a largest lower bound a since this is a fundamental property of $\mathbb{R}$. It could be that $a \in U$ or $a \notin U$. In the former case we have that any basis element (c, d) containing a is not a subset of U. To see this, we have that $c < a < d$, which means that d is not a lower bound of A since a is the largest lower bound. Hence there is a $z \in A$ where $d > z$. We then have $c < a \leq z < d$ (noting that $a \leq z$ since a is a lower bound of A) so $z \in (c, d)$ and $z \in A$ so that $z \notin U$. Hence (c, d) is not a subset of U, which contradicts the fact that U is open since the basis element (c, d) was arbitrary.

In the latter case where $a \notin U$ then it has to be that $x < a$ since x is a lower bound of A and a is the largest lower bound (and it cannot be that $a = x$ since $x \in U$ but $a \notin U$). We clearly have that $a \in \mathbb{R} - U$, which is open since U is closed. Now consider any basis element (c, d) containing a so that $c < a < d$. Let $b = \max(x, c)$ so that $b < a$ and hence there is a real z where $c \leq b < z < a < d$ and hence $z \in (c, d)$. Now, since $z < a$ it has to be that $z \notin A$ since otherwise a would not be a lower bound of A. We also have that $x \leq b < z$ so that it has to be that $z \in U$ since otherwise it would be that $z \in A$. Thus $z \notin \mathbb{R} - U$, which shows that (c, d) is not a subset of $\mathbb{R} - U$ since $z \in (c, d)$. Since (c, d) was an arbitrary basis element, this contradicts the fact that $\mathbb{R} - U$ is open.

It was thus shown that in either case a contradiction arises. Analogous arguments also show contradictions when $x > y$, this time using the set $A = \{z < x \mid z \notin U\}$ and its least upper bound. Hence it has to be that $U = \mathbb{R}$, which shows the desired result. □

Main Problem.

(a) Recall that by the definition of the one-sided limit, $f : \mathbb{R} \to \mathbb{R}$ is continuous from the right if, for every $a \in \mathbb{R}$ and every $\epsilon > 0$, there is a $\delta > 0$ such that $|f(x) - f(a)| < \epsilon$ for every $x > a$ where $|x - a| < \delta$.

Proof. So suppose that f is continuous from the right and consider any $a \in \mathbb{R}$. Let V be neighborhood of $f(a)$ in $\mathbb{R}$. Then there is a basis element (c, d) of $\mathbb{R}$ that contains $f(a)$ and is a subset of V. Hence $c < f(a) < d$, so let $\epsilon = \min[f(a) - c, d - f(a)]$ so that clearly $\epsilon > 0$ and if $|y - f(a)| < \epsilon$, then $y \in (c, d)$ so that also $y \in V$. Now, since f is continuous from the right, there is a $\delta > 0$ such that $|f(x) - f(a)| < \epsilon$ for every $x > a$ where $|x - a| < \delta$. So let $U = [a, a + \delta)$ which is clearly a basis element of $\mathbb{R}_\ell$ and contains a so that it is a neighborhood of a.

Now consider any $y \in f(U)$ so that there is an $x \in U$ where $y = f(x)$. If $x = a$ then clearly $|f(x) - f(a)| = |f(a) - f(a)| = |0| = 0 < \epsilon$ so that $f(x) \in V$. If $x \neq a$ then it has to be that $x > a$ and also that $|x - a| = x - a < \delta$ since $U = [a, a + \delta)$. It then follows that $|f(x) - f(a)| < \epsilon$ so that again $f(x) \in V$. Hence in both cases $y = f(x) \in V$, which shows that $f(U) \subset V$ since y was arbitrary. We have thus shown part (4) of Theorem 18.1, from which the topological continuity of f follows. □

(b) We claim that only constant functions are continuous from $\mathbb{R}$ to $\mathbb{R}_\ell$.

Proof. First, it was shown in Theorem 18.2 part (a) that constant functions are always continuous regardless of the topologies. Hence we must show that any continuous function from $\mathbb{R}$ to $\mathbb{R}_\ell$ is constant. So suppose that f is such a function. Now consider any real x where $x \neq 0$. Clearly if $f(x) = f(0)$ then f is a constant function since x was arbitrary. So suppose that this is not the case so that $f(x) \neq f(0)$. Without loss of generality we can assume that $f(0) < f(x)$. So consider the basis element $B = [f(0), f(x))$ of $\mathbb{R}_\ell$, which clearly contains $f(0)$ but not $f(x)$.

Since f is continuous and B is both open and closed by Lemma 18.7.1, it follows from the definition of continuity and from Theorem 18.1 part (3) that $f^{-1}(B)$ must be both open and closed in $\mathbb{R}$. However the only sets that are both open and closed in $\mathbb{R}$ are $\varnothing$ and $\mathbb{R}$ itself by Lemma 18.7.2. Thus either $f^{-1}(B) = \varnothing$ or $f^{-1}(B) = \mathbb{R}$. It cannot be that $f^{-1}(B) = \varnothing$ since we have that $f(0) \in B$ so that $0 \in f^{-1}(B)$. Hence it must be that $f^{-1}(B) = \mathbb{R}$, but then we would have $x \in f^{-1}(B)$ so that $f(x) \in B$, which we know it not the case. We therefore have a contradiction so that it must be that $f(x) = f(0)$ so that f is constant. □

Lastly, we claim that the only functions that are continuous from $\mathbb{R}_\ell$ to $\mathbb{R}_\ell$ are those that are *continuous and non-decreasing from the right*. For a function $f : \mathbb{R} \to \mathbb{R}$ this means that for every $x \in \mathbb{R}$ and every $\epsilon > 0$ there is a $\delta > 0$ such that $|f(y) - f(x)| < \epsilon$ and $f(y) \geq f(x)$ for every $x \leq y < x + \delta$.

Proof. First we show that such functions are in fact continuous. So suppose that f is continuous and non-decreasing from the right and consider any real x. Let V be any neighborhood of $f(x)$ so that there is a basis element $B = [c, d)$ containing $f(x)$ such that $B \subset V$. Let $\epsilon = d - f(x)$ so that $\epsilon > 0$ since $f(x) < d$. Hence there is a $\delta > 0$ such that $x < y < x + \delta$ implies that $|f(y) - f(x)| < \epsilon$ and $f(y) \geq f(x)$. We then have that $U = [x, x + \delta)$ is a basis element and therefore an open set of $\mathbb{R}_\ell$ that contains x. Consider any $z \in f(U)$ so that $z = f(y)$ for some $y \in U$. Then $x \leq y < x + \delta$ so that $z = f(y) \geq f(x)$ and $|z - f(x)| = |f(y) - f(x)| < \epsilon$. It then follows that $0 \leq z - f(x) < \epsilon$ so that $c \leq f(x) \leq z < f(x) + \epsilon = d$, and hence $z \in [c, d) = B$. Thus also $z \in V$ since $B \subset V$. This shows that $f(U) \subset V$ since z was arbitrary, and hence that f is continuous by Theorem 18.1.

Now we show that a continuous function *must* be continuous and non-decreasing from the right by showing the contrapositive. So suppose that f is not continuous and non-decreasing from the right. Then there exists a real x and an $\epsilon > 0$ such that, for any $\delta > 0$, there is a $x \leq y < x + \delta$ where $f(y) < f(x)$ or $|f(y) - f(x)| \geq \epsilon$. Clearly we have that $V = [f(x), f(x) + \epsilon)$ is basis element and therefore open set of $\mathbb{R}_\ell$ that contains $f(x)$. Consider any neighborhood U of x so that there is a basis element $B = [a, b)$ containing x where $B \subset U$. Then $x < b$ so that $\delta = b - x > 0$. It then follows that there is a $x \leq y < x + \delta = b$ such that $f(y) < f(x)$ or $|f(y) - f(x)| \geq \epsilon$. Clearly we have that $y \in B$ so that also $y \in U$ and $f(y) \in f(U)$. However, if $f(y) < f(x)$ then clearly $f(y) \notin V$. On the other hand if $f(y) \geq f(x)$ then it has to be that $|f(y) - f(x)| \geq \epsilon$. Then we have that $f(y) - f(x) \geq 0$ so that $f(y) - f(x) = |f(y) - f(x)| \geq \epsilon$, and hence $f(y) \geq f(x) + \epsilon$ so that again $f(y) \notin V$. This suffices to show that $f(U)$ is not a subset of V, which shows that f is not continuous by Theorem 18.1 since U was an arbitrary neighborhood of x. □

Exercise 18.8

Let Y be an ordered set in the order topology. Let $f, g : X \to Y$ be continuous.

(a) Show that the set $\{x \mid f(x) \leq g(x)\}$ is closed in X.

(b) Let $h : X \to Y$ be the function

$$h(x) = \min \{f(x), g(x)\} .$$

Show that h is continuous. [Hint: Use the pasting lemma.]

Solution:

(a) First let $C = \{x \in X \mid f(x) \leq g(x)\}$ so that we must show that C is closed in X.

Proof. We prove this by showing that the complement $X - C$ is open in X. So first let S be the set of all $y \in Y$ where y has an immediate successor, and denote that successor by $y + 1$. Then clearly $y + 1$ is well defined for all $y \in S$. Now define

$$A_{>y} = \{z \in Y \mid z > y\} \qquad\qquad A_{<y} = \{z \in Y \mid z < y + 1\}$$

for $y \in S$. As these are both rays in the order topology Y, they are both basis elements and therefore open. It then follows that $f^{-1}(A_{>y})$ and $g^{-1}(A_{<y})$ are both open in X since f and g are continuous. Hence their intersection $U_y = f^{-1}(A_{>y}) \cap g^{-1}(A_{<y})$ is also open in X.

Similarly the rays

$$B_{>y} = \{z \in Y \mid z > y\} \qquad\qquad B_{<y} = \{z \in Y \mid z < y\}$$

for $y \in Y$ are also open so that the intersection $V_y = f^{-1}(B_{>y}) \cap g^{-1}(B_{<y})$ is open in X. Then clearly the union of unions

$$D = \bigcup_{y \in S} U_y \cup \bigcup_{y \in Y} V_y$$

is also open in X. We claim that $X - C = D$ so that the complement is open in X and hence C is closed as desired.

($\subset$) First consider any $x \in X - C$ so that clearly $f(x) > g(x)$. If $g(x)$ has an immediate successor $g(x) + 1$ then $g(x) \in S$ and we have $f(x) \in A_{>g(x)}$ so that $x \in f^{-1}(A_{>g(x)})$. We also have that $g(x) \in A_{<g(x)}$ since $g(x) < g(x) + 1$, and hence $x \in g^{-1}(A_{<g(x)})$. It then follows that $x \in U_{g(x)}$ and hence $\bigcup_{y \in S} U_y$ and $x \in D$ since $g(x) \in S$. If $g(x)$ does not have an immediate successor then there must be a $y \in Y$ where $g(x) < y < f(x)$. We then have that clearly $f(x) \in B_{>y}$ and $g(x) \in B_{<y}$ so that $x \in f^{-1}(B_{>y})$ and $x \in g^{-1}(B_{<y})$. Thus $x \in V_y$ so that $x \in \bigcup_{y \in Y} V_y$ and $x \in D$. This shows that $X - C \subset D$ since either way $x \in D$ and x was arbitrary.

($\supset$) Now suppose that $x \in D$. If $x \in \bigcup_{y \in S} U_y$ when there is a $y \in S$ where $x \in U_y$. Hence $x \in f^{-1}(A_{>y})$ and $x \in g^{-1}(A_{<y})$ so that $f(x) \in A_{>y}$ and $g(x) \in A_{<y}$. From this it follows that $f(x) > y$ and $g(x) < y + 1$. Then it has to be that $g(x) \leq y$ so that $f(x) > y \geq g(x)$. If $x \in \bigcup_{y \in Y} V_y$ then there is a $y \in Y$ where $x \in V_y$. Hence $x \in f^{-1}(B_{>y})$ and $x \in g^{-1}(B_{<y})$ so that $f(x) \in B_{>y}$ and $g(x) \in B_{<y}$. It then clearly follows that $f(x) > y$ and $g(x) < y$ so that $f(x) > y > g(x)$. Therefore in either case we have $f(x) > g(x)$ so that $x \in X - C$. This of course shows that $X - C \supset D$ since x was arbitrary. $\square$

(b)

Proof. Let $A = \{x \in X \mid f(x) \leq g(x)\}$ and $B = \{x \in X \mid g(x) \leq f(x)\}$, which are clearly both closed by part (a). It is easy to see that $X = A \cup B$. First, clearly $X \supset A \cup B$ since both $A \subset X$ and $B \subset X$. Then, for any $x \in X$, it has to be that either $f(x) \leq g(x)$ or $f(x) > g(x)$ since $<$ is a total order on Y. In the former case of course $x \in A$, and in the latter $x \in B$ so that either way $x \in A \cup B$. Hence $X \subset A \cup B$. It is also easy to see that $f(x) = g(x)$ for every $x \in A \cap B$. For any such x, we have that $x \in A$ so that $f(x) \leq g(x)$, and $x \in B$ so that $g(x) \leq f(x)$. From this it clearly must be that $f(x) = g(x)$.

Since f and g are continuous, it then follows from the pasting lemma that the function

$$h(x) = \begin{cases} f(x) & x \in A \\ g(x) & x \in B \end{cases}$$

for $x \in X$ is continuous as well. Based on the definitions of A and B it is then easy to see and trivial to show that $h(x) = \min\{f(x), g(x)\}$ for all $x \in X$, which of course shows the desired result. □

Exercise 18.9

Let $\{A_\alpha\}$ be a collection of subsets of X; let $X = \bigcup_\alpha A_\alpha$. Let $f : X \to Y$; suppose that $f \restriction A_\alpha$, is continuous for each α.

(a) Show that if the collection $\{A_\alpha\}$ is finite and each set A_α is closed, then f is continuous.
(b) Find an example where the collection $\{A_\alpha\}$ countable and each A_α is closed, but f is not continuous.
(c) An indexed family of sets $\{A_\alpha\}$ is said to be ***locally finite*** if each point x of X has a neighborhood that intersects A_α for only finitely many values of α. Show that if the family $\{A_\alpha\}$ is locally finite and each A_α is closed, then f is continuous.

Solution:

(a)

Proof. We show using induction that f is continuous for any collection $\{A_\alpha\}_{\alpha=1}^n$, for any $n \in \mathbb{Z}_+$, where each A_α is closed. This of course shows the desired result since the collection is $\{A_\alpha\}_{\alpha=1}^n$ for some $n \in \mathbb{Z}_+$ if it is finite. So first, for $n = 1$, we have that $A_1 = \bigcup_{\alpha=1}^n A_\alpha = X$ so that of course $f = f \restriction X = f \restriction A_1$ is continuous.

Now suppose that f is continuous for any collection of size n and suppose we have the collection $\{A_\alpha\}_{\alpha=1}^{n+1}$ of size $n+1$. Let $A = \bigcup_{\alpha=1}^n A_\alpha$, which is closed by Theorem 17.1 since each A_α is closed and it is a finite union, and let $B = A_{n+1}$ so that B is also closed. We then have that $A \cup B = \bigcup_{\alpha=1}^n A_\alpha \cup A_{n+1} = \bigcup_{\alpha=1}^{n+1} A_\alpha = X$. We know that $g = f \restriction B = f \restriction A_{n+1}$ is continuous. Considering the set A as a subspace of X, then each A_α for $\alpha \in \{1, \dots, n\}$ is closed in A by Theorem 17.2 since they are subsets of A and closed in X. Since by definition $\bigcup_{\alpha=1}^n A_\alpha = A$, it follows from the induction hypothesis that $f' = f \restriction A$ is continuous. Clearly also for any $x \in A \cap B$ we have that $x \in A$ and $x \in B = A_{n+1}$ so that $f'(x) = (f \restriction A)(x) = f(x) = (f \restriction A_{n+1})(x) = g(x)$.

Then by the pasting lemma the function $h : X \to Y$ defined by

$$h(x) = \begin{cases} f'(x) & x \in A \\ g(x) & x \in B \end{cases}$$

is continuous. However, consider any $x \in X$. If $x \in A$ then $h(x) = f'(x) = (f \restriction A)(x) = f(x)$. Similarly if $x \in B$ then $h(x) = g(x) = (f \restriction B)(x) = f(x)$ as well. This suffices to show that $h = f$ since x was arbitrary. Thus f is continuous, which completes the induction. □

(b) Consider the standard topology on $\mathbb{R}$ and define the countable collection of set $\{A_n\}$ by

$$A_n = \begin{cases} (-\infty, 0] & n = 1 \\ [1, \infty) & n = 2 \\ \left[\frac{1}{n-1}, \frac{1}{n-2}\right] & n > 2 \end{cases}$$

for $n \in \mathbb{Z}_+$. Also define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & x \le 0 \\ 0 & x > 0 \end{cases}$$

for real x. We claim that this collection and function have the desired properties.

Proof. First, it is trivial to show that the collection covers all of $\mathbb{R}$, i.e. that $\bigcup_{n=1}^{\infty} A_n = \mathbb{R}$. It is also obvious by this point that each A_n is closed in the standard topology. Clearly f is not a continuous function since there is a discontinuity at $x = 0$, which is trivial to prove. Lastly, consider any $n \in \mathbb{Z}_+$. If $n = 1$ then for any $x \in A_n = A_1 = (-\infty, 0]$ we have that $x \leq 0$ and hence $f(x) = 1$. Likewise if $n = 2$ then for any $x \in A_n = A_2 = [1, \infty)$ it follows that $x \geq 1 > 0$, and hence $f(x) = 0$. Lastly, if $n > 2$ then for any $x \in A_n = [1/(n-1), 1/(n-2)]$ we have that $0 < 1/(n-1) \leq x$ so that $f(x) = 0$ again. Thus in all cases $f \restriction A_n$ is constant and therefore continuous. This shows the desired properties. □

(c)

Proof. Consider any $x \in X$ so that there is a neighborhood U' of x that intersects a finite subcollection $\{A_k\}_{k=1}^{n}$ of the full collection $\{A_\alpha\}$. Consider $A = \bigcup_{k=1}^{n} A_k$ as a subspace of X, from which it follows from Theorem 17.2 that each A_k is closed in A since it is a subset of A and closed in X. It is then easy to show that $U' \subset A$. It also follows from part (a) that $f \restriction A$ is continuous with the domain being the subspace topology on A.

Now consider any neighborhood V of $f(x)$, noting that of course $x \in A$ since $x \in U'$ and $U' \subset A$. Thus $f(x)$ is in the image of $f \restriction A$ so that there is a neighborhood U_A of x in the subspace topology such that $(f \restriction A)(U_A) \subset V$ by Theorem 18.1 since $f \restriction A$ is continuous. Since U_A is open in the subspace topology, there is an open set U_X in X where $U_A = A \cap U_X$. Now let $U = U' \cap U_X$, which is open in X since U' and U_X are both open in X. Then also $x \in U_X$ since $x \in U_A$ and $U_A = A \cap U_X$, and hence $x \in U$ since also $x \in U'$ and $U = U' \cap U_X$. Thus U is a neighborhood of x in X.

Let z be any element of $f(U)$ so that $z = f(y)$ for some $y \in U$. Then $y \in U'$ and $y \in U_X$ since $U = U' \cap U_X$. Then also $y \in A$ since $U' \subset A$, and hence $y \in A \cap U_X = U_A$. From this it follows that $z = f(y) = (f \restriction A)(y) \in (f \restriction A)(U_A)$ so that $z \in V$ since $(f \restriction A)(U_A) \subset V$. Since z was arbitrary, this shows that $f(U) \subset V$, which in turn shows that f is continuous by Theorem 18.1 since V was an arbitrary neighborhood of $f(x)$ and x was an arbitrary element of X. □

We note that the example in part (b) is not locally finite since any neighborhood of $x = 0$ intersects infinitely many A_n in the collection. This fact is easy to see and would be easy to prove formally, though a bit tedious.

Exercise 18.10

Let $f : A \to B$ and $g : C \to D$ be continuous functions. Let us define a map $f \times g : A \times C \to B \times D$ by the equation

$$(f \times g)(a \times c) = f(a) \times g(c) .$$

Show that $f \times g$ is continuous.

Solution:

Proof. Consider any $x \times y \in A \times C$ and any neighborhood V of $(f \times g)(x \times y)$ in $B \times D$. Since V is open in $B \times D$, there is a basis element $U_B \times U_D$ of the product topology that contains $(f \times g)(x \times y)$ where $U_B \times U_D \subset V$. Then U_B and U_D are open in B and D, respectively. Since f is continuous, we then have that $U_A = f^{-1}(U_B)$ is open in A. Likewise $U_C = g^{-1}(U_D)$ is open in C since g is continuous. Then the set $U = U_A \times U_C$ is a basis element of the product topology $A \times C$ and therefore open.

Since $U_B \times U_D$ contains $(f \times g)(x \times y) = f(x) \times g(y)$ we have that $f(x) \in U_B$ and $g(y) \in U_D$. From this it follows that $x \in f^{-1}(U_B) = U_A$ and $y \in g^{-1}(U_D) = U_C$. Therefore $x \times y \in U_A \times U_C = U$ so that U is a neighborhood of $x \times y$ in $A \times C$. Now consider any $w \times z \in (f \times g)(U)$ so that there is an $x' \times y' \in U = U_A \times U_C$ where $w \times z = (f \times g)(x' \times y') = f(x') \times g(y')$. Hence $w = f(x')$ and $x' \in U_A = f^{-1}(U_B)$ so that $w = f(x') \in U_B$. Similarly $z = g(y')$ and $y' \in U_C = g^{-1}(U_D)$ so that $z = g(y') \in U_D$. Thus $w \times z \in U_B \times U_D$ so that also $w \times z \in V$ since $U_B \times U_D \subset V$. This shows that $(f \times g)(U) \subset V$ since $w \times z$ was arbitrary.

This suffices to show that $f \times g$ is continuous by Theorem 18.1 as desired. □

Exercise 18.11

Let $F : X \times Y \to Z$. We say that F is ***continuous in each variable separately*** if for each y_0 in Y, the map $h : X \to Z$ defined by $h(x) = F(x \times y_0)$ is continuous, and for each $x_0 \in X$, the map $k : Y \to Z$ defined by $k(y) = F(x_0 \times y)$ is continuous. Show that if F is continuous, then F is continuous in each variable separately.

Solution:

Proof. To show that F is continuous in x, consider any $y_0 \in Y$ and define $h : X \to Z$ by $h(x) = F(x \times y_0)$. Now consider any $x \in X$ and any neighborhood V of $h(x) = F(x \times y_0)$. Then V is an open set containing $h(x) = F(x \times y_0)$ so that it is a neighborhood of $F(x \times y_0)$. Since F is continuous, this means that there is neighborhood U' of $x \times y_0$ in $X \times Y$ such that $F(U') \subset V$ by Theorem 18.1. It then follows that there is a basis element $U_X \times U_Y$ of $X \times Y$ containing $x \times y_0$ where $U_X \times U_Y \subset U'$. Since $X \times Y$ is a product topology, we have that U_X is open in X and U_Y is open in Y. Then, since $x \times y_0 \in U_X \times U_Y$ we have that $x \in U_X$ and $y_0 \in U_Y$ so that U_X is a neighborhood of x in X.

So consider any $z \in h(U_X)$ so that $z = h(x')$ for some $x' \in U_X$. Then $x' \times y_0 \in U_X \times U_Y$ so that also $x' \times y_0$ in U' since $U_X \times U_Y \subset U'$. It then also follows that $z = h(x') = F(x' \times y_0) \in F(U')$ so that $z \in V$ since $F(U') \subset V$. This shows that $h(U_X) \subset V$ since z was arbitrary. It then follows that h is continuous by Theorem 18.1.

The proof that F is continuous in y is directly analogous. □

Exercise 18.12

Let $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by defined by the equation

$$F(x \times y) = \begin{cases} xy/(x^2 + y^2) & \text{if } x \times y \neq 0 \times 0. \\ 0 & \text{if } x \times y = 0 \times 0. \end{cases}$$

(a) Show that F is continuous in each variable separately.
(b) Compute the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = F(x \times x)$.
(c) Show that F is not continuous.

Solution:

(a)

Proof. It is easy to see that F is continuous in x. For any real y_0 we generally have that

$$h(x) = F(x \times y_0) = \frac{xy_0}{x^2 + y_0^2}$$

so long as one of x and y_0 are nonzero. If $y_0 = 0$ then $x = 0$ implies that $x \times y_0 = 0 \times 0$ so that $h(x) = F(0 \times 0) = 0$ by definition. If $x \neq 0$ then we have $h(x) = 0/x^2 = 0$ again. Thus h is the constant function $h(x) = 0$ and so is continuous when $y_0 = 0$. If $y_0 \neq 0$ then $y_0^2 > 0$ so that $x^2 + y_0^2 > 0$ since also $x \geq 0$. Thus the denominator is never zero that the $h(x)$ is given by the expression above, which is continuous by elementary calculus. Hence h is always continuous. The same arguments show that F is continuous in y as well. □

(b) We clearly have

$$g(x) = F(x \times x) = \begin{cases} \frac{x^2}{x^2+x^2} = \frac{x^2}{2x^2} = \frac{1}{2} & x \neq 0 \\ 0 & x = 0 \,. \end{cases}$$

(c)

Proof. First consider the function $f : \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ defined simply by $f(x) = x \times x$. This function is clearly continuous by Theorem 18.4 since it can be expressed as $f(x) = f_1(x) \times f_2(x)$ where the identical functions $f_1(x) = f_2(x) = x$ are obviously continuous. Then $g = F \circ f$, where g is the function from part (b) since we have $g(x) = F(x \times x) = F(f(x))$ for any real x. Now, clearly g as calculated in part (b) has a discontinuity at $x = 0$ so that it is not continuous. It then follows from Theorem 18.2 part (c) that either F or f is not continuous since $g = F \circ f$. As we know that the trivial function f is continuous, it must then be that F is not as desired. □

Exercise 18.13

Let $A \subset X$; let $f : A \to Y$ be continuous; let Y be Hausdorff. Show that if f may be extended to a continuous function $g : \overline{A} \to Y$, then g is uniquely determined by f.

Solution:

Proof. Suppose that g_1 and g_2 are both continuous functions from $\overline{A}$ to Y that extend f so that $g_1(x) = g_2(x) = f(x)$ for all $x \in A$. Clearly $g_1 = g_2$ if and only if $g_1(x) = g_2(x)$ for all $x \in \overline{A}$. So suppose that this is *not* the case so that there is an $x_0 \in \overline{A}$ where $g_1(x_0) \neq g_2(x_0)$. Since Y is a Hausdorff space and $g_1(x_0)$ and $g_2(x_0)$ are distinct, there are disjoint neighborhoods V_1 and V_2 of $g_1(x_0)$ and $g_2(x_0)$, respectively. Then there are also neighborhoods U_1 and U_2 of x_0 such that $g_1(U_1) \subset V_1$ and $g_2(U_2) \subset V_2$ by Theorem 18.1 since both g_1 and g_2 are continuous.

Now let $U = U_1 \cap U_2$ so that U is also a neighborhood of x_0. Since $x_0 \in \overline{A}$, it follows that U intersects A so that there is a $y \in U$ where also $y \in A$ by Theorem 17.5. Since $y \in A$ we have that $g_1(y) = g_2(y) = f(y)$. We also have that $y \in U_1$ and $y \in U_2$ since $U = U_1 \cap U_2$. Thus $g_1(y) \in g_1(U_1)$ so that $f(y) = g_1(y) \in V_1$ since $g_1(U_1) \subset V_1$. Similarly $f(y) = g_2(y) \in V_2$, but then we have that $f(y) \in V_1 \cap V_2$, which contradicts the fact that V_1 and V_2 are disjoint! Hence it must be that $g_1 = g_2$, which shows uniqueness. □

§19 The Product Topology

Exercise 19.1

Prove Theorem 19.2

Solution:

Let $\mathcal{C}$ be the collection of sets that are alleged to be a basis for the box or product topologies in Theorem 19.2.

Proof. We show that $\mathcal{C}$ is a basis of the box or product topology using Lemma 13.2. First, it is easy to see that $\mathcal{C}$ is a collection of open sets. Consider any $B \in \mathcal{C}$ so that $B = \prod B_\alpha$ where each $B_\alpha \in \mathcal{B}_\alpha$ (for a finitely many $\alpha \in J$ and $B_\alpha = X_\alpha$ for the rest in the product topology). Since each B_α is a basis element of X_α (or X_α itself), they are open so that B is a basis element of the box or product topology by definition and therefore open. Note that the basis for the product topology is given directly by Theorem 19.1.

Now suppose that U is an any open set of the box topology and consider any $x \in U$. Then it follows that there is a basis element $\prod_{\alpha \in J} U_\alpha$ of the box or product topology containing x where $\prod_{\alpha \in J} U_\alpha \subset U$. Thus each U_α is an open set of X_α (or $U_\alpha = X_\alpha$ for all but finitely many $\alpha \in J$ for the product topology). Also $x \in \prod_{\alpha \in J} U_\alpha$ so that $x = (x_\alpha)_{\alpha \in J}$ where each $x_\alpha \in U_\alpha$. It then follows that there is basis element $B_\alpha \in \mathcal{B}_\alpha$ of X_α containing x_α where $B_\alpha \subset U_\alpha$ (for $U_\alpha = X_\alpha$ we simply set $B_\alpha = X_\alpha$ as well).

Then clearly $x \in \prod_{\alpha \in J} B_\alpha$ and $\prod_{\alpha \in J} B_\alpha \in \mathcal{C}$. Consider also any $y \in \prod_{\alpha \in J} B_\alpha$ so that $y = (y_\alpha)_{\alpha \in J}$ where each $y_\alpha \in B_\alpha$. Then also each $y_\alpha \in U_\alpha$ since $B_\alpha \subset U_\alpha$. This suffices to show that $y \in \prod_{\alpha \in J} U_\alpha \subset U$. Since y was arbitrary this shows that $\prod_{\alpha \in J} B_\alpha \subset U$. Therefore $\mathcal{C}$ is a basis of the box topology by Lemma 13.2. □

Exercise 19.2

Prove Theorem 19.3.

Solution:

Proof. The basis of the box or product topologies on $\prod A_\alpha$ is the collection of sets $\prod V_\alpha$, where each V_α is open in A_α and, in the case of the product topology, $V_\alpha = A_\alpha$ for all but finitely many $\alpha \in J$ (by Theorem 19.1). Denote this basis collection by $\mathcal{C}$. By Lemma 16.1, the collection

$$\mathcal{B}_A = \left\{ B \cap \prod A_\alpha \mid B \in \mathcal{B} \right\}$$

is a basis of the subspace topology on $\prod A_\alpha$, where $\mathcal{B}$ is the basis of $\prod X_\alpha$. To prove that $\prod A_\alpha$ is a subspace of $\prod X_\alpha$, it therefore suffices to show that $\mathcal{C} = \mathcal{B}_A$.

($\subset$) First consider any element $B \in \mathcal{C}$ so that $B = \prod V_\alpha$ for open sets V_α in A_α (and $V_\alpha = A_\alpha$ for all but finite many $\alpha \in J$ for the product topology). For each $\alpha \in J$, we then have that $V_\alpha = U_\alpha \cap A_\alpha$ for some open set U_α in X_α since A_α is a subspace of X_α. Note that this is true even for those α where $V_\alpha = A_\alpha$ in the product topology since then $V_\alpha = A_\alpha = X_\alpha \cap A_\alpha$. In fact, for these α we need to choose $U_\alpha = X_\alpha$ as will become apparent. We then have the following:

$$\begin{aligned}
x \in B &\Leftrightarrow x \in \prod V_\alpha \\
&\Leftrightarrow \forall \alpha \in J (x_\alpha \in V_\alpha) \\
&\Leftrightarrow \forall \alpha \in J (x_\alpha \in U_\alpha \cap A_\alpha) \\
&\Leftrightarrow \forall \alpha \in J (x_\alpha \in U_\alpha \wedge x_\alpha \in A_\alpha) \\
&\Leftrightarrow \forall \alpha \in J (x_\alpha \in U_\alpha) \wedge \forall \alpha \in J (x_\alpha \in A_\alpha) \\
&\Leftrightarrow x \in \prod U_\alpha \wedge x \in \prod A_\alpha
\end{aligned}$$

$$\Leftrightarrow x \in \left(\prod U_\alpha\right) \cap \left(\prod A_\alpha\right) ,$$

Since $U_\alpha = X_\alpha$ for all but a finitely many $\alpha \in J$ for the product topology, we have that $\prod U_\alpha$ is a basis element of $\prod X_\alpha$, i.e. $\prod U_\alpha \in \mathcal{B}$. This shows that $B \in \mathcal{B}_A$ so that $\mathcal{C} \subset \mathcal{B}_A$ since B was arbitrary.

($\supset$) Now suppose that $B \in \mathcal{B}_A$ so that $B = B_X \cap \prod A_\alpha$ for some basis element $B_X \in \mathcal{B}$ of $\prod X_\alpha$. We then have that $B_X = \prod U_\alpha$ where each U_α is an open set of X_α (and $U_\alpha = X_\alpha$ for all but finitely many $\alpha \in J$ for the product topology). Then let $V_\alpha = U_\alpha \cap A_\alpha$ for each $\alpha \in J$, noting that $V_\alpha = X_\alpha \cap A_\alpha = A_\alpha$ when $U_\alpha = X_\alpha$. Following the above chain of logical equivalences in reverse order then shows that $B = \prod V_\alpha$ so that $B \in \mathcal{C}$ since clearly each V_α is open in the subspace topology A_α. Hence $\mathcal{C} \supset \mathcal{B}_A$ since B was arbitrary. $\square$

Exercise 19.3

Prove Theorem 19.4.

Solution:

Proof. Suppose that x and y are distinct points of $\prod X_\alpha$. Then $x = (x_a)$ and $y = (y_\alpha)$ where each $x_\alpha, y_\alpha \in X_\alpha$, and there must be a β where $x_\beta \neq y_\beta$ since $x \neq y$. Thus x_β and y_β are distinct points of X_β, so that there are neighborhoods W_x and W_y of x_β and y_β, respectively, that are disjoint since X_β is a Hausdorff space. So define the sets

$$U_\alpha = \begin{cases} W_x & \alpha = \beta \\ X_\alpha & \alpha \neq \beta \end{cases} \qquad V_\alpha = \begin{cases} W_y & \alpha = \beta \\ X_\alpha & \alpha \neq \beta \end{cases}$$

so that clearly $x \in \prod U_\alpha$ and $y \in \prod V_\alpha$. Then since each U_α and V_α are open, we have that $\prod U_\alpha$ and $\prod V_\alpha$ are both basis elements of $\prod X_\alpha$ and therefore open. Note that this is true for both the box and product topologies since, in the case of the latter, U_α and V_α are not all of X_α for only one α, namely $\alpha = \beta$. Thus $\prod U_\alpha$ is a neighborhood of x and $\prod V_\alpha$ is a neighborhood of y in $\prod X_\alpha$.

We also assert that $\prod U_\alpha$ and $\prod V_\alpha$ are disjoint, which of course completes the proof that $\prod X_\alpha$ is Hausdorff. To see this, suppose to the contrary that there is a z in both $\prod U_\alpha$ and $\prod V_\alpha$. Then $z = (z_\alpha)$ and in particular we would have that $z_\beta \in U_\beta = W_x$ and $z_\beta \in V_\beta = W_y$. But then $z_\beta \in W_x \cap W_y$, which contradicts the fact that W_x and W_y are disjoint! So it must be that in fact $\prod U_\alpha$ and $\prod V_\alpha$ are disjoint. $\square$

Exercise 19.4

Show that $(X_1 \times \cdots \times X_{n-1}) \times X_n$ is homeomorphic to $X_1 \times \cdots \times X_n$.

Solution:

Proof. First we note that since we are dealing with finite products, the box and product topologies are the same; we shall find it most convenient to use the box topology definition. Also, as there are no intervals involved here, we use the traditional tuple notation using parentheses. So define $f : X_1 \times \cdots \times X_n \to (X_1 \times \cdots \times X_{n-1}) \times X_n$ by

$$f(x_1, \ldots, x_{n-1}) = ((x_1, \ldots, x_{n-1}), x_n) .$$

It is obvious that this is a bijection, and it is trivial to prove. Also obvious and trivial to prove based on the definition of f is that $f(A_1 \times \cdots \times A_n) = (A_1 \times \cdots \times A_{n-1}) \times A_n$ when each $A_k \subset X_k$.

First we show that f is continuous by showing that the inverse image of every basis element in $(X_1 \times \cdots \times X_{n-1}) \times X_n$ is open in $X_1 \times \cdots \times X_n$. So consider any basis element C of $(X_1 \times \cdots \times X_{n-1}) \times X_n$ and let $U = f^{-1}(C)$ so that of course $f(U) = C$ and $U \subset X_1 \times \cdots \times X_n$. We then have that $C = V' \times V_n$ where V' is open in $X_1 \times \cdots \times X_{n-1}$ and V_n is open in X_n by the definition of the box/product topology. Now consider any $x \in U$ so that $x = (x_1, \ldots, x_n)$ and we have that $f(x) = ((x_1, \ldots, x_{n-1}), x_n) \in f(U) = C$. Hence $x' = (x_1, \ldots, x_{n-1}) \in V'$ and $x_n \in V_n$. Since V' is open in $X_1 \times \cdots \times X_{n-1}$ there is a basis element C' containing x' that is a subset of V'. By the definition of the box topology, we then have that $C' = V_1 \times \cdots \times V_{n-1}$ where each V_k is open in X_k.

We then have that $B = V_1 \times \cdots \times V_n$ is a basis element of $X_1 \times \cdots \times X_n$ and also clearly B contains x since $(x_1, \ldots, x_{n-1}) = x' \in C' = V_1 \times \cdots \times V_{n-1}$ and $x_n \in V_n$. Now suppose that $y = (y_1, \ldots, y_n) \in B$ so that each $y_k \in V_k$. Then we have that $y' = (y_1, \ldots, y_{n-1}) \in C'$ so that also $y' \in V'$ since $C' \subset V'$. Since also of course $y_n \in V_n$, we have that $(y', y_n) \in V' \times V_n = C$. Also clearly $f(y) = (y', y_n) \in C = f(U)$ so that $y \in U$. Since y was arbitrary this shows that $B \subset U$, which suffices to show that U is open since x was arbitrary. This completes the proof that f is continuous.

Next we show that f^{-1} is continuous, which is a little simpler. Let B be any basis element of $X_1 \times \cdots \times X_n$ so that $B = U_1 \times \cdots \times U_n$ where each U_k is open in X_k by the definition of the box topology. Then we have that $f(B) = (U_1 \times \cdots \times U_{n-1}) \times U_n$. By the definition of the box topology, we then have that $U' = U_1 \times \cdots \times U_{n-1}$ is a basis element of $X_1 \times \cdots \times X_{n-1}$ and is therefore open. Since U_n is also open, we have that $f(B) = U' \times U_n$ is a basis element of $(X_1 \times \cdots \times X_{n-1}) \times X_n$ by the definition of the box/product topology, and is therefore open. Since $f(B) = (f^{-1})^{-1}(B)$ is the inverse image of B under f^{-1}, this shows that f^{-1} is also continuous.

We have shown that both f and f^{-1} are continuous, which proves that f is a homeomorphism by definition. □

Exercise 19.5

One of the implications stated in Theorem 19.6 holds for the box topology. Which one?

Solution:

Example 19.2 gives a function f that is not continuous in the box topology even though all of its constituent functions f_α are continuous. Hence the only implication that can be generally true in the box topology is that f being continuous implies that each f_α is continuous. A proof of this is straightforward.

Proof. As in Theorem 19.6 suppose that $f : A \to \prod_{\alpha \in J} X_\alpha$ be given by

$$f(a) = (f_\alpha(a))_{\alpha \in J},$$

where $f_\alpha : A \to X_\alpha$ for each $\alpha \in J$. Here $\prod X_\alpha$ has the box topology. Suppose that f is continuous and consider any $\beta \in J$. We show that f_β is continuous, which of course shows the desired result.

So let V be any open set of X_β and define

$$B_\alpha = \begin{cases} V & \alpha = \beta \\ X_\alpha & \alpha \neq \beta \,. \end{cases}$$

Then, since each B_α is clearly open in X_α, we have that $B = \prod B_\alpha$ is a basis element of the box topology by definition and is therefore open. Hence $U = f^{-1}(B)$ is open in A since f is continuous.

We claim that $U = f_\beta^{-1}(V)$, which shows that f_β is continuous since U is open in A and V was an arbitrary open set of X_β.

($\subset$) If $x \in U = f^{-1}(B)$ then of course $f(x) \in B$ so that each $f_\alpha(x) \in B_\alpha$ since $f(x) = (f_\alpha(x))_{\alpha \in J}$ and $B = \prod B_\alpha$. In particular $f_\beta(x) \in B_\beta = V$ so that $x \in f_\beta^{-1}(V)$. Hence $U \subset f_\beta^{-1}(V)$ since x was arbitrary.

($\supset$) If $x \in f_\beta^{-1}(V)$ then $f_\beta(x) \in V = B_\beta$. Since of course every other $f_\alpha(x) \in X_\alpha = B_\alpha$ we have that $f(x) \in \prod B_\alpha = B$. Hence $x \in f^{-1}(B) = U$ so that $f_\beta^{-1}(V) \subset U$ since x was arbitrary. □

Exercise 19.6

Let $\mathbf{x}_1, \mathbf{x}_2, \ldots$ be a sequence of the points of the product space $\prod X_\alpha$. Show that the sequence converges to the point $\mathbf{x}$ if and only if the sequence $\pi_\alpha(\mathbf{x}_1), \pi_\alpha(\mathbf{x}_2), \ldots$ converges to $\pi_\alpha(\mathbf{x})$ for each α. Is this fact true if one uses the box topology instead of the product topology?

Solution:

Proof. ($\Rightarrow$) First suppose that the sequence $\mathbf{x}_1, \mathbf{x}_2, \ldots$ converges to $\mathbf{x}$ and consider any β. Also suppose that U is any neighborhood of $\pi_\beta(\mathbf{x})$. Define

$$B_a = \begin{cases} U & \alpha = \beta \\ X_\alpha & \alpha \neq \beta \end{cases}$$

so that $B = \prod B_\alpha$ is a basis element of $\prod X_\alpha$ since each B_α is open. Note that B is a basis element of both the box and product topologies since possibly $B_\alpha \neq X_\alpha$ for only one α (i.e. for $\alpha = \beta$). We also clearly have that $\mathbf{x} \in B$ so that B is a neighborhood of $\mathbf{x}$ in $\prod X_\alpha$. Since the sequence $\mathbf{x}_1, \mathbf{x}_2, \ldots$ converges to $\mathbf{x}$, we have that there is an $N \in \mathbb{Z}_+$ where $\mathbf{x}_n \in B$ for all $n \geq N$. So consider any such $n \geq N$ so that $\mathbf{x}_n \in B = \prod B_\alpha$. Hence $\pi_\alpha(\mathbf{x}_n) \in B_\alpha$ for all α, and in particular $\pi_\beta(\mathbf{x}_n) \in B_\beta = U$. This suffices to show that the sequence $\pi_\beta(\mathbf{x}_1), \pi_\beta(\mathbf{x}_2), \ldots$ converges to $\pi_\beta(\mathbf{x})$ as desired since U was an arbitrary neighborhood.

($\Leftarrow$) Now suppose that the sequence $\pi_\alpha(\mathbf{x}_1), \pi_\alpha(\mathbf{x}_2), \ldots$ converges to $\pi_\alpha(\mathbf{x})$ for every α. Let U be any neighborhood of $\mathbf{x}$ in $\prod X_\alpha$. Then there is a basis element $B = \prod U_\alpha$ of $\prod X_\alpha$ where $\mathbf{x} \in B$ and $B \subset U$. Since $\prod X_\alpha$ is the product topology, each U_α is open but only a finite number of them are different from X_α. Suppose then that J is the index set of α and that $I \subset J$ is the finite subset where $U_\alpha = X_\alpha$ for all $\alpha \notin I$.

Then for any $\beta \in I$ we have that $\pi_\beta(\mathbf{x}) \in U_\beta$ since $\mathbf{x} \in B = \prod U_\alpha$, hence U_β is a neighborhood of $\pi_\beta(\mathbf{x})$. Then, since $\pi_\beta(\mathbf{x}_1), \pi_\beta(\mathbf{x}_2), \ldots$ converges to $\pi_\beta(\mathbf{x})$, there is an $N_\beta \in \mathbb{Z}_+$ where $\pi_\beta(\mathbf{x}_n) \in U_\beta$ for all $n \geq N_\beta$. So let $N = \max_{\alpha \in I} N_\alpha$, noting that this exists since I is finite. Consider any $n \geq N$ and any $\alpha \in J$. If $\alpha \in I$ then we have that $n \geq N \geq N_\alpha$ so that $\pi_\alpha(\mathbf{x}_n) \in U_\alpha$. If $\alpha \notin J$ then of course we have that $\pi_a(\mathbf{x}_n) \in X_\alpha = U_\alpha$. Hence either way we have that $\pi_\alpha(\mathbf{x}_n) \in U_\alpha$ so that $\mathbf{x}_n \in \prod U_\alpha = B$ and hence also $\mathbf{x}_n \in U$ since $B \subset U$. Since $n \geq N$ was arbitrary and U was an arbitrary neighborhood of $\mathbf{x}$, this shows that $\mathbf{x}_1, \mathbf{x}_2, \ldots$ converges to $\mathbf{x}$ as desired. □

As noted there, the forward direction of the preceding proof works for the product or the box topology. However, then reverse direction was proved only for the product topology, with the critical point being where we took $\max_{\alpha \in I} N_\alpha$, which was only guaranteed to exist since I is finite in the product topology. The provides a hint as to how to construct a counterexample that proves that this direction is not generally true for the box topology.

Proof. Define

$$x_{ij} = \begin{cases} 1 & j \leq i \\ \frac{1}{j-i} & j > i \end{cases}$$

for $i, j \in \mathbb{Z}_+$. Now define a sequence $\mathbf{x}_1, \mathbf{x}_2, \ldots$ in $\prod_{i \in \mathbb{Z}_+} \mathbb{R} = \mathbb{R}^\omega$ by $\pi_i(\mathbf{x}_j) = x_{ij}$. With the box topology on $\mathbb{R}^\omega$ we claim that each coordinate sequence $\pi_i(\mathbf{x}_1), \pi_i(\mathbf{x}_2), \ldots$ converges to 0 but that the sequence $\mathbf{x}_1, \mathbf{x}_2, \ldots$ does not converge to the point $\mathbf{0} = (0, 0, \ldots)$.

First, it is easy to see that each coordinate sequence $\pi_i(\mathbf{x}_1), \pi_i(\mathbf{x}_2), \ldots$ converges to 0 since, for fixed i, there is always an $N \in \mathbb{Z}_+$ large enough such that $j > i$ and $\pi_i(\mathbf{x}_j) = x_{ij} = 1/(j-i)$ is small enough to be within any fixed neighborhood of 0 for all $j \geq N$. To show that the sequence $\mathbf{x}_1, \mathbf{x}_2, \ldots$ does not converge to $\mathbf{0}$ though, consider the neighborhood $U = \prod U_k$ of $\mathbf{0}$ where every $U_k = (-1, 1)$. We note that clearly U is open in the box topology since each U_k is a basis element of $\mathbb{R}$ and therefore open. For any $N \in \mathbb{Z}_+$ we then have that $\pi_N(\mathbf{x}_N) = x_{NN} = 1$ so that clearly $\pi_N(\mathbf{x}_N) \notin (-1, 1) = U_N$ and hence $\mathbf{x}_N \notin \prod U_k = U$. This suffices to show that the sequence does not converge, but it does not even come close to converging since there are actually no points in the sequence that are even in this quite large neighborhood of $\mathbf{0}$! □

Exercise 19.7

Let $\mathbb{R}^\infty$ be the subset of $\mathbb{R}^\omega$ consisting of all sequences that are "eventually zero," that is all sequences $(x_1, x_2, \ldots)$ such that $x_i \neq 0$ for only finitely many values of i. What is the closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ in the box and product topologies? Justify your answer.

Solution:

First we claim that $\mathbb{R}^\infty$ is dense in $\mathbb{R}^\omega$ in the product topology in the sense that its closure is all of $\mathbb{R}^\omega$.

Proof. We show that any point of $\mathbb{R}^\omega$ is in $\overline{\mathbb{R}^\omega}$. So consider any point $x = (x_1, x_2, \ldots) \in \mathbb{R}^\omega$ and any neighborhood U of x. Then there is a basis element $B = \prod U_n$ containing x where $B \subset U$. By the definition of the product topology each U_n is open and $U_n = \mathbb{R}$ for all but finitely many values of n. So let I be a finite subset of $\mathbb{Z}_+$ such that $U_n = \mathbb{R}$ for all $n \notin I$ and U_n is merely just open for $n \in I$.

Consider now the sequence $y = (y_1, y_2, \ldots)$ defined by

$$y_n = \begin{cases} x_n & n \in I \\ 0 & n \notin I \end{cases}$$

for $n \in \mathbb{Z}_+$. Since I is finite clearly $y \in \mathbb{R}^\infty$. Also $y_n = x_n \in U_n$ when $n \in I$ since $B = \prod U_n$ contains x. We also have $y_n = 0 \in \mathbb{R} = U_n$ when $n \notin I$ so that either way $y_n \in U_n$ and hence $y \in \prod U_n = B$. Thus also $y \in U$ since $B \subset U$. Since U was an arbitrary neighborhood and U intersects $\mathbb{R}^\infty$ (with y being a point in the intersection), this shows that $x \in \overline{\mathbb{R}^\infty}$ by Theorem 17.5. This of course shows the desired result since x was any element of $\mathbb{R}^\omega$. □

For the box topology, we claim that $\mathbb{R}^\infty$ is already closed.

Proof. We show this by showing that any point not in $\mathbb{R}^\infty$ is not a limit point of $\mathbb{R}^\infty$ so that $\mathbb{R}^\infty$ must already contain all its limit points. So consider any $x = (x_1, x_2, \ldots) \notin \mathbb{R}^\infty$ so that $x_n \neq 0$ for infinitely many values of n. Now define the sets

$$U_n = \begin{cases} (-1,1) & x_n = 0 \\ (x_n/2, 2x_n) & x_n > 0 \\ (2x_n, x_n/2) & x_n < 0 \end{cases}$$

for $n \in \mathbb{Z}_+$. Clearly each U_n is a basis element of $\mathbb{R}$ and is therefore open. Also clearly each $x_n \in U_n$. It therefore follows that $B = \prod U_n$ is a basis element of $\mathbb{R}^\omega$ and is therefore open, and that $x \in B$. Hence B is a neighborhood of x.

Then, for any $y = (y_1, y_2, \ldots) \in B$ we have that each $y_n \in U_n$. For infinitely many $n \in \mathbb{Z}_+$ we then have that $x_n \neq 0$ and hence $x_n > 0$ or $x_n < 0$. In the former case $y_n \in U_n = (x_n/2, 2x_n)$ so that $0 < x_n/2 < y_n$. In the latter case $y_n \in U_n = (2x_n, x_n/2)$ so that $y_n < x_n/2 < 0$. Hence either way $y_n \neq 0$ so that $y \notin \mathbb{R}^\infty$ since this is true for infinitely many n. Since $y \in B$ was arbitrary, this shows that B cannot not intersect $\mathbb{R}^\infty$. Therefore x is not a limit point of $\mathbb{R}^\infty$ since B is a neighborhood of x. □

Exercise 19.8

Given sequences $(x_1, x_2, \ldots)$ and $(b_1, b_2, \ldots)$ of reals numbers with $a_i > 0$ for all i, define $h : \mathbb{R}^\omega \to \mathbb{R}^\omega$ by the equation

$$h((x_1, x_2, \ldots)) = (a_1 x_1 + b_1, a_2 x_2 + b_2, \ldots).$$

Show that if $\mathbb{R}^\omega$ is given the product topology, h is a homeomorphism of $\mathbb{R}^\omega$ with itself. What happens if $\mathbb{R}^\omega$ is given the box topology?

Solution:

Lemma 19.8.1. *Consider the spaces $\prod X_\alpha$ and $\prod Y_\alpha$ in the box topologies over the index set J. If $f : \prod X_\alpha \to \prod Y_\alpha$ is defined by*

$$f((x_\alpha)_{\alpha \in J}) = (f_\alpha(x_\alpha))_{\alpha \in J}$$

and each $f_\alpha : X_\alpha \to Y_\alpha$ is continuous, then f is continuous.

Proof. Consider any basis element $B = \prod V_\alpha$ in $\prod Y_\alpha$ so that each V_α is open in Y_α since we are in the box topology. For each $\alpha \in J$ then define $U_\alpha = f_\alpha^{-1}(V_\alpha)$, which is open in X_α since f_α is continuous. Hence the set $U = \prod U_\alpha$ is a basis element of $\prod X_\alpha$ in the box topology and is therefore open. We claim that $U = f^{-1}(B)$, which shows that f is continuous since U is open and B was arbitrary.

($\subset$) Consider any $\mathbf{x} \in U = \prod U_\alpha$. Then, for any $\alpha \in J$, we have $x_\alpha \in U_\alpha = f_\alpha^{-1}(V_\alpha)$ so that $f(x_\alpha) \in V_\alpha$. Hence $f(\mathbf{x}) = (f_\alpha(x_\alpha))_{\alpha \in J} \in \prod V_\alpha = B$ so that $\mathbf{x} \in f^{-1}(B)$. this shows that $U \subset f^{-1}(B)$ since $\mathbf{x}$ was arbitrary.

($\supset$) Now consider any $\mathbf{x} \in f^{-1}(B)$ so that $f(\mathbf{x}) \in B = \prod V_\alpha$ and hence each $f_\alpha(x_\alpha) \in V_\alpha$ by the definition of f. Then $x_\alpha \in f_\alpha^{-1}(V_\alpha) = U_\alpha$ so that clearly $\mathbf{x} \in \prod U_\alpha = U$. Since $\mathbf{x}$ was arbitrary this shows that $f^{-1}(B) \subset U$ as well. □

Main Problem.

Proof. First note that clearly $h(\mathbf{x}) = (h_1(\mathbf{x}), h_2(\mathbf{x}), \ldots)$ for $\mathbf{x} \in \mathbb{R}^\omega$, where each $h_i : \mathbb{R}^\omega \to \mathbb{R}$ is defined by

$$h_i(\mathbf{x}) = a_i \pi_i(\mathbf{x}) + b_i \, .$$

This can further be broken down as $h_i(\mathbf{x}) = f_i(\pi_i(\mathbf{x})) = (f_i \circ \pi_i)(\mathbf{x})$, where each $f_i : \mathbb{R} \to \mathbb{R}$ is defined by $f_i(x) = a_i x + b_i$. As discussed in the proof of Theorem 19.6, each π_i is continuous and we have that each f_i is continuous by elementary calculus, noting that this is true whether each $a_i > 0$ or not. It then follows from Theorem 18.2 part (c) that each $f_i \circ \pi_i = h_i$ is continuous. Then we have that h is continuous by Theorem 19.6 since each coordinate function is continuous and we are using the product topology.

Now define the functions $g_i : \mathbb{R} \to \mathbb{R}$ by $g_i(x) = (x - b_i)/a_i$ for $i \in \mathbb{Z}_+$, noting that this is defined since each $a_i > 0$. Define also the functions $k_i : \mathbb{R}^\omega \to \mathbb{R}$ by $k_i = g_i \circ \pi_i$, and finally define $k : \mathbb{R}^\omega \to \mathbb{R}^\omega$ by $k(\mathbf{x}) = (k_1(\mathbf{x}), k_2(\mathbf{x}), \ldots)$. Now again we have that each π_i and g_i are continuous by the proof of Theorem 19.6 and elementary calculus. Hence $k_i = g_i \circ \pi_i$ and k are continuous by Theorem 18.2 part (c), and Theorem 19.6, respectively, as before.

Now consider any $\mathbf{x} = (x_1, x_2, \ldots) \in \mathbb{R}^\omega$ so that we have, for any $i \in \mathbb{Z}_+$,

$$\begin{aligned} k_i(h(\mathbf{x})) &= [g_i \circ \pi_i](h(\mathbf{x})) = g_i(\pi_i(h(\mathbf{x}))) = g_i(h_i(\mathbf{x})) \\ &= g_i([f_i \circ \pi_i](\mathbf{x})) = g_i(f_i(\pi_i(\mathbf{x}))) = g_i(f_i(x_i)) \\ &= \frac{f_i(x_i) - b_i}{a_i} = \frac{(a_i x_i + b_i) - b_i}{a_i} = \frac{a_i x_i}{a_i} \\ &= x_i \, . \end{aligned}$$

Therefore

$$k(h(\mathbf{x})) = (k_1(h(\mathbf{x})), k_2(h(\mathbf{x})), \ldots) = (x_1, x_2, \ldots) = \mathbf{x} \, .$$

We also have that

$$\begin{aligned} h_i(k(\mathbf{x})) &= [f_i \circ \pi_i](k(\mathbf{x})) = f_i(\pi_i(k(\mathbf{x}))) = f_i(k_i(\mathbf{x})) \\ &= f_i([g_i \circ \pi_i](\mathbf{x})) = f_i(g_i(\pi_i(\mathbf{x}))) = f_i(g_i(x_i)) \\ &= a_i g_i(x_i) + b_i = a_i \left(\frac{x_i - b_i}{a_i} \right) + b_i = (x_i - b_i) + b_i \\ &= x_i \, . \end{aligned}$$

for each $i \in \mathbb{Z}_+$ so that

$$h(k(\mathbf{x})) = (h_1(k(\mathbf{x})), h_2(k(\mathbf{x})), \ldots) = (x_1, x_2, \ldots) = \mathbf{x} \, .$$

Since $\mathbf{x}$ was arbitrary, it thus follows from Lemma 2.1 that h is bijective and $k = h^{-1}$. Since we have already shown that h and $k = h^{-1}$ are continuous, this suffices to prove that h is a homeomorphism as desired. □

We claim that h is also a homeomorphism in the box topology.

Proof. First, h is still a bijection as the proof of this above does not depend on the topology at all. However, Theorem 19.6 was used in the proofs that h and h^{-1} are continuous, and we know that this theorem is not generally true for the box topology. On the other hand h can be formulated as $h(\mathbf{x}) = (f_1(x_1), f_2(x_2), \ldots)$, where as before each $f_i(x) = a_i x + b_i$. Since each f_i is continuous by elementary calculus, it follows from Lemma 19.8.1 that h is continuous in the box topology. The same argument applies to the inverse function h^{-1} since $h^{-1}(\mathbf{x}) = (g_1(x_1), g_2(x_2), \ldots)$ and each g_i is continuous. □

Exercise 19.9

Show that the choice axiom is equivalent to the statement that for any indexed family $\{A_\alpha\}_{\alpha\in J}$ of nonempty sets, with $J \neq 0$, the cartesian product

$$\prod_{\alpha\in J} A_\alpha$$

is not empty.

Solution:

Proof. For the following denote the collection $\{A_\alpha\}_{\alpha\in J}$ by $\mathcal{A}$.

($\Rightarrow$) First suppose that the choice axiom is true. Then by Lemma 9.2 there exists a choice function

$$c : \mathcal{A} \to \bigcup_{A\in\mathcal{A}} A$$

where $c(A) \in A$ for each $A \in \mathcal{A}$, noting that this is true since $\mathcal{A}$ is a collection of nonempty sets. Then consider, set $x_\alpha = c(A_\alpha)$ for each $\alpha \in J$ so that $x_\alpha = c(A_\alpha) \in A_\alpha$. Therefore clearly $\mathbf{x} = (x_\alpha)_{\alpha\in J} \in \prod A_\alpha$ so that $\prod A_\alpha$ is not empty.

($\Leftarrow$) Now suppose that $\prod_{\alpha\in J} A_\alpha$ is nonempty for any indexed family $\{A_\alpha\}_{\alpha\in J}$ of nonempty sets when $J \neq \varnothing$. Let $\mathcal{A}$ be a collection of disjoint nonempty sets where $\mathcal{A} \neq \varnothing$. Then the $\{A\}_{A\in\mathcal{A}}$ is a nonempty family of nonempty sets. Hence $\prod_{A\in\mathcal{A}} A$ is nonempty so that there is an $\mathbf{x} = (x_A)_{A\in\mathcal{A}} \in \prod_{A\in\mathcal{A}} A$, and thus $x_A \in A$ for every $A \in \mathcal{A}$. Now let $C = \{x_A\}_{A\in\mathcal{A}}$ so that clearly $C \subset \bigcup \mathcal{A}$. Consider any $A \in \mathcal{A}$ so that $x_A \in C$ and $x_A \in A$, and hence $x_A \in C \cap A$. Suppose that $y \in C \cap A$ so that $y \in C$ and hence there is a $B \in \mathcal{A}$ where $y = x_B$. We also have that $x_B = y \in A$. If $B \neq A$ then $x_B \in B$ and $x_B \in A$, which is not possible since B and A are disjoint as they are distinct elements of $\mathcal{A}$. So it must be that $B = A$ and hence $y = x_B = x_A$. Since y was arbitrary, this shows that $C \cap A$ has only a single element x_A. This suffices to show the choice axiom. $\square$

Exercise 19.10

Let A be a set; let $\{X_\alpha\}_{\alpha\in J}$ be an indexed family of spaces; and let $\{f_\alpha\}_{\alpha\in J}$ be an indexed family of functions $f_\alpha : A \to X_\alpha$.

(a) Show there is a unique coarsest topology $\mathcal{T}$ on A relative to which each of the functions f_α is continuous.

(b) Let

$$\mathcal{S}_\beta = \left\{ f_\beta^{-1}(U_\beta) \mid U_\beta \text{ is open in } X_\beta \right\},$$

and let $\mathcal{S} = \bigcup \mathcal{S}_\beta$. Show that $\mathcal{S}$ is a subbasis for $\mathcal{T}$.

(c) Show that a map $g : Y \to A$ is continuous relative to $\mathcal{T}$ if and only if each map $f_\alpha \circ g$ is continuous.

(d) Let $f : A \to \prod X_\alpha$ be defined by the equation

$$f(a) = (f_\alpha(a))_{\alpha\in J};$$

let Z denote the subspace $f(A)$ of the product space $\prod X_\alpha$. Show that the image under f of each element of $\mathcal{T}$ is an open set of Z.

Solution:

(a)

Proof. Let $\mathcal{C}$ be the collection of topologies on A relative to which each of the functions f_α is continuous. Clearly $\mathcal{C}$ is nonempty as the discrete topology is in $\mathcal{C}$ since every subset of A is open in it so that $f_\alpha(V_\alpha)$ is always open when V_α is open in X_α. Let $\mathcal{T} = \bigcap \mathcal{C}$, which is a topology on A by what was shown in Exercise 13.4 part (a). We claim that this is the unique coarsest topology such that each f_α is continuous relative to it. To see this suppose that $\mathcal{T}'$ is any topology in such that each f_α is continuous relative to it, hence $\mathcal{T}' \in \mathcal{C}$. Then, for any open $U \in \mathcal{T} = \bigcap \mathcal{C}$ we of course have that $U \in \mathcal{T}'$ since $\mathcal{T}' \in \mathcal{C}$. Hence $\mathcal{T} \subset \mathcal{T}'$ since U was arbitrary so that $\mathcal{T}$ is courser than $\mathcal{T}'$, noting that it could of course be that $\mathcal{T} = \mathcal{T}'$ as well. Since $\mathcal{T}'$ was artbitary, this shows the desired result.

Of course it also must be that $\mathcal{T}$ is unique since, for any other $\mathcal{T}'$ that is a coarsest element of $\mathcal{C}$, we just showed above that $\mathcal{T} \subset \mathcal{T}'$ since $\mathcal{T}' \in \mathcal{C}$. But also $\mathcal{T} \supset \mathcal{T}'$ since $\mathcal{T}'$ must be coarser than $\mathcal{T}$ since $\mathcal{T} \in \mathcal{C}$. This shows that $\mathcal{T} = \mathcal{T}'$ so that $\mathcal{T}$ is unique since $\mathcal{T}'$ was arbitrary. This also follows from the more general fact that any smallest element in an order or partial order is always unique, and inclusion is always at least a partial order. □

(b)

Proof. We show that $\mathcal{C}$ from part (a) is exactly the set of topologies on A that contain the subbasis $\mathcal{S}$. That is, we show that $\mathcal{T}' \in \mathcal{C}$ if and only if $\mathcal{S} \subset \mathcal{T}'$ when $\mathcal{T}'$ is a topology on A. Since the coarsest topology $\mathcal{T}$ from part (a) is defined as $\bigcap \mathcal{C}$, this shows that $\mathcal{T}$ is the topology generated from the subbasis $\mathcal{S}$ by Exercise 13.5.

($\Rightarrow$) Suppose that $\mathcal{T}' \in \mathcal{C}$ so that every f_α is continuous relative to $\mathcal{T}'$. Now consider any subbasis element $S \in \mathcal{S}$ so that $S = f_\beta^{-1}(U_\beta)$ for some $\beta \in J$ and some open set U_β in X_β. Then f_β is continuous relative to $\mathcal{T}'$ so that S is open with respect to $\mathcal{T}'$, and hence $S \in \mathcal{T}'$. This shows that $\mathcal{S} \subset \mathcal{T}'$ since S was arbitrary, hence $\mathcal{T}'$ contains $\mathcal{S}$.

($\Leftarrow$) Now suppose that $\mathcal{T}'$ is a topology on A that contains $\mathcal{S}$ so that $\mathcal{S} \subset \mathcal{T}'$. Consider any $\alpha \in J$ and any open set U_α of X_α. Then clearly $f_\alpha^{-1}(U_\alpha)$ is in $\mathcal{S}_\alpha$ so that it is also clearly in $\mathcal{S} = \bigcup \mathcal{S}_\beta$. Hence also $f_\alpha^{-1}(U_\alpha) \in \mathcal{T}'$ since $\mathcal{S} \subset \mathcal{T}'$. Therefore $f_\alpha^{-1}(U_\alpha)$ is open with respect to $\mathcal{T}'$, which shows that f_α is continuous relative to $\mathcal{T}'$ since U_α was an arbitrary open set of X_α. Since $\alpha \in J$ was also arbitrary, this shows that every f_α is continuous relative to $\mathcal{T}'$ so that $\mathcal{T}' \in \mathcal{C}$ by definition. □

(c)

Proof. ($\Rightarrow$) Suppose that $g : Y \to A$ is continuous relative to $\mathcal{T}$. Consider any $\alpha \in J$ and any open set U_α of X_α. Then $f_\alpha^{-1}(U_\alpha)$ is open with respect to $\mathcal{T}$ since f_α is continuous relative to $\mathcal{T}$ since every f_α is. It then follows that $g^{-1}(f_\alpha^{-1}(U_\alpha))$ is open in Y since g is continuous relative to $\mathcal{T}$. From Exercise 2.4 part (a) we have that $g^{-1}(f_\alpha^{-1}(U_\alpha)) = (f_\alpha \circ g)^{-1}(U_\alpha)$, which shows that $f_\alpha \circ g$ is continuous since U_α was an arbitrary open set of X_α. Since $\alpha \in J$ was arbitrary, this shows the desired result.

($\Leftarrow$) Now suppose that every $f_\alpha \circ g$ is continuous and consider any open set U of A with respect to $\mathcal{T}$. Then by part (b) we have that U is an arbitrary union of finite intersections of subbasis elements $f_\alpha^{-1}(U_\alpha)$ for $\alpha \in J$ and open U_α in X_α. It then follows from Exercise 2.2 parts (b) and (c) that $g^{-1}(U)$ is an arbitrary union of finite intersections of sets $g^{-1}(f_\alpha^{-1}(U_\alpha))$. Again we have that each $g^{-1}(f_\alpha^{-1}(U_\alpha)) = (f_\alpha \circ g)^{-1}(U_\alpha)$ by Exercise 2.4 part (a) so that each of these sets is open in Y since every $f_\alpha \circ g$ is continuous. Hence $g^{-1}(U)$ is open as well since it is the arbitrary union of finite intersections of these open sets and Y is a topological space. Since U was an arbitrary open set of A with respect to $\mathcal{T}$, this shows that g is continuous relative to $\mathcal{T}$ as desired. □

(d)

Proof. Suppose that U is any open set of A with respect to $\mathcal{T}$. Consider any $\mathbf{y} = (y_\alpha)_{\alpha \in J} \in f(U)$ so that there is an $a \in U$ where $f(a) = \mathbf{y}$. Since $a \in U$ and U is open in A, we have that there is a basis element B_A containing a where $B_A \subset U$. It then follows from part (b) that this basis element is a finite intersection of subbasis elements, hence $B_A = \bigcap_{\beta \in I} f_\beta^{-1}(U_\beta)$, where $I \subset J$ is finite and each U_β is open in X_β. Now define

$$V_\alpha = \begin{cases} U_\beta & \alpha \in I \\ X_\beta & \alpha \notin I \end{cases}$$

so that clearly the set $B_p = \prod V_\alpha$ is a basis element of $\prod X_\alpha$ in the product topology by Theorem 19.1 since I is finite. We then have that $B_Z = Z \cap B_p$ is a basis element of the subspace Z by Lemma 16.1.

Now, we have that $a \in U$ and $U \subset A$ so that $a \in A$ as well. It then follows that $\mathbf{y} = f(a) \in f(A) = Z$. For $\beta \in I$, we also have that $a \in f_\beta^{-1}(U_\beta)$ since the basis element $B_A = \bigcap_{\beta \in I} f_\beta^{-1}(U_\beta)$ contains a. Hence $f_\beta(a) \in U_\beta$. Since of course every other $f_\alpha(a) \in X_\alpha$ when $\alpha \notin I$, we have that $f_\alpha(a) \in V_\alpha$ for all $\alpha \in J$ and thus $\mathbf{y} = f(a) = (f_\alpha(a))_{\alpha \in J} \in \prod V_\alpha = B_p$. We therefore have that $\mathbf{y} \in Z \cap B_p = B_Z$ so that B_Z contains $\mathbf{y}$.

Lastly, consider any $\mathbf{z} = (z_\alpha)_{\alpha \in J} \in B_Z = Z \cap B_p$. Then $\mathbf{z} \in Z = f(A)$ so that there is an $x \in A$ where $f(x) = (f_\alpha(x))_{\alpha \in J} = \mathbf{z}$ and hence each $f_\alpha(x) = z_\alpha$. We also have that $\mathbf{z} \in B_p = \prod V_\alpha$ so that $z_\alpha \in V_\alpha$ for every $\alpha \in J$. In particular $f_\beta(x) = z_\beta \in V_\beta = U_\beta$ for all $\beta \in I$ so that $x \in f_\beta^{-1}(U_\beta)$. Therefore $x \in \bigcap_{\beta \in I} f_\beta^{-1}(U_\beta) = B_A$ so that also $x \in U$ since $B_A \subset U$. Then we have that $\mathbf{z} = f(x) \in f(U)$. Since $\mathbf{z}$ was arbitrary this shows that $B_Z \subset f(U)$.

We have thus shown that B_Z is a basis element of the subspace Z that contains $\mathbf{y}$ where $B_Z \subset f(U)$. Since $\mathbf{y}$ was an arbitrary element of $f(U)$, this suffices to show that $f(U)$ is open in the subspace Z as desired. □

§20 The Metric Topology

Exercise 20.1

(a) In $\mathbb{R}^n$, define

$$d'(\mathbf{x}, \mathbf{y}) = |x_1 - y_1| + \cdots + |x_n - y_n| \; .$$

Show that d' is a metric that induces the usual topology of $\mathbb{R}^n$. Sketch the basis elements under d' when $n = 2$.

(b) More generally, given $p \geq 1$, define

$$d'(\mathbf{x}, \mathbf{y}) = \left[\sum_{i=1}^{n} |x_i - y_i|^p \right]^{1/p}$$

for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Assume that d' is a metric. Show that it induces the usual topology on $\mathbb{R}^n$.

Solution:

Lemma 20.1.1. *If x and y are real and $x, y \geq 0$ then $x^p < y^p$ if and only if $x < y$, for all integers $p \geq 1$.*

Proof. First, if $x = 0$ then of course

$$x < y \Leftrightarrow 0 < y \Leftrightarrow 0 < y^p \Leftrightarrow 0^p < y^p \Leftrightarrow x^p < y^p$$

for any $p \geq 1$, so assume it what follows that $x > 0$. We show this by induction on p. First, for $p = 1$ we clearly have that $x^p = x$ and $y^p = y$ so that of course the biconditional holds. Now suppose that $x^p < y^p$ if and only if $x < y$. Suppose that $x < y$ so that $x^p < y^p$ follows by the induction hypothesis. We also have that $y > 0$ since $0 < x < y$ so that $y^p > 0$. Then

$$\begin{aligned} x^p &< y^p \\ x \cdot x^p &< x \cdot y^p && \text{(since } x > 0\text{)} \\ x \cdot x^p &< x \cdot y^p < y \cdot y^p && \text{(since } x < y \text{ and } y^p > 0\text{)} \\ x^{p+1} &< y^{p+1} \,. \end{aligned}$$

Now suppose that it is not true that $x < y$ so that $x \geq y$. It then follows from the induction hypothesis that $x^p \geq y^p$. Then we have

$$\begin{aligned} x^p &\geq y^p \\ x \cdot x^p &\geq x \cdot y^p && \text{(since } x > 0\text{)} \\ x \cdot x^p &\geq x \cdot y^p \geq y \cdot y^p && \text{(since } x \geq y \text{ and } y^p \geq 0 \text{ since } y \geq 0\text{)} \\ x^{p+1} &\geq y^{p+1} \,. \end{aligned}$$

Hence by the contrapositive we have that $x^{p+1} < y^{p+1}$ implies that $x < y$. This completes the induction. □

Corollary 20.1.2. *If x and y are real and $x, y \geq 0$ then $x^{1/p} < y^{1/p}$ if and only if $x < y$, for all integers $p \geq 1$.*

Proof. Consider any $p \geq 1$ and let $u = x^{1/p}$ and $v = y^{1/p}$. Then clearly we have $u, v \geq 0$ since $x, y \geq 0$. We then have by Lemma 20.1.1 that

$$\begin{aligned} u^p < v^p &\Leftrightarrow u < v \\ (x^{1/p})^p < (y^{1/p})^p &\Leftrightarrow x^{1/p} < y^{1/p} \\ x < y &\Leftrightarrow x^{1/p} < y^{1/p} \,, \end{aligned}$$

which is of course the desired result. □

Lemma 20.1.3. *For any $n, p \in \mathbb{Z}_+$ and a finite sequence $(x_i)_{i=1}^n$ where each $x_i \geq 0$,*

$$\sum_{i=1}^n x_i^p \leq \left(\sum_{i=1}^n x_i \right)^p \,.$$

Proof. For every $n \in \mathbb{Z}_+$, we show this by induction on p. For $p = 1$ we clearly have

$$\sum_{i=1}^n x_i^p = \sum_{i=1}^n x_i \leq \sum_{i=1}^n x_i = \left(\sum_{i=1}^n x_i \right)^p \,.$$

Now suppose that the hypothesis is true for p. Then we have

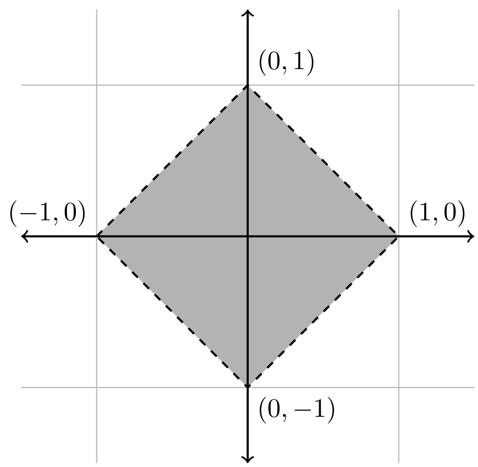
$$\left(\sum_{i=1}^n x_i \right)^{p+1} = \left(\sum_{i=1}^n x_i \right) \left(\sum_{i=1}^n x_i \right)^p$$

$$
\begin{aligned}
&\geq \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} x_i^p\right) && \text{(by the induction hypothesis since } \sum_{i=1}^{n} x_i \geq 0\text{)} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j^p \\
&= \sum_{i=1}^{n}\left(x_i x_i^p + \sum_{j\neq i} x_i x_j^p\right) \\
&= \sum_{i=1}^{n} x_i^{p+1} + \sum_{i=1}^{n}\sum_{j\neq i} x_i x_j^p \\
&\geq \sum_{i=1}^{n} x_i^{p+1}
\end{aligned}
$$

since each $x_i x_j^p \geq 0$ so that the double sum is as well. This completes the induction. □

Main Problem.

(a) First, the basis elements of the metric topology induced by d' are open intervals in $\mathbb{R}$, open diamonds in $n = 2$, open octahedrons for $n = 3$, and the higher dimensional analogues for $n > 3$. A sketch of the ball $B_{d'}(0 \times 0, 1)$ in $\mathbb{R}^2$ is shown below:



Now we show that d' is a metric and induces the usual topology of $\mathbb{R}^n$.

Proof. It is easy to see that d' meets the properties required of a metric. Clearly $d'(\mathbf{x}, \mathbf{y}) \geq 0$ since each $|x_i - y_i| \geq 0$, and $d'(\mathbf{x}, \mathbf{y}) = 0$ if and only if each $x_i = y_i$ so that $\mathbf{x} = \mathbf{y}$. Also it is obvious that $d'(\mathbf{x}, \mathbf{y}) = d'(\mathbf{y}, \mathbf{x})$ since each $|x_i - y_i| = |y_i - x_i|$. For the triangle inequality we simply have that

$$
\begin{aligned}
d'(\mathbf{x}, \mathbf{z}) &= \sum_{i=1}^{n} |x_i - z_i| \\
&\leq \sum_{i=1}^{n} (|x_i - y_i| + |y_i - z_i|) && \text{(since each } |x_i - z_i| \leq |x_i - y_i| + |y_i - z_i|\text{)} \\
&= \sum_{i=1}^{n} |x_i - y_i| + \sum_{i=1}^{n} |y_i - z_i|
\end{aligned}
$$

$$= d'(\mathbf{x}, \mathbf{y}) + d'(\mathbf{y}, \mathbf{z}) \,.$$

We now show that the metric topology induced by d' is the same as that induced by the square metric ρ, which shows the desired result since the square metric induces the standard product topology on $\mathbb{R}^n$ by Theorem 20.3. First consider any $\mathbf{x} \in \mathbb{R}^n$ and any $\epsilon > 0$. Let $\delta = \epsilon$ and consider any $\mathbf{y} \in B_{d'}(\mathbf{x}, \delta)$. Suppose also that j is an index in $\{1, \ldots, n\}$ where

$$\rho(\mathbf{x}, \mathbf{y}) = \max \{|x_1 - y_1|, \ldots, |x_n - y_n|\} = |x_j - y_j| \,.$$

Since $\mathbf{y} \in B_{d'}(\mathbf{x}, \delta)$, we have

$$\begin{aligned} d'(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} |x_i - y_i| < \delta = \epsilon \\ |x_j - y_j| + \sum_{i \neq j} |x_i - y_i| < \epsilon \\ |x_j - y_j| < \epsilon - \sum_{i \neq j} |x_i - y_i| \leq \epsilon \\ \rho(\mathbf{x}, \mathbf{y}) < \epsilon \end{aligned}$$

since of course $\sum_{i \neq j} |x_i - y_i| \geq 0$. Therefore $\mathbf{y} \in B_\rho(\mathbf{x}, \epsilon)$, which shows $B_{d'}(\mathbf{x}, \delta) \subset B_\rho(\mathbf{x}, \epsilon)$ so that the metric topology of d' is finer the the metric topology of ρ by Lemma 20.2.

Now again consider and $\mathbf{x} \in \mathbb{R}^n$ and $\epsilon > 0$, and this time let $\delta = \epsilon/n$. Consider any $\mathbf{y} \in B_\rho(\mathbf{x}, \delta)$ and again suppose also that j is an index in $\{1, \ldots, n\}$ where

$$\rho(\mathbf{x}, \mathbf{y}) = \max \{|x_1 - y_1|, \ldots, |x_n - y_n|\} = |x_j - y_j| \,.$$

We then have

$$\begin{aligned} |x_j - y_j| = \rho(\mathbf{x}, \mathbf{y}) < \delta = \epsilon/n \\ n\,|x_j - y_j| < \epsilon \,. \end{aligned}$$

We also have

$$\begin{aligned} d'(\mathbf{x}, \mathbf{y}) &= \sum_{i=1}^{n} |x_i - y_i| \\ &\leq \sum_{i=1}^{n} |x_j - y_j| && \text{(since each } |x_i - y_i| \leq |x_j - y_j|\text{)} \\ &= n\,|x_j - y_j| \\ &< \epsilon \end{aligned}$$

so that $\mathbf{y} \in B_{d'}(\mathbf{x}, \epsilon)$. Hence $B_\rho(\mathbf{x}, \delta) \subset B_{d'}(\mathbf{x}, \epsilon)$ so that the metric topology of ρ is also finer than that of d' again by Lemma 20.2. Therefore it must be that the two topologies are equal since each is finer than the other. □

(b) Let d denote the metric defined in part (a), that is

$$d(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} |x_i - y_i| \,.$$

First we show that the metric topology induced by d' is finer than that induced by ρ. So consider any $\mathbf{x} \in \mathbb{R}^n$ and $\epsilon > 0$. Let $\delta = \epsilon$ and suppose that $\mathbf{y} \in B_{d'}(\mathbf{x}, \delta)$ so that

$$d'(\mathbf{x}, \mathbf{y}) = \left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{1/p} < \delta = \epsilon \,.$$

Suppose that j is an index in $\{1, \ldots, n\}$ where

$$\rho(\mathbf{x}, \mathbf{y}) = \max \{|x_1 - y_1|, \ldots, |x_n - y_n|\} = |x_j - y_j| \,.$$

Then

$$|x_j - y_j|^p \le |x_j - y_j|^p + \sum_{i \ne j} |x_i - y_i|^p = \sum_{i=1}^{n} |x_i - y_i|^p$$

so that, by Corollary 20.1.2, we have

$$\begin{aligned} (|x_j - y_j|^p)^{1/p} \le \left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{1/p} &< \epsilon \\ |x_j - y_j| &< \epsilon \\ \rho(\mathbf{x}, \mathbf{y}) &< \epsilon \,. \end{aligned}$$

Therefore $\mathbf{y} \in B_\rho(\mathbf{x}, \epsilon)$ so that $B_{d'}(\mathbf{x}, \delta) \subset B_\rho(\mathbf{x}, \epsilon)$. This suffices to show that the metric topology induced by d' is finer than that induced by ρ by Lemma 20.2.

Now we show that the metric topology induced by d is finer than that induced by d'. So again consider any $\mathbf{x} \in \mathbb{R}^n$ and $\epsilon > 0$. Again let $\delta = \epsilon$ and suppose that $\mathbf{y} \in B_d(\mathbf{x}, \delta)$ so that

$$d(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} |x_i - y_i| < \delta = \epsilon \,.$$

Then, since each $|x_i - y_i| \ge 0$, we have by Lemma 20.1.3 that

$$\begin{aligned} \sum_{i=1}^{n} |x_i - y_i|^p &\le \left(\sum_{i=1}^{n} |x_i - y_i| \right)^p \\ \left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{1/p} &\le \left[\left(\sum_{i=1}^{n} |x_i - y_i| \right)^p \right]^{1/p} \\ d'(\mathbf{x}, \mathbf{y}) &\le \sum_{i=1}^{n} |x_i - y_i| < \epsilon \,, \end{aligned}$$

where we have used Corollary 20.1.2 in the second step. Thus $\mathbf{y} \in B_{d'}(\mathbf{x}, \epsilon)$ so that $B_d(\mathbf{x}, \delta) \subset B_{d'}(\mathbf{x}, \epsilon)$. This of course shows that the metric topology induced by d is finer than that induced by d' by Lemma 20.2 again.

Thus we have shown that the metric topology induced by d' is finer than that induced by ρ, and also that that induced by d is finer than that induced by d'. But it was shown in part (a) and Theorem 20.3 that those induced by d and ρ are the same topology, which is is the usual product topology on $\mathbb{R}^n$. Hence if $\mathcal{T}_p$ denotes this usual product topology, we have

$$\mathcal{T}_p = \mathcal{T}_\rho \subset \mathcal{T}_{d'} \subset \mathcal{T}_d = \mathcal{T}_\rho = \mathcal{T}_p \,.$$

So it must be that the metric topology induced by d' is this topology as well as desired.

Exercise 20.2

Show that $\mathbb{R} \times \mathbb{R}$ in the dictionary order topology is metrizable.

Solution:

Proof. In what follows let

$$\bar{d}(x, y) = \min\{|x - y|, 1\}$$

be the standard bounded metric on $\mathbb{R}$, noting that this is a metric by Theorem 20.1. Now define the function $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by

$$d(\mathbf{x}, \mathbf{y}) = \begin{cases} 1 & x_1 \neq y_1 \\ \bar{d}(x_2, y_2) & x_1 = y_1 \, . \end{cases}$$

We claim that this is a metric on $\mathbb{R}^2$ that induces the dictionary order topology.

First we show that d is a metric on $\mathbb{R}^2$. Clearly $d(\mathbf{x}, \mathbf{y}) \geq 0$ since both $1 \geq 0$ and $\bar{d}(x_2, y_2) \geq 0$ since $\bar{d}$ is a metric. Moreover if $\mathbf{x} = \mathbf{y}$ then $x_1 = y_1$ and $x_2 = y_2$ so that $d(\mathbf{x}, \mathbf{y}) = \bar{d}(x_2, y_2) = 0$. Conversely if $d(\mathbf{x}, \mathbf{y}) = 0$ then clearly $d(\mathbf{x}, \mathbf{y}) \neq 1$ so that it must be that $x_1 = y_1$ and $d(\mathbf{x}, \mathbf{y}) = \bar{d}(x_2, y_2) = 0$ so that $x_2 = y_2$ since $\bar{d}$ is a metric. From this it follows that $\mathbf{x} = \mathbf{y}$ since $x_1 = y_1$ and $x_2 = y_2$, which shows property (1) of a metric.

It is also obvious that $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x})$ since if $x_1 \neq y_1$ then $d(\mathbf{x}, \mathbf{y}) = 1 = d(\mathbf{y}, \mathbf{x})$. If $x_1 = x_2$ then $d(\mathbf{x}, \mathbf{y}) = \bar{d}(x_2, y_2) = \bar{d}(y_2, x_2) = d(\mathbf{y}, \mathbf{x})$ since $\bar{d}$ is a metric. This shows property (2) of a metric. Lastly, consider $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ in $\mathbb{R}^2$.

Case: $x_1 \neq z_1$. Then $d(\mathbf{x}, \mathbf{z}) = 1$ and it must be that either $y_1 \neq x_1$ or $y_1 \neq z_1$ since otherwise we would have that $x_1 = y_1 = z_1$. Thus either $d(\mathbf{x}, \mathbf{y}) = 1$ or $d(\mathbf{y}, \mathbf{z}) = 1$ and hence

$$d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z}) \geq 1 = d(\mathbf{x}, \mathbf{z})$$

since both $d(\mathbf{x}, \mathbf{y}) \geq 0$ and $d(\mathbf{y}, \mathbf{z}) \geq 0$.

Case: $x_1 = z_1$. Then $d(\mathbf{x}, \mathbf{z}) = \bar{d}(x_2, z_2)$. If $y_1 = x_1$ then $x_1 = y_1 = z_1$ so that

$$d(\mathbf{x}, \mathbf{z}) = \bar{d}(x_2, z_2) \leq \bar{d}(x_2, y_2) + \bar{d}(y_2, z_2) = d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z})$$

since $\bar{d}$ is a metric. If $y_1 \neq x_1$ then $y_1 \neq x_1 = z_1$, and hence $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{z}) = 1$ so that

$$d(\mathbf{x}, \mathbf{z}) = \bar{d}(x_2, z_2) \leq 1 \leq 2 = 1 + 1 = d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z})$$

since $\bar{d}$ is the bounded metric so that it is always at most 1.

Thus in all cases we have shown property (3) of a metric.

In what follows let $\prec$ denote the dictionary order on $\mathbb{R}^2$. To show that d induces the dictionary order topology, first consider any point $\mathbf{x} \in \mathbb{R}^2$ and any basis element B of the dictionary order topology that contains $\mathbf{x}$. Then of course $B = (\mathbf{a}, \mathbf{b})$, where $\mathbf{a} \prec \mathbf{x} \prec \mathbf{b}$ since the dictionary order has no largest or smallest elements in $\mathbb{R}^2$. Now define

$$\delta_a = \begin{cases} 1 & x_2 = a_2 \\ |x_2 - a_2| & x_2 \neq a_2 \end{cases}$$

and

$$\delta_b = \begin{cases} 1 & x_2 = b_2 \\ |x_2 - b_2| & x_2 \neq b_2 \, , \end{cases}$$

and let $\delta = \min\{1, \delta_a, \delta_b\}$. Clearly the set $B_d(\mathbf{x}, \delta)$ is a basis element of the topology induced by d, and we claim that $\mathbf{x} \in B_d(\mathbf{x}, \delta) \subset B$.

That $\mathbf{x} \in B_d(\mathbf{x}, \delta)$ is obvious. So now consider any $\mathbf{y} \in B_d(\mathbf{x}, \delta)$ so that $d(\mathbf{x}, \mathbf{y}) < \delta \leq 1$. Hence it cannot be that $x_1 \neq y_1$ by definition, since $d(\mathbf{x}, \mathbf{y}) = 1$ in that case, and so $x_1 = y_1$. If $x_2 = a_2$ then it has to be that $a_1 < x_1$ since otherwise it would not be the case that $\mathbf{a} \prec \mathbf{x}$. Thus we have $a_1 < x_1 = y_1$ so that $\mathbf{a} \prec \mathbf{y}$.

On the other hand if $x_2 \neq a_2$ then it must be that $a_1 \leq y_1$ since otherwise we would have $x_1 = y_1 < a_1$ so that $\mathbf{x} \prec \mathbf{a}$. If $a_1 < y_1 = x_1$ then of course $\mathbf{a} \prec \mathbf{y}$ so assume that $a_1 = y_1 = x_1$. The it must be that $a_2 < x_2$ since $\mathbf{a} \prec \mathbf{x}$, and so $|x_2 - a_2| = x_2 - a_2$. Then, since $x_1 = y_1$, we have that $\bar{d}(x_2, y_2) = d(\mathbf{x}, \mathbf{y}) < \delta \leq 1$ so it must be that $d(\mathbf{x}, \mathbf{y}) = \bar{d}(x_2, y_2) = |x_2 - y_2|$. Also $\delta_a = |x_2 - a_2| = x_2 - a_2$ since $x_2 \neq a_2$. Hence we have $|x_2 - y_2| = d(\mathbf{x}, \mathbf{y}) < \delta \leq \delta_a = x_2 - a_2$, from which it readily follows that $a_2 < y_2$ so that again $\mathbf{a} \prec \mathbf{y}$.

Therefore in all cases $\mathbf{a} \prec \mathbf{y}$. Analogous arguments show that $\mathbf{y} \prec \mathbf{b}$ so that $\mathbf{y} \in (\mathbf{a}, \mathbf{b}) = B$, which shows that $B_d(\mathbf{x}, \delta) \subset B$ as desired since $\mathbf{y}$ was arbitrary. This shows that the topology induced by d is finer than the dictionary order topology by Lemma 13.3.

Now again suppose that $\mathbf{x} \in \mathbb{R}^2$, and that $\epsilon' > 0$ and $\mathbf{x}' \in \mathbb{R}^2$ such that $B_d(\mathbf{x}', \epsilon')$ is an arbitrary basis element of the metric topology induced by d that contains $\mathbf{x}$. It was shown after the definition of a metric topology in the text that there is another ball $B_d(\mathbf{x}, \epsilon)$ centered at $\mathbf{x}$ such that $B_d(\mathbf{x}, \epsilon) \subset B_d(\mathbf{x}', \epsilon')$. Let $\delta = \min\{1, \epsilon\}$ and define $\mathbf{a} = x_1 \times (x_2 - \delta)$ and $\mathbf{b} = x_1 \times (x_2 + \delta)$. Set $B = (\mathbf{a}, \mathbf{b})$, which is clearly a basis element of the dictionary order topology. So consider any $\mathbf{y} \in B$ so that $\mathbf{a} \prec \mathbf{y} \prec \mathbf{b}$. Clearly it must be that $a_1 = y_1 = b_1 = x_1$ since otherwise we would have that $\mathbf{y} \prec \mathbf{a}$ or $\mathbf{b} \prec \mathbf{y}$. From this it follows that $a_2 = x_2 - \delta < y_2 < x_2 + \delta = b_2$ so that $-\delta < y_2 - x_2 < \delta$ and hence $|x_2 - y_2| < \delta$. Moreover, since $y_1 = x_1$ and $\delta \leq 1$, it follows that $d(\mathbf{x}, \mathbf{y}) = \bar{d}(x_2, y_2) = |x_2 - y_2| < \delta \leq \epsilon$. This shows that $\mathbf{y} \in B_d(\mathbf{x}, \epsilon)$, which shows that $B \subset B_d(\mathbf{x}, \epsilon) \subset B_d(\mathbf{x}', \epsilon')$ since $\mathbf{y}$ was arbitrary. This proves that the dictionary order topology is finer than the topology induced by d again by Lemma 13.3.

Since each is finer than the other the topologies must be the same, which shows that the dictionary order topology is metrizable as desired. □

Exercise 20.3

Let X be metric space with metric d.

(a) Show that $d : X \times X \to \mathbb{R}$ is continuous.

(b) Let X' denote a space having the same underlying set as X. Show that if $d : X' \times X' \to \mathbb{R}$ is continuous, then the topology of X' is finer than the topology of X.

One can summarize the result of this exercise as follows: If X has a metric d, then the topology induced by d is the coarsest topology relative to which the function d is continuous.

Solution:

(a) We use Theorem 18.1 part (4) to show that d is continuous. So consider any $x_1 \times x_2 \in X \times X$ and any neighborhood V of $z = d(x_1, x_2)$, noting that $V \subset \mathbb{R}$ since $\mathbb{R}$ is the range of d. Since V is open in $\mathbb{R}$, there is a basis element $B = (a, b)$ containing z where $B \subset V$. Hence $a < z < b$. Now let $\epsilon = \min\{(z - a)/2, (b - z)/2\}$, noting that $\epsilon > 0$ since $z > a$ and $b > z$. Next define $U_1 = B_d(x_1, \epsilon)$ and $U_2 = B_d(x_2, \epsilon)$ so that they are both basis elements and therefore open sets of the metric space X. It then follows that $U = U_1 \times U_2$ is a a basis element and therefore an open set of the product space $X \times X$. Clearly we have that $x_1 \in B_d(x_1, \epsilon) = U_1$ and $x_2 \in B_d(x_2, \epsilon) = U_2$ so that U contains $x_1 \times x_2$ and so is a neighborhood of $x_1 \times x_2$.

We claim that $d(U) \subset B$. To see this, consider any $w \in d(U)$ so that there is a $y_1 \times y_2 \in U = U_1 \times U_2$ such that $w = d(y_1, y_2)$. Therefore $y_1 \in U_1 = B_d(x_1, \epsilon)$ so that $d(y_1, x_1) < \epsilon$, and similarly $d(y_2, x_2) < \epsilon$ since $y_2 \in U_2 = B_d(x_2, \epsilon)$. Then, since d is a metric, we have

$$\begin{aligned} z = d(x_1, x_2) &\leq d(x_1, y_1) + d(y_1, x_2) \\ &\leq d(x_1, y_1) + d(y_1, y_2) + d(y_2, x_2) \\ &= d(y_1, x_1) + d(y_1, y_2) + d(y_2, x_2) \\ &< \epsilon + w + \epsilon \\ z &< w + 2\epsilon \leq w + 2(z-a)/2 = w + z - a \\ a &< w\,. \end{aligned}$$

Similarly, we have

$$\begin{aligned} w = d(y_1, y_2) &\leq d(y_1, x_1) + d(x_1, y_2) \\ &\leq d(y_1, x_1) + d(x_1, x_2) + d(x_2, y_2) \\ &= d(y_1, x_1) + d(x_1, x_2) + d(y_2, x_2) \\ &< \epsilon + z + \epsilon \\ w &< z + 2\epsilon \leq z + 2(b-z)/2 = z + b - z \\ w &< b\,. \end{aligned}$$

We therefore have that $a < w < b$ so that $w \in (a, b) = B$. This of course shows that $d(U) \subset B$ since w was arbitrary. Moreover, we have that $B \subset V$ so that clearly $d(U) \subset V$, which completes the proof of Theorem 18.1 part (4) so that d is continuous.

(b) Let U be any open set of X and consider any $x \in U$. Then clearly there is a basis element $B_d(y, \epsilon)$, for some $\epsilon > 0$ and $y \in U$, of the metric topology X that contains x and where $B_d(y, \epsilon) \subset U$. Now, since d is continuous with respect to $X' \times X'$, it follows from Exercise 18.11 that the function $d_y(z) = d(y, z)$ is a continuous function from X' to $\mathbb{R}$. Since clearly the interval $(-\infty, \epsilon)$ is open in $\mathbb{R}$, it then follows that the set

$$d_y^{-1}((-\infty, \epsilon)) = \{z \in X' \mid d_y(z) < \epsilon\} = \{z \in X \mid d(y, z) < \epsilon\} = B_d(y, \epsilon)$$

is also open in X'. Thus $B_d(y, \epsilon)$ is an open set in X' containing x such that $B_d(y, \epsilon) \subset U$. This shows that U is also open in X' by Exercise 13.1 since the point $x \in U$ was arbitrary. This suffices to show the desired result.

Exercise 20.4

Consider the product, uniform, and box topologies on $\mathbb{R}^\omega$,

(a) In which topologies are the following functions from $\mathbb{R}$ to $\mathbb{R}^\omega$ continuous?

$$\begin{aligned} f(t) &= (t, 2t, 3t, \ldots), \\ g(t) &= (t, t, t, \ldots), \\ h(t) &= (t, \tfrac{1}{2}t, \tfrac{1}{3}t, \ldots). \end{aligned}$$

(b) In which topologies do the following sequences converge?

$$\begin{aligned} \mathbf{w}_1 &= (1, 1, 1, 1, \ldots), & \mathbf{x}_1 &= (1, 1, 1, 1, \ldots), \\ \mathbf{w}_2 &= (0, 2, 2, 2, \ldots), & \mathbf{x}_2 &= (0, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \ldots), \\ \mathbf{w}_3 &= (0, 0, 3, 3, \ldots), & \mathbf{x}_3 &= (0, 0, \tfrac{1}{3}, \tfrac{1}{3}, \ldots), \end{aligned}$$

$$\begin{aligned} &\dots & &\dots \\ \mathbf{y}_1 &= (1,0,0,0,\dots), & \mathbf{z}_1 &= (1,1,0,0,\dots), \\ \mathbf{y}_2 &= (\tfrac{1}{2},\tfrac{1}{2},0,0,\dots), & \mathbf{z}_2 &= (\tfrac{1}{2},\tfrac{1}{2},0,0,\dots), \\ \mathbf{y}_3 &= (\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3},0,\dots), & \mathbf{z}_3 &= (\tfrac{1}{3},\tfrac{1}{3},0,0,\dots), \\ &\dots & &\dots \end{aligned}$$

Solution:

Lemma 20.4.1. *Suppose that X is a metric space with metric d. If U is an open set of X containing a point x then there is a ball $B_d(x, \epsilon)$ centered at x that is contained in U.*

Proof. The main part of this proof was given after the definition of a metric topology in the text, but we repeat it here for completeness.

By the definition of the metric topology, there is a $\delta > 0$ and $y \in X$ such that the basis element $B_d(y, \delta)$ contains x and is contained in U. Let $\epsilon = \delta - d(x, y)$ so that $d(x, y) = \delta - \epsilon$, noting that $\epsilon > 0$ since $x \in B_d(y, \delta)$ so that $d(x, y) < \delta$. Then, for any $z \in B_d(x, \epsilon)$, we have that $d(z, x) < \epsilon$ and so

$$d(z, y) \leq d(z, x) + d(x, y) = d(z, x) + \delta - \epsilon < \epsilon + \delta - \epsilon = \delta$$

since d is a metric. Hence $z \in B_d(y, \delta)$ so that $B_d(x, \epsilon) \subset B_d(y, \delta) \subset U$ as desired since z was arbitrary. □

Lemma 20.4.2. *Suppose that X is a topological space and Y and Y' are topological spaces on the same set, and that Y' is finer than Y. Suppose also that $f : X \to Y$ so that of course it is also a function from X to Y'. We assert the following:*

(1) If f is continuous with respect to Y' then it is also continuous with respect to Y.

(2) If f is not *continuous with respect to Y then it is also not continuous with respect to Y'.*

(3) If a sequence in Y' converges to a point y_0, then it also converges to y_0 in Y.

(4) If a sequence in Y does not converge to a point y_0, then it also does not converge to y_0 in Y'.

(5) If a sequence in Y does not converge at all, then it also does not converge at all in Y'.

(6) If Y is a Hausdorff space, then so is Y'.

Proof. For assertion (1) suppose that f is continuous with respect to Y' and let U be any open set of Y. Since Y' is finer than Y, it follows that U is also open in Y'. Then, since f is continuous with respect to Y' we have that $f^{-1}(U)$ is open in X, which suffices to show that f is continuous with respect to Y since U was an arbitrary open set. Assertion (2) follows immediately from the contrapositive of (1).

Regarding (3), suppose that a sequence $(y_1, y_2, \dots)$ converges to y_0 in Y' and let U be any neighborhood of y_0 in Y. Then U is also open in Y' since it is finer than Y, hence U is a neighborhood of y_0 in Y'. Thus there is an $N \in \mathbb{Z}_+$ such that $x_n \in U$ for all $n \geq N$, since the sequence converges to y_0 in Y'. Since U was an arbitrary neighborhood of Y, this shows that the sequence converges to y_0 in Y. Assertion (4) follows immediately from the contrapositive of (3). Assertion (5) then immediately follows from (4) since, if a sequence does not converge at all in Y then for any point $y_0 \in Y$, it does not converge to y_0. Then it also does not converge to y_0 in Y' by (4). Since y_0 was arbitrary, this shows that it does not converge at all in Y'.

For (6), suppose that Y is a Hausdorff space and let x and y be distinct points of Y' so that they are of course also points of Y. Hence there are neighborhoods U and V of x and y, respectively, in Y that are disjoint. Since Y' is finer than Y, we have that U and V are also open sets of Y' and thus are disjoint neighborhoods of x and y in Y' as well. This suffices to show that Y' is Hausdorff as desired. □

Main Problem.

(a) Regarding whether or not the functions are continuous in the various topologies, we claim the following:

	Product	Uniform	Box
f	Yes	No	No
g	Yes	Yes	No
h	Yes	Yes	No

Proof. First, the functions f, g, and h can all be considered as special cases of the more general function

$$s(t) = (s_n(t))_{n\in\mathbb{Z}_+} ,$$

where each $s_n(t) = \alpha_n t$, and $\alpha_n = n$ for f, $\alpha_n = 1$ for g, and $\alpha_n = 1/n$ for h.

Clearly each s_n is continuous for the three α_n by elementary calculus so that s is continuous in the product topology by Theorem 19.6 for all three α_n. We can show that s is *not* continuous in the box topology for all three α_n with a single example. Consider the set $B = \prod_{n\in\mathbb{Z}_+}(-1/n^2, 1/n^2)$, which is clearly a basis element of the box topology and so is open. Similar to Example 19.2, if s were continuous then there would be an interval $(-\delta, \delta)$ about the point 0 such that $s((-\delta, \delta)) \subset B$, where of course $\delta > 0$. This would of course mean that

$$s_n((-\delta, \delta)) = (-\alpha_n\delta, \alpha_n\delta) \subset (-1/n^2, 1/n^2)$$

for all $n \in \mathbb{Z}_+$. However, since clearly there is an $n \in \mathbb{Z}_+$ large enough that

$$n^3\delta \geq n^2\delta \geq n\delta > 1 ,$$

we have that

$$n\delta \geq \delta \geq \delta/n > 1/n^2 ,$$

and hence for all three functions we have that $\alpha_n\delta > 1/n^2$ so that

$$s_n((-\delta, \delta)) = (-\alpha_n\delta, \alpha_n\delta) \not\subset (-1/n^2, 1/n^2) .$$

This shows that s cannot be continuous with respect to the box topology for all three α_n.

Next we show that f is *not* continuous in the uniform topology. First, suppose that $\bar{\rho}$ is the metric that induces the uniform topology, i.e.

$$\bar{\rho}(\mathbf{x}, \mathbf{y}) = \sup\left\{\bar{d}(x_n, y_n) \mid n \in \mathbb{Z}_+\right\} .$$

Now consider the basis element and open set $B_{\bar{\rho}}(\mathbf{0}, 1)$ in the uniform topology. If f were continuous then there would be a $\delta > 0$ such that

$$f((-\delta, \delta)) = \prod_{n\in\mathbb{Z}_+}(-\delta n, \delta n) \subset B_{\bar{\rho}}(\mathbf{0}, 1) .$$

Clearly there is an $n \in \mathbb{Z}_+$ large enough such that $n > 1/\delta$ so that $\delta n > 1$. Then consider the point $\mathbf{x} \in \mathbb{R}^\omega$ defined by

$$x_m = \begin{cases} 0 & m \neq n+1 \\ \delta n & m = n+1 \,. \end{cases}$$

We then have of course that $-(n+1)\delta < 0 < n\delta = x_{n+1} < (n+1)\delta$ so that $x_{n+1} \in (-(n+1)\delta, (n+1)\delta)$. It then follows that $\mathbf{x} \in f((-\delta, \delta)$. However, we also have that $\bar{d}(\delta n, 0) = \max\{|\delta n - 0|, 1\} = \max\{\delta n, 1\} = 1$ since $\delta n > 1$. Hence it is not true that $\bar{\rho}(\mathbf{x}, \mathbf{0}) < 1$ so that $\mathbf{x} \notin B_{\bar{\rho}}(\mathbf{0}, 1)$. Thus $f((-\delta, \delta)) \not\subset B_{\bar{\rho}}(\mathbf{0}, 1)$ so that f is not continuous in the uniform topology.

Next we show that g and h *are* continuous in the uniform topology at the same time, which we show using Theorem 18.1 part (4). Consider any real u and any neighborhood V of $\mathbf{x} = g(u)$ (or $\mathbf{x} = h(u)$) in the uniform topology. Then by Lemma 20.4.1 there is an $\epsilon > 0$ such that the basis element $B_{\bar{\rho}}(\mathbf{x}, \epsilon)$ is a subset of V. Now consider the basis element and open set $U = B_d(u, \epsilon/2)$, where d denotes the usual metric on $\mathbb{R}$. Obviously U contains u but we also claim that $g(U) \subset V$ (or $h(U) \subset V$), thereby completing the proof.

So consider any $\mathbf{y} \in g(U)$ (or $\mathbf{y} \in h(U)$) so that there is some $v \in U$ such that $\mathbf{y} = g(v)$ (or $\mathbf{y} = h(v)$) In the case of g we have that $\mathbf{x} = g(u) = (u, u, u, \ldots)$, which is to say that $x_n = u$ for all $n \in \mathbb{Z}_+$. Similarly $y_n = v$ for all $n \in \mathbb{Z}_+$ since $\mathbf{y} = g(v)$. Now, since $v \in U = B_d(u, \epsilon/2)$, we have that

$$\bar{d}(y_n, x_n) \leq d(y_n, x_n) = d(v, u) < \epsilon/2$$

for all $n \in \mathbb{Z}_+$. From this it follows that

$$\bar{\rho}(\mathbf{y}, \mathbf{x}) = \sup\left\{\bar{d}(y_n, x_n) \mid n \in \mathbb{Z}_+\right\} \leq \epsilon/2 < \epsilon \,.$$

Likewise in the case of h we have that $x_n = u/n$ and $y_n = v/n$ for all $n \in \mathbb{Z}_+$ since $\mathbf{x} = h(u)$ and $\mathbf{y} = h(v)$. We therefore have that

$$\bar{d}(y_n, x_n) \leq d(y_n, x_n) = |y_n - x_n| = |v/n - u/n| = \left|\frac{v-u}{n}\right| = \frac{1}{n}|v - u| = \frac{1}{n} d(v, u) < \frac{\epsilon/2}{n} \leq \epsilon/2$$

for all $n \in \mathbb{Z}_+$ since every $n \geq 1$. Hence again

$$\bar{\rho}(\mathbf{y}, \mathbf{x}) = \sup\left\{\bar{d}(y_n, x_n) \mid n \in \mathbb{Z}_+\right\} \leq \epsilon/2 < \epsilon \,.$$

Therefore for both functions we have $\bar{\rho}(\mathbf{y}, \mathbf{x}) < \epsilon$ so that $\mathbf{y} \in B_{\bar{\rho}}(\mathbf{x}, \epsilon)$. This shows that $g(U) \subset B_{\bar{\rho}}(\mathbf{x}, \epsilon) \subset V$ (or $h(U) \subset B_{\bar{\rho}}(\mathbf{x}, \epsilon) \subset V$) as desired since $\mathbf{y}$ was arbitrary. □

(b) First we note that, since $\mathbb{R}$ is a Hausdorff space, $\mathbb{R}^\omega$ is as well in both the box and product topologies by Theorem 19.4. Therefore the uniform topology on $\mathbb{R}^\omega$ is also Hausdorff by Lemma 20.4.2 part (6) since it is finer than the product topology. It then follows from Theorem 17.10 that if any of the sequences converge in any of the three topologies, then they converge to a unique point.

Regarding whether the sequences converge in the various topologies then, we claim

	Product	Uniform	Box
$\mathbf{w}$	Yes	No	No
$\mathbf{x}$	Yes	Yes	No
$\mathbf{y}$	Yes	Yes	No
$\mathbf{z}$	Yes	Yes	Yes

Proof. Now, regarding the **w** sequence, each element in the sequence is defined as

$$\mathbf{w}_n = (w_{n,1}, w_{n,2}, w_{n,3}, \ldots),$$

where

$$w_{n,m} = \begin{cases} 0 & m < n \\ n & m \geq n \end{cases}$$

for $n, m \in \mathbb{Z}_+$.

First we show that the **w** sequence converges to the point $\mathbf{0}$ in the product topology. So consider any neighborhood U of $\mathbf{0}$ in the product topology so that there is a basis element B containing $\mathbf{0}$ where $B \subset U$. Then $B = \prod_{m=1}^{\infty} B_m$ where each B_m is open and B_m is all of $\mathbb{R}$ for all but finitely many values of m. Let J then be a finite subset of $\mathbb{Z}_+$ where each $B_m = \mathbb{R}$ for $m \notin J$. Of course we also have that $0 \in B_m$ for all $m \in \mathbb{Z}_+$ since B contains $\mathbf{0}$.

Then J has a largest element N since it is a finite set of positive integers. Now consider any $n \geq N+1$ and any $m \in \mathbb{Z}_+$. If $m \in J$ then we have that $m \leq N < N+1 \leq n$ since N is the largest element of J, and hence $w_{n,m} = 0 \in B_m$. If $m \notin J$ then of course $B_m = \mathbb{R}$ so that of course $w_{n,m} \in \mathbb{R} = B_m$ regardless of whether $w_{n,m} = 0$ or $w_{n,m} = n$. Hence either way we have $w_{n,m} \in B_m$, which shows that $\mathbf{w}_n \in \prod_{m=1}^{\infty} B_m = B$ since m was arbitrary. Thus also $\mathbf{w}_n \in U$ since $B \subset U$. Since $n \geq N+1$ was arbitrary and U was an arbitrary neighborhood of $\mathbf{0}$, this shows that the sequence converges to $\mathbf{0}$ as desired.

Next we show that the **w** sequence does not converge in the uniform topology. It suffices to show that the sequence does not converge to $\mathbf{0}$, since if it converged to any other point $\mathbf{x}$, then by Lemma 20.4.2 part (3) it would also converge to $\mathbf{x}$ in the product topology since it is coarser than the uniform topology. However, this would violate the fact that the sequence converges to $\mathbf{0}$ in the product topology (just shown above), and so cannot also converge to $\mathbf{x} \neq \mathbf{0}$ since the convergence point is unique as noted above.

So consider the neighborhood $B_{\bar{\rho}}(\mathbf{0}, 1)$ of $\mathbf{0}$ in the uniform topology. We claim that no elements of the sequence are in this neighborhood so that it clearly cannot converge to $\mathbf{0}$. So consider any $n \in \mathbb{Z}_+$ so that we clearly have $w_{n,n} = n \geq 1 > 0$. Therefore $d(w_{n,n}, 0) = |w_{n,n} - 0| = |w_{n,n}| = w_{n,n} \geq 1$, from which it follows that it has to be that $\bar{d}(w_{n,n}, 0) = 1$. This of course implies that

$$\bar{\rho}(\mathbf{w}_n, \mathbf{0}) = \sup\left\{\bar{d}(w_{n,m}, 0) \mid m \in \mathbb{Z}_+\right\} \geq 1.$$

Hence it is not true that $\bar{\rho}(\mathbf{w}_n, \mathbf{0}) < 1$ so that $\mathbf{w}_n \notin B_{\bar{\rho}}(\mathbf{0}, 1)$. This shows the desired result since n was arbitrary.

It then follows that the **w** sequence also does not converge at all in the box topology by Lemma 20.4.2 part (5) since it is finer than the uniform topology.

Regarding the **x** sequence, the definition is that each

$$\mathbf{x}_n = (x_{n,1}, x_{n,2}, x_{n,3}, \ldots),$$

where

$$x_{n,m} = \begin{cases} 0 & m < n \\ 1/n & m \geq n \end{cases}$$

for $n, m \in \mathbb{Z}_+$.

First we show that this sequence converges to $\mathbf{0}$ in the uniform topology, which is of course is the unique convergence point. So consider any neighborhood U of $\mathbf{0}$ in the uniform topology so that by

Lemma 20.4.1 there is an $\epsilon > 0$ where $B_{\bar{\rho}}(\mathbf{0}, \epsilon) \subset U$. Then there is a positive integer N large enough so that $N > 2/\epsilon$ so that, for any $n \geq N$ we have $1/n \leq 1/N < \epsilon/2$. Next consider any such $n \geq N$ and any $m \in \mathbb{Z}_+$. Since $x_{n,m}$ is either 0 or $1/n$ we have that $|x_{n,m}| = x_{n,m} \leq 1/n$ so that

$$\bar{d}(x_{n,m}, 0) \leq d(x_{n,m}, 0) = |x_{n,m} - 0| = |x_{n,m}| = x_{n,m} \leq 1/n < \epsilon/2 .$$

Thus, since m was arbitrary, it follows that

$$\bar{\rho}(\mathbf{x}_n, \mathbf{0}) = \sup \left\{ \bar{d}(x_{n,m}, 0) \mid m \in \mathbb{Z}_+ \right\} \leq \epsilon/2 < \epsilon ,$$

and hence $\mathbf{x}_n \in B_{\bar{\rho}}(\mathbf{0}, \epsilon)$. Thus also $\mathbf{x}_n \in U$ since $B_{\bar{\rho}}(\mathbf{0}, \epsilon) \subset U$. Since $n \geq N$ was arbitrary as was the neighborhood U, this shows that the sequence converges to $\mathbf{0}$ as desired.

Since it is coarser than the uniform topology, it follows that the $\mathbf{x}$ sequence also converges to $\mathbf{0}$ in the product topology as well by Lemma 20.4.2 part (3).

Next we show that the $\mathbf{x}$ sequence does not converge in the box topology, for which it suffices to show that it does not converge to $\mathbf{0}$. Again, this is because, if it were to converge to some other point $\mathbf{y} \neq 0$ in the box topology, then it would also converge to $\mathbf{y}$ in the uniform topology since it is coarser (Lemma 20.4.2 part (3)), but this would violate the fact that it converges to the unique point $\mathbf{0}$ by what was just shown. So consider the basis element and open set of the box topology $U = \prod_{n=1}^{\infty} U_n$ where each $U_n = (-1/n, 1/n)$. Clearly U contains $\mathbf{0}$ so that it is a neighborhood of $\mathbf{0}$. We claim that no element of the sequence is in U, which of course suffices to show that it cannot converge to $\mathbf{0}$. So consider any $n \in \mathbb{Z}_+$ and so that $x_{n,n} = 1/n \geq 1/n$ so that $x_{n,n} \notin (-1/n, 1/n) = U_n$. From this it follows that $\mathbf{x}_n \notin \prod_{n=1}^{\infty} U_n = U$. Since n was arbitrary this shows no element of the sequence is in U so that the sequence cannot converge to $\mathbf{0}$.

Regarding the $\mathbf{y}$ sequence, it is defined as

$$\mathbf{y}_n = (y_{n,1}, y_{n,2}, y_{n,3}, \ldots) ,$$

where

$$y_{n,m} = \begin{cases} 1/n & m \leq n \\ 0 & m > n \end{cases}$$

for $n, m \in \mathbb{Z}_+$. Since $y_{n,m}$ is always either 0 or $1/n$, the same argument that shows that the $\mathbf{x}$ sequence converges to $\mathbf{0}$ in the uniform topology shows that the $\mathbf{y}$ sequence does as well. Of course this also mean that it converges to $\mathbf{0}$ in the product topology as well since it is coarser. Similarly, the same argument that shows that the $\mathbf{x}$ sequence does not converge in the box topology applies to $\mathbf{y}$ as well since we have that $y_{n,n} = x_{n,n} = 1/n$ for all $n \in \mathbb{Z}_+$.

Now, the $\mathbf{z}$ sequence is defined by

$$\mathbf{z}_n = (z_{n,1}, z_{n,2}, z_{n,3}, \ldots) ,$$

where

$$z_{n,m} = \begin{cases} 1/n & m \leq 2 \\ 0 & m > 2 \end{cases}$$

for $n, m \in \mathbb{Z}_+$.

We show that this sequence converges to $\mathbf{0}$ in the box topology. So consider any neighborhood U of $\mathbf{0}$ in the box topology so that there is a basis element $B = \prod_{m=1}^{\infty} B_m$ containing $\mathbf{0}$ where $B \subset U$. Of course then each B_m is open in $\mathbb{R}$ and $0 \in B_m$. Considering the standard topology of $\mathbb{R}$ using the metric topology basis, there is then an $\epsilon_1 > 0$ such that $B_d(0, \epsilon_1) \subset B_1$ by Lemma 20.4.1 since

B_1 is open and contains 0. Likewise there is an $\epsilon_2 > 0$ where $B_d(0, \epsilon_2) \subset B_2$. So set $\epsilon = \min\{\epsilon_1, \epsilon_2\}$ so that of course there is a positive integer N large enough that $N > 1/\epsilon$. Then, for any $n \geq N$ we have that $n \geq N > 1/\epsilon$ so that $1/n < \epsilon \leq \epsilon_1$, and similarly $1/n < \epsilon \leq \epsilon_2$. Now consider any $m \in \mathbb{Z}_+$. If $m \leq 2$ then of course either $m = 1$ or $m = 2$ so that, either way, we have

$$d(z_{n,m}, 0) = |z_{n,m} - 0| = |z_{n,m}| = |1/n| = 1/n < \epsilon \leq \epsilon_m$$

so that $z_{n,m} \in B_d(0, \epsilon_m) \subset B_m$. If $m > 2$ then we clear have $z_{n,m} = 0 \in B_m$ as well. Since m was arbitrary, this shows that $\mathbf{z}_n \in \prod_{m=1}^{\infty} B_m = B \subset U$. This shows that the sequence converges to $\mathbf{0}$ since $n \geq N$ was arbitrary and U was any neighborhood of $\mathbf{0}$.

Of course this also shows that the $\mathbf{z}$ sequence converges to $\mathbf{0}$ in the uniform and product topologies as well by Lemma 20.4.2 part (3) since they are both coarser than the box topology. □

Exercise 20.5

Let $\mathbb{R}^\infty$ be the subset of $\mathbb{R}^\omega$ consisting of all sequences that are eventually zero. What is the closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ in the uniform topology? Justify your answer.

Solution:

Let $\mathbb{R}^0$ denote the subset of $\mathbb{R}^\omega$ consisting of all sequences that converge to zero. We then claim that $\overline{\mathbb{R}^\infty} = \mathbb{R}^0$, i.e. the closure of $\mathbb{R}^\infty$ is $\mathbb{R}^0$.

Proof. ($\subset$) We show this by contrapositive. So suppose that $\mathbf{x} \notin \mathbb{R}^0$ so that the sequence $\mathbf{x}$ does *not* converge to zero. Then there is a neighborhood V of 0 in the standard topology on $\mathbb{R}$ such that, for any $N \in \mathbb{Z}_+$, there is an $n \geq N$ where $x_n \notin V$. It also follows from Lemma 20.4.1 that there is an $\epsilon > 0$ such that $B_d(0, \epsilon) \subset V$. So let $\delta = \min\{\epsilon, 1\}$ and consider the set $B_{\bar{\rho}}(\mathbf{x}, \delta)$, which is clearly a neighborhood of $\mathbf{x}$ in the uniform topology. Also suppose that $\mathbf{y}$ is any element of $\mathbb{R}^\infty$ so that $\mathbf{y}$ is eventually zero. Then there must be an N where $y_n = 0$ for all $n \geq N$. From before we have that there is a specific $n \geq N$ where $x_n \notin V$ so that also $x_n \notin B_d(0, \epsilon)$, and thus

$$d(y_n, x_n) = d(0, x_n) = d(x_n, 0) \geq \epsilon \geq \delta\,.$$

Then we have that both $d(y_n, x_n) \geq \delta$ and $1 \geq \delta$ so that

$$\bar{d}(y_n, x_n) = \min\{d(y_n, x_n), 1\} \geq \delta\,.$$

From this it clearly follows that

$$\bar{\rho}(\mathbf{y}, \mathbf{x}) = \sup\left\{\bar{d}(y_n, x_n) \mid n \in \mathbb{Z}_+\right\} \geq \delta$$

so that $\mathbf{y} \notin B_{\bar{\rho}}(\mathbf{x}, \delta)$. Since $\mathbf{y}$ was arbitrary, this shows that $B_{\bar{\rho}}(\mathbf{x}, \delta)$ does not intersect $\mathbb{R}^\infty$. This in turn shows that $\mathbf{x}$ is not a limit point of $\mathbb{R}^\infty$. Now, clearly also $\mathbf{x}$ cannot be eventually zero since then it would converge to zero, hence $\mathbf{x} \notin \mathbb{R}^\infty$ either. Therefore $\mathbf{x}$ cannot be in the closure of $\mathbb{R}^\infty$. Hence by the contrapositive we have that $\overline{\mathbb{R}^\infty} \subset \mathbb{R}^0$.

($\supset$) Now consider any $\mathbf{x} \in \mathbb{R}^0$ so that the sequence $\mathbf{x}$ converges to zero. Consider any neighborhood U of $\mathbf{x}$ in the uniform topology so that, by Lemma 20.4.1, there is an $\epsilon > 0$ such that $B_{\bar{\rho}}(\mathbf{x}, \epsilon) \subset U$. Now, since $B_d(0, \epsilon/2)$ is a neighborhood of 0, it follows that there is an $N \in \mathbb{Z}_+$ where $x_n \in B_d(0, \epsilon/2)$ for all $n \geq N$ since $\mathbf{x}$ converges to 0. Now define the sequence $\mathbf{y}$ where

$$y_n = \begin{cases} x_n & n < N \\ 0 & n \geq N \end{cases}$$

for $n \in \mathbb{Z}_+$. Clearly $\mathbf{y}$ is eventually zero so that $\mathbf{y} \in \mathbb{R}^\infty$.

We also claim that $\mathbf{y} \in U$. To see this consider any $n \in \mathbb{Z}_+$. If $n < N$ then clearly $y_n = x_n$ so that

$$\bar{d}(y_n, x_n) \leq d(y_n, x_n) = d(x_n, x_n) = 0 < \epsilon/2 \,.$$

If $n \geq N$ then $y_n = 0$ and we have from before that $x_n \in B_d(0, \epsilon/2)$ and hence

$$\bar{d}(y_n, x_n) \leq d(y_n, x_n) = d(0, x_n) = d(x_n, 0) < \epsilon/2 \,.$$

Hence it follows that

$$\bar{\rho}(\mathbf{y}, \mathbf{x}) = \sup \left\{ \bar{d}(y_n, x_n) \mid n \in \mathbb{Z}_+ \right\} \leq \epsilon/2 < \epsilon$$

so that $\mathbf{y} \in B_{\bar{\rho}}(\mathbf{x}, \epsilon) \subset U$. This shows that $\mathbf{y} \in \mathbb{R}^\infty \cap U$. If $\mathbf{y} = \mathbf{x}$ then of course $\mathbf{x} = \mathbf{y} \in \mathbb{R}^\infty$ itself. If $\mathbf{y} \neq \mathbf{x}$ then we have shown that $\mathbf{x}$ is a limit point of $\mathbb{R}^\infty$ since U was an arbitrary neighborhood. Thus either way $\mathbf{y} \in \overline{\mathbb{R}^\infty}$ so that $\overline{\mathbb{R}^\infty} \supset \mathbb{R}^0$ since $\mathbf{x}$ was arbitrary. □

Lastly, we note that $\mathbb{R}^\infty$ is a proper subset of its closure $\overline{\mathbb{R}^\infty} = \mathbb{R}^0$ since, for example, the sequence $\mathbf{x}$ defined by $x_n = 1/n$ for all $n \in \mathbb{Z}_+$ clearly converges to zero so is in $\mathbb{R}^0$ but is not eventually zero so is not in $\mathbb{R}^\infty$.

Exercise 20.6

Let $\bar{\rho}$ be the uniform metric on $\mathbb{R}^\omega$. Given $\mathbf{x} = (x_1, x_2, \ldots) \in \mathbb{R}^\omega$ and given $0 < \epsilon < 1$, let

$$U(\mathbf{x}, \epsilon) = (x_1 - \epsilon, x_1 + \epsilon) \times \cdots \times (x_n - \epsilon, x_n + \epsilon) \times \cdots \,.$$

(a) Show that $U(\mathbf{x}, \epsilon)$ is not equal to the ϵ-ball $B_{\bar{\rho}}(\mathbf{x}, \epsilon)$.

(b) Show that $U(\mathbf{x}, \epsilon)$ is not even open in the uniform topology.

(c) Show that

$$B_{\bar{\rho}}(\mathbf{x}, \epsilon) = \bigcup_{\delta < \epsilon} U(\mathbf{x}, \delta) \,.$$

Solution:

(a)

Proof. We show that $B_{\bar{\rho}}(\mathbf{x}, \epsilon)$ is not a subset of $U(\mathbf{x}, \epsilon)$, which of course suffices to show that they cannot be equal. To this end we define the point $\mathbf{y}$ in $\mathbb{R}^\omega$ by

$$y_n = x_n + \epsilon \left(1 - \frac{1}{n}\right)$$

for any $n \in \mathbb{Z}_+$. Then, for any such n, we clearly have that

$$\begin{aligned} n &\geq 1 \\ -n &\leq -1 < 0 \\ -1 &\leq -\frac{1}{n} < 0 && \text{(since } n > 0\text{)} \\ 0 &\leq 1 - \frac{1}{n} < 1 \end{aligned}$$

$$
\begin{aligned}
0 \le \epsilon\left(1-\frac{1}{n}\right) &< \epsilon && \text{(since } \epsilon > 0\text{)}\\
x_n \le x_n + \epsilon\left(1-\frac{1}{n}\right) &< x_n + \epsilon\\
x_n - \epsilon < x_n \le y_n &< x_n + \epsilon
\end{aligned}
$$

so that $y_n \in (x_n - \epsilon, x_n + \epsilon)$. Since n was arbitrary, this shows that $\mathbf{y} \in \prod_{n=1}^{\infty}(x_n - \epsilon, x_n + \epsilon) = U(\mathbf{x}, \epsilon)$. However, again for any $n \in \mathbb{Z}_+$, it was shown that $x_n \le y_n$ so that

$$
d(y_n, x_n) = |y_n - x_n| = y_n - x_n = x_n + \epsilon\left(1-\frac{1}{n}\right) - x_n = \epsilon\left(1-\frac{1}{n}\right).
$$

It was shown above that

$$
\begin{aligned}
0 \le \epsilon\left(1-\frac{1}{n}\right) &< \epsilon < 1\\
0 \le d(y_n, x_n) &< \epsilon < 1
\end{aligned}
$$

so that

$$
\bar{d}(y_n, x_n) = \min\{d(y_n, x_n), 1\} = d(y_n, x_n).
$$

We then clearly have

$$
\lim_{n\to\infty} \bar{d}(y_n, x_n) = \lim_{n\to\infty} d(y_n, x_n) = \lim_{n\to\infty} \epsilon\left(1-\frac{1}{n}\right) = \epsilon
$$

so that

$$
\bar{\rho}(\mathbf{y}, \mathbf{x}) = \sup_{n\in\mathbb{Z}_+} \bar{d}(y_n, x_n) = \epsilon \ge \epsilon
$$

since the sequence $\mathbf{y}$ is clearly monotonically increasing. This shows that $\mathbf{y} \notin B_{\bar{\rho}}(\mathbf{x}, \epsilon)$ so that $B_{\bar{\rho}}(\mathbf{x}, \epsilon)$ cannot be a subset of $U(\mathbf{x}, \epsilon)$. □

(b)

Proof. Let $\mathbf{y}$ be the point in $\mathbb{R}^\omega$ defined in part (a) so that we know that $\mathbf{y} \in U(\mathbf{x}, \epsilon)$. Now if $U(\mathbf{x}, \epsilon)$ were open in the uniform topology then there would be a basis element $B_{\bar{\rho}}(\mathbf{y}, \delta)$ that is contained in $U(\mathbf{x}, \epsilon)$. We shall show that any such basis element cannot be contained within $U(\mathbf{x}, \epsilon)$, from which the desired result follows.

So consider any $\delta > 0$ so that there is an $n \in \mathbb{Z}_+$ large enough that $n > \epsilon/\delta$. Then we have

$$
\begin{aligned}
n &> \frac{\epsilon}{\delta}\\
\delta &> \epsilon\frac{1}{n} && \text{(since both } \delta > 0 \text{ and } n > 0\text{)}\\
-\epsilon\frac{1}{n} + \delta &> 0\\
\epsilon - \epsilon\frac{1}{n} + \delta &> \epsilon\\
x_n + \epsilon\left(1-\frac{1}{n}\right) + \delta &> x_n + \epsilon\\
y_n + \delta &> x_n + \epsilon\,.
\end{aligned}
$$

Now define the point $\mathbf{z}$ by

$$z_m = \begin{cases} y_m & m \neq n \\ \frac{(x_n+\epsilon)+(y_n+\delta)}{2} & m = n\,. \end{cases}$$

It then follows that $x_n + \epsilon < z_n < y_n + \delta$. The fact that $x_n + \epsilon < z_n$ means that of course $z_n \notin (x_n - \epsilon, x_n + \epsilon)$ so that $\mathbf{z} \notin \prod_{m=1}^{\infty}(x_m - \epsilon, x_m + \epsilon) = U(\mathbf{x}, \epsilon)$. However, for $m \neq n$ we have that

$$d(z_m, y_m) = d(y_m, y_m) = 0$$

and so $\bar{d}(z_m, y_m) = 0$ as well. For $m = n$ we have

$$\begin{gathered} x_m + \epsilon = x_n + \epsilon < z_m = z_n < y_n + \delta = y_m + \delta \\ x_m + \epsilon - y_m < z_m - y_m < \delta \\ x_m + \epsilon - x_m - \epsilon\left(1 - \frac{1}{n}\right) < z_m - y_m < \delta \\ 0 < \epsilon\frac{1}{n} < z_m - y_m < \delta \\ |z_m - y_m| < \delta \\ \bar{d}(z_m, y_m) \leq d(z_m, y_m) < \delta\,. \end{gathered}$$

From these facts it follows that

$$\bar{\rho}(\mathbf{z}, \mathbf{y}) = \sup_{m \in \mathbb{Z}_+} \bar{d}(z_m, y_m) = \bar{d}(z_n, y_n) < \delta$$

so that $\mathbf{z} \in B_{\bar{\rho}}(\mathbf{y}, \delta)$. This shows that $B_{\bar{\rho}}(\mathbf{y}, \delta)$ is not a subset of $U(\mathbf{x}, \epsilon)$, which shows the desired result as explained before. □

(c)

Proof. ($\subset$) Let $\mathbf{y}$ be any element of $B_{\bar{\rho}}(\mathbf{x}, \epsilon)$ so that $\bar{\rho}(\mathbf{y}, \mathbf{x}) < \epsilon$. Then there is a δ where $\bar{\rho}(\mathbf{y}, \mathbf{x}) < \delta < \epsilon$ since the reals are order-dense. For any $n \in \mathbb{Z}_+$ it must be that $\bar{d}(y_n, x_n) < \delta < \epsilon < 1$ since $\bar{\rho}$ is the supremum of these. From this it has to be that $\bar{d}(y_n, x_n) = d(y_n, x_n) < \delta$ so that

$$\begin{gathered} d(y_n, x_n) = |y_n - x_n| < \delta \\ -\delta < y_n - x_n < \delta \\ x_n - \delta < y_n < x_n + \delta\,. \end{gathered}$$

Hence $y_n \in (x_n - \delta, x_n + \delta)$. Since n was arbitrary, this shows that $\mathbf{y} \in \prod_{n=1}^{\infty}(x_n - \delta, x_n + \delta) = U(\mathbf{x}, \delta)$. Thus obviously $\mathbf{y} \in \bigcup_{\delta < \epsilon} U(\mathbf{x}, \delta)$, which shows the desired result.

($\supset$) Now suppose that $\mathbf{y} \in \bigcup_{\delta < \epsilon} U(\mathbf{x}, \delta)$ so that there is a $\delta < \epsilon$ where $\mathbf{y} \in U(\mathbf{x}, \delta)$. Consider any $n \in \mathbb{Z}_+$ so that we have $y_n \in (x_n - \delta, x_n + \delta)$. Then of course

$$\begin{gathered} x_n - \delta < y_n < x_n + \delta \\ -\delta < y_n - x_n < \delta \end{gathered}$$

so that $d(y_n, x_n) = |y_n - x_n| < \delta < \epsilon < 1$ so that it must be that $\bar{d}(y_n, x_n) = d(y_n, x_n) < \delta$. Since n was arbitrary, it follows that

$$\bar{\rho}(\mathbf{y}, \mathbf{x}) = \sup\left\{\bar{d}(y_n, x_n) \mid n \in \mathbb{Z}_+\right\} \leq \delta < \epsilon\,,$$

and hence $\mathbf{y} \in B_{\bar{\rho}}(\mathbf{x}, \epsilon)$. Since $\mathbf{y}$ was arbitrary, this shows that $B_{\bar{\rho}}(\mathbf{x}, \epsilon) \supset \bigcup_{\delta < \epsilon} U(\mathbf{x}, \delta)$, which completes the proof. □

Exercise 20.7

Consider the map $h : \mathbb{R}^\omega \to \mathbb{R}^\omega$ defined in Exercise 8 of §19; give $\mathbb{R}^\omega$ the uniform topology. Under what conditions on the numbers a_i and b_i is h continuous? a homeomorphism?

Solution:

First some discussion. We know that the product topology is strictly finer than the uniform topology in $\mathbb{R}^\omega$, and the that box topology is strictly finer than the uniform topology. By Lemma 20.4.2 part (1), when the topology on the range of a function becomes coarser, the function remains continuous. It is similarly easy to show that if the topology on the *domain* of a function becomes *finer*, it also remains continuous. However, nothing can be said for sure if the range becomes finer and/or the domain becomes coarser.

It was shown in Exercise 19.8 that h is a homeomorphism (for $a_i > 0$ as in the exercise) if both the domain and range have the product topology, or if they both have the box topology. By what was just discussed then, h is at least continuous with box topology on the domain, and the uniform topology on the range, or likewise with the uniform topology on the domain and the product topology on the range. However, the relative "fineness" of these topologies does not allow us to conclude anything about whether h is continuous or a homeomorphism when both the domain and range are the uniform topologies, which is unfortunately what we are interested in.

In fact, we claim that h is continuous with the uniform topology as the domain and range if and only if the set of numbers $\{a_i\}_{i\in\mathbb{Z}_+}$ is bounded (and of course each $a_i > 0$).

Proof. ($\Rightarrow$) We show this direction by contrapositive. So suppose that $\{a_i\}$ is not bounded. We then show that h is not continuous by showing the negation of Theorem 18.1 part (4). So consider the point $\mathbf{0}$ and the neighborhood $V = B_{\bar{\rho}}(h(\mathbf{0}), 1)$ in the uniform topology. Consider also any neighborhood U of $\mathbf{0}$ so that by Lemma 20.4.1 there is an $\epsilon > 0$ where $B_{\bar{\rho}}(\mathbf{0}, \epsilon) \subset U$. Now define the point $\mathbf{x} \in \mathbb{R}^\omega$ by $x_i = \epsilon/2$ for all $i \in \mathbb{Z}_+$. Then, for any $i \in \mathbb{Z}_+$, we have

$$\bar{d}(x_i, 0) \le d(x_i, 0) = |x_i - 0| = |x_i| = |\epsilon/2| = \epsilon/2$$

so that clearly

$$\bar{\rho}(\mathbf{x}, \mathbf{0}) = \sup\left\{\bar{d}(x_i, 0) \mid i \in \mathbb{Z}_+\right\} \le \epsilon/2 < \epsilon\,,$$

and hence $\mathbf{x} \in B_{\bar{\rho}}(\mathbf{0}, \epsilon)$ so that also $\mathbf{x} \in U$. Then clearly $h(\mathbf{x}) \in h(U)$.

Now, since the a_i coefficients are unbounded, there is a specific $i \in \mathbb{Z}_+$, where $a_i \ge 2/\epsilon$ regardless of how small ϵ is. We then have that

$$d(h_i(x_i), h_i(0)) = d(a_i x_i + b_i, b_i) = |a_i x_i + b_i - b_i| = |a_i x_i| = a_i\,|x_i| = a_i \frac{\epsilon}{2} \ge \frac{2}{\epsilon}\frac{\epsilon}{2} = 1\,,$$

from which we have $\bar{d}(h_i(x_i), h_i(0)) = 1$ and so

$$\bar{\rho}(h(\mathbf{x}), h(\mathbf{0})) = \sup\left\{\bar{d}(h_i(x_i), h_i(0)) \mid i \in \mathbb{Z}_+\right\} \ge 1\,.$$

Then of course $h(\mathbf{x}) \notin B_{\bar{\rho}}(h(\mathbf{0}), 1) = V$. This shows that $h(U) \not\subset V$, which in turn shows that h is not continuous, since U was an arbitrary neighborhood of $\mathbf{0}$.

($\Leftarrow$) Now suppose that the coefficients a_i are bounded so that there is a real a where $0 < a_i \le a$ for all $i \in \mathbb{Z}_+$. Consider any $\mathbf{x} \in \mathbb{R}^\omega$ and any neighborhood V of $h(\mathbf{x})$ in the uniform topology. Then there is an $\epsilon > 0$ where $B_{\bar{\rho}}(h(\mathbf{x}), \epsilon) \subset V$ by Lemma 20.4.1. So let $\delta = \min\{\epsilon/2a, 1\}$, noting that $\delta > 0$ since both $\epsilon > 0$ and $a > 0$. Then $U = B_{\bar{\rho}}(\mathbf{x}, \delta)$ is of course a neighborhood of $\mathbf{x}$ in the uniform topology. We claim that $h(U) \subset V$.

To see this, suppose that $\mathbf{z} \in h(U)$ so that there is a $\mathbf{y} \in U$ where $\mathbf{z} = h(\mathbf{y})$. Then $\bar{\rho}(\mathbf{y}, \mathbf{x}) < \delta \leq 1$ since $\mathbf{y} \in U$, from which it follows that each $\bar{d}(y_i, x_i) < \delta \leq 1$, and hence

$$\bar{d}(y_i, x_i) = d(y_i, x_i) = |y_i - x_i| < \delta\,.$$

We then have that

$$\begin{aligned}\bar{d}(h_i(y_i), h_i(x_i)) &\leq d(h_i(y_i), h_i(x_i)) = d(a_i y_i + b_i, a_i x_i + b_i) = |a_i y_i + b_i - a_i x_i - b_i| \\ &= |a_i y_i - a_i x_i| = |a_i(y_i - x_i)| = a_i\,|y_i - x_i| \leq a\,|y_i - x_i| \\ &< a\delta \leq a\epsilon/2a = \epsilon/2\end{aligned}$$

for each $i \in \mathbb{Z}_+$. From this it follows that

$$\bar{\rho}(\mathbf{z}, h(\mathbf{x})) = \bar{\rho}(h(\mathbf{y}), h(\mathbf{x})) = \sup\left\{\bar{d}(h_i(y_i), h_i(x_i)) \mid i \in \mathbb{Z}_+\right\} \leq \epsilon/2 < \epsilon$$

so that $\mathbf{z} \in B_{\bar{\rho}}(h(\mathbf{x}), \epsilon) \subset V$. Since $\mathbf{z}$ was arbitrary, this shows that $f(U) \subset V$, which shows that h is continuous by Theorem 18.1 part (4) since V was an arbitrary neighborhood. □

The function h is a homeomorphism if and only if there are real $a, a_0 > 0$ where $0 < a_0 \leq a_i \leq a$ for all $i \in \mathbb{Z}_+$.

Proof. First, it was just shown that h is continuous if and only if $\{a_i\}$ is bounded above. It was shown in Exercise 19.8 that h is bijective (so long as each $a_i > 0$) and that its inverse function h^{-1} has the same form as h:

$$h^{-1}(\mathbf{y}) = (c_i y_i + d_i)_{i \in \mathbb{Z}_+}\,,$$

where each $c_i = 1/a_i$ and $d_i = -b_i/a_i$. Since the topologies of the domain and range of h^{-1} are both the uniform topology, as with h, it follows that the same conditions on c_i and d_i will make h^{-1} continuous. That is to say that h^{-1} is continuous (and thus h is a homeomorphism) if and only if also $\{c_i\} = \{1/a_i\}$ is bounded above. Of course $\{1/a_i\}$ being bounded above means that $\{a_i\}$ cannot get arbitrarily close to zero and so must have some nonzero lower bound a_0. □

Exercise 20.8

Let X be the subset of $\mathbb{R}^\omega$ consisting of all sequences $\mathbf{x}$ such that $\sum x_i^2$ converges. Then the formula

$$d(\mathbf{x}, \mathbf{y}) = \left[\sum_{i=1}^{\infty} (x_i - y_i)^2\right]^{1/2}$$

defines a metric on X. (See Exercise 10) On X we have the three topologies it inherits from the box, uniform, and product topologies on $\mathbb{R}^\omega$. We have also the topology given by the metric d, which we call the ℓ^2*-topology.* (Read "little ell two.")

(a) Show that on X, we have the inclusions

$$\text{box topology} \supset \ell^2\text{-topology} \supset \text{uniform topology}\,.$$

(b) The set $\mathbb{R}^\infty$ of all sequences that are eventually zero is contained in X. Show that the four topologies that $\mathbb{R}^\infty$ inherits as a subspace of X are all distinct.

(c) The set

$$H = \prod_{n \in \mathbb{Z}_+} [0, 1/n]$$

is contained in X; it is called the ***Hilbert cube***. Compare the four topologies that H inherits as a subspace of X.

Solution:

Lemma 20.8.1. *Suppose that $(a_1, a_2, \ldots)$ and $(b_1, b_2, \ldots)$ are two real sequences that converge to a and b, respectively. Then, if $a_n \leq b_n$ for every $n \in \mathbb{Z}_+$, then $a \leq b$.*

Proof. Suppose to the contrary that $a > b$, and let $\epsilon = (a-b)/2$ so that clearly $\epsilon > 0$. Since $(a_1, a_2, \ldots)$ converges to a there is an $N_a \in \mathbb{Z}_+$ where $|a_n - a| < \epsilon$ for all $n \geq N_a$. Similarly there is an $N_b \in \mathbb{Z}_+$ where $|b_n - b| < \epsilon$ for all $n \geq N_b$ since (b_n) converges to b. So let $N = \max\{N_a, N_b\}$ and consider any $n \geq N$. Then $n \geq N \geq N_a$ so that $|a_n - a| < \epsilon$ and hence

$$\begin{aligned} -\epsilon < a_n - a &< \epsilon \\ a - \epsilon < a_n &< a + \epsilon \\ a - \frac{a-b}{2} &< a_n \\ \frac{a+b}{2} &< a_n \,. \end{aligned}$$

Analogously, we have that $n \geq N \geq N_b$ so that $|b_n - b| < \epsilon$ and so

$$\begin{aligned} -\epsilon < b_n - b &< \epsilon \\ b - \epsilon < b_n &< b + \epsilon \\ b_n &< b + \frac{a-b}{2} \\ b_n &< \frac{a+b}{2} \,. \end{aligned}$$

Therefore we have $b_n < (a+b)/2 < a_n$, which contradicts the supposition that $b_n \geq a_n$. So it has to be that in fact $a \leq b$ as desired. □

Corollary 20.8.2. *Suppose that $\sum a_n$ and $\sum b_n$ are two real series that converge to a and b, respectively. Then, if $a_n \leq b_n$ for every $n \in \mathbb{Z}_+$, then $a \leq b$.*

Proof. Since we have that $a_n \leq b_n$ for every $n \in \mathbb{Z}_+$, it follows that we have

$$s_n = \sum_{i=1}^{n} a_i \leq \sum_{i=1}^{n} b_i = t_n$$

for any $n \in \mathbb{Z}_+$ for the partial sums. Then we have by definition of series that

$$a = \sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n \leq \lim_{n\to\infty} t_n = \sum_{n=1}^{\infty} b_n = b$$

by Lemma 20.8.1, as desired. □

The following is a corollary of Lemma 20.4.1:

Corollary 20.8.3. *Suppose that X is a subspace of metric space Y with metric d. If U is open in the subspace topology on X and contains x, then there is a ball $B_d(x, \epsilon)$ in Y such that $X \cap B_d(x, \epsilon) \subset U$.*

Proof. Consider open U in X and any $x \in U$. Then there is an open set V in Y such that $U = X \cap V$ by the definition of the subspace topology. Then of course $x \in X$ and $x \in V$ since $x \in U$. It then follows that there is an $\epsilon > 0$ such that $B_d(x, \epsilon) \subset V$ by Lemma 20.4.1. Now consider any $y \in X \cap B_d(x, \epsilon)$ so that $y \in X$ and $y \in B_d(x, \epsilon)$. Then also $y \in V$ since $B_d(x, \epsilon) \subset V$. Hence $y \in X \cap V = U$, which shows that $X \cap B_d(x, \epsilon) \subset U$ as desired since y was arbitrary. □

Definition 20.8.4. *If $(x_1, x_2, \ldots)$ is a sequence whose series $\sum x_i$ converges, we define the partial series starting at $n \in \mathbb{Z}_+$ as*

$$\sum_{i=n}^{\infty} x_i = \sum_{i=1}^{\infty} x_i - \sum_{i=1}^{n-1} x_i \,,$$

where we adopt the standard convention that $\sum_{i=a}^{b} x_i = 0$ when $b < a$. According to this the partial series starting at $n = 1$ is just the series itself as expected.

Lemma 20.8.5. *If $(x_1, x_2, \ldots)$ is a series of non-negative real numbers such that the series $\sum x_i$ converges, then the sequence of partial series defined by*

$$p_n = \sum_{i=n}^{\infty} x_i$$

is non-increasing and converges to zero. Also each $p_n \geq 0$.

Proof. Since the terms are all non-negative, clearly the sequence of partial sums is non-decreasing. Thus we have

$$\begin{aligned}
\sum_{i=1}^{n} x_i &\leq \sum_{i=1}^{n+1} x_i \\
-\sum_{i=1}^{n} x_i &\geq -\sum_{i=1}^{n+1} x_i \\
\sum_{i=1}^{\infty} x_i - \sum_{i=1}^{n} x_i &\geq \sum_{i=1}^{\infty} x_i - \sum_{i=1}^{n+1} x_i \\
\sum_{i=n+1}^{\infty} x_i &\geq \sum_{i=n+2}^{\infty} x_i \\
p_{n+1} &\geq p_{n+2}
\end{aligned}$$

for any $n \in \mathbb{Z}_+$. Of course we also have

$$\begin{aligned}
0 \leq x_1 &= \sum_{i=1}^{1} x_i \\
0 &\geq -\sum_{i=1}^{1} x_i \\
\sum_{i=1}^{\infty} x_i &\geq \sum_{i=1}^{\infty} x_i - \sum_{i=1}^{1} x_i \\
p_1 &\geq p_2 \,.
\end{aligned}$$

Together these show that the sequence of partial series is non-increasing. Also, since the series of partial sums is non-decreasing, we have that that the infinite sum cannot be less than any of the partial sums, that is

$$\sum_{i=1}^{\infty} x_i \geq \sum_{i=1}^{n-1} x_i$$
$$\sum_{i=1}^{\infty} x_i - \sum_{i=1}^{n-1} x_i \geq 0$$
$$p_n \geq 0$$

for any $n \in \mathbb{Z}_+$.

To show that the sequence of partial series converges to zero, consider any $\epsilon > 0$. We know that the sequence of partial sums converges to the infinite sum by the definition of a series. Hence there is an $N \in \mathbb{Z}_+$ such that

$$\left| \sum_{i=1}^{n} x_i - \sum_{i=1}^{\infty} x_i \right| < \epsilon$$
$$|-p_{n+1}| < \epsilon$$
$$|p_{n+1}| < \epsilon$$
$$|p_{n+1} - 0| < \epsilon$$

for all $n \geq N$, and hence $|p_n - 0| < \epsilon$ for all $n \geq N + 1$. This of course shows convergence to zero as desired. □

Main Problem.

(a)

Proof. First we show that the ℓ^2-topology is finer than the topology inherited from the uniform topology using Lemma 20.2 since they are both metric topologies. So consider any $\mathbf{x} \in X$ and let $B = X \cap B_{\bar{\rho}}(\mathbf{x}, \epsilon)$ for any $\epsilon > 0$, which is of course a basis element of the subspace topology inherited by the uniform topology. Then the set $C = B_d(\mathbf{x}, \epsilon/2)$ (where d is the metric defined above for the ℓ^2-topology instead of the usual metric on $\mathbb{R}$) is a basis element of the ℓ^2-topology. We claim that $\mathbf{x} \in C \subset B$, which completes the proof that the ℓ^2-topology is finer.

First, it is obvious that $\mathbf{x} \in C$. Now consider any $\mathbf{y} \in C = B_d(\mathbf{x}, \epsilon/2)$. Then we have that

$$d(\mathbf{y}, \mathbf{x}) = \left[\sum_{i=1}^{\infty} (y_i - x_i)^2 \right]^{1/2} < \epsilon/2$$

For $n \in \mathbb{Z}_+$ let

$$s_n = \sum_{i=1}^{n} (y_i - x_i)^2$$

be the partial sums of the infinite sum $\sum (y_i - x_i)^2$. Clearly each term in the sum is non-negative so that the sequence of partial sums is non-decreasing. It then follows that $s_n \leq \sum (y_i - x_i)^2$ for any $n \in \mathbb{Z}_+$. We clearly then have, for any $n \in \mathbb{Z}_+$, that

$$|y_n - x_n|^2 = (y_n - x_n)^2 \leq \sum_{i=1}^{n-1} (y_i - x_i)^2 + (y_n - x_n)^2 = \sum_{i=1}^{n} (y_i - x_i)^2 = s_n \leq \sum_{i=1}^{\infty} (y_i - x_i)^2$$

since again each term is non-negative. Hence by Corollary 20.1.2 we have

$$|y_n - x_n| = \left[|y_n - x_n|^2\right]^{1/2} \leq \left[\sum_{i=1}^{\infty}(y_i - x_i)^2\right]^{1/2} < \epsilon/2\,.$$

It then follows that

$$\bar{p}(y_n, x_n) \leq p(y_n, x_n) = |y_n - x_n| < \epsilon/2\,,$$

where we have let p and $\bar{p}$ denote the standard metric and standard bounded metric, respectively, on $\mathbb{R}$. Since this is true for any $n \in \mathbb{Z}_+$, it follows that

$$\bar{\rho}(\mathbf{y}, \mathbf{x}) = \sup\{\bar{p}(y_n, x_n) \mid n \in \mathbb{Z}_+\} \leq \epsilon/2 < \epsilon$$

so that $\mathbf{y} \in B_{\bar{\rho}}(\mathbf{x}, \epsilon)$. Thus clearly $\mathbf{y} \in X \cap B_{\bar{\rho}}(\mathbf{x}, \epsilon) = B$ so that $C \subset B$ as desired since $\mathbf{y}$ was arbitrary.

Now we show that the topology inherited from the box topology is finer the ℓ^2-topology using Lemma 13.3. So consider any $\mathbf{x} \in X$ and any basis element B containing $\mathbf{x}$ of the ℓ^2-topology. Then by Lemma 20.4.1 there is an $\epsilon > 0$ where $B_d(\mathbf{x}, \epsilon) \subset B$ since B is of course open. Now consider the set

$$C = X \cap \prod_{i \in \mathbb{Z}_+} B_p(x_i, \epsilon/\sqrt{2^{i+1}})\,,$$

where again p denotes the usual metric on $\mathbb{R}$. Then clearly C is a basis element of the subspace topology inherited by the box topology and contains $\mathbf{x}$, noting that clearly each $\epsilon/\sqrt{2^{i+1}} > 0$. We claim that $C \subset B_d(\mathbf{x}, \epsilon) \subset B$, which shows the desired result.

To see this suppose that $\mathbf{y} \in C$ so that $\mathbf{y} \in X$ and $\mathbf{y} \in \prod B_p(x_i, \epsilon/\sqrt{2^{i+1}})$. Then, for any $i \in \mathbb{Z}_+$, we have that

$$p(y_i, x_i) = |y_i - x_i| \leq \epsilon/\sqrt{2^{i+1}}$$

is true. It then follows from Lemma 20.1.1 that

$$(y_i - x_i)^2 = |y_i - x_i|^2 \leq \left(\epsilon/\sqrt{2^{i+1}}\right)^2\,.$$

Since this is true for any $i \in \mathbb{Z}_+$, we have by Corollary 20.8.2 that

$$\begin{aligned}
\sum_{i=1}^{\infty}(y_i - x_i)^2 &\leq \sum_{i=1}^{\infty}\left(\frac{\epsilon}{\sqrt{2^{i+1}}}\right)^2 = \sum_{i=1}^{\infty}\frac{\epsilon^2}{2^{i+1}} = \epsilon^2\sum_{i=1}^{\infty}\left(\frac{1}{2}\right)^{i+1} \\
&= \epsilon^2\sum_{i=2}^{\infty}\left(\frac{1}{2}\right)^i = \epsilon^2\left[\sum_{i=0}^{\infty}\left(\frac{1}{2}\right)^i - \left(\frac{1}{2}\right)^0 - \left(\frac{1}{2}\right)^1\right] \\
&= \epsilon^2\left[\frac{1}{1-\frac{1}{2}} - 1 - \frac{1}{2}\right] = \epsilon^2\left[2 - 1 - \frac{1}{2}\right] \\
&= \frac{\epsilon^2}{2}
\end{aligned}$$

since $\sum(1/2)^i$ is a geometric series. It then follows from Corollary 20.1.2 that

$$d(\mathbf{y}, \mathbf{x}) = \left[\sum_{i=1}^{\infty}(y_i - x_i)^2\right]^{1/2} \leq \left(\frac{\epsilon^2}{2}\right)^{1/2} = \frac{\epsilon}{\sqrt{2}} < \epsilon$$

so that $\mathbf{y} \in B_d(\mathbf{x}, \epsilon)$. Since $\mathbf{y}$ was arbitrary, this shows that $C \subset B_d(\mathbf{x}, \epsilon) \subset B$ as desired, thereby completing the proof. □

(b)

Proof. Since relative topological "fineness" is preserved when inherited by subspace topologies (which is trivial to show), we have from part (a) and what was shown in the text that

$$\text{box topology} \supset \ell^2\text{-topology} \supset \text{uniform topology} \supset \text{product topology}$$

on $\mathbb{R}^\infty$. To show that they are all distinct, it then suffices to show that each of the box, ℓ^2, and uniform topologies (or more properly the subspace topologies inherited from them) have open sets that are not open in the ℓ^2, uniform, and product topologies, respectively.

First we show that the inherited box topology has an open set that is not open in the inherited ℓ^2-topology. Consider the set $U = \mathbb{R}^\infty \cap \prod_{n\in\mathbb{Z}_+}(-1, 1/n)$, which is clearly a basis element and open set in the inherited box topology. We show that U is not open in the inherited ℓ^2-topology using the contrapositive of Corollary 20.8.3. So consider the point $\mathbf{0}$, which is clearly contained in U and any $\epsilon > 0$ and the arbitrary ball $B_d(\mathbf{0}, \epsilon)$ of X. Of course, there is positive integer N large enough that $N \geq 2/\epsilon$, and hence $1/N \leq \epsilon/2$.

Now, consider the sequence $\mathbf{x}$ defined by

$$x_n = \begin{cases} 0 & n \neq N \\ \epsilon/2 & n = N \end{cases}$$

for $n \in \mathbb{Z}_+$. Clearly $\mathbf{x}$ is eventually zero so that $\mathbf{x} \in \mathbb{R}^\infty$ since $x_n = 0$ for all $n \geq N + 1$. We also clearly have

$$d(\mathbf{x}, \mathbf{0}) = \left[\sum_{i=1}^{\infty}(x_i - 0)^2\right]^{1/2} = \left[\sum_{i=1}^{\infty} x_i^2\right]^{1/2} = \left[\left(\frac{\epsilon}{2}\right)^2\right]^{1/2} = \frac{\epsilon}{2} < \epsilon$$

so that $\mathbf{x} \in B_d(\mathbf{0}, \epsilon)$. Therefore $\mathbf{x} \in \mathbb{R}^\infty \cap B_d(\mathbf{0}, \epsilon)$. However, we have that $1/N \leq \epsilon/2 = x_N$ so that $x_N \notin (-1, 1/N)$, and hence $\mathbf{x} \notin \prod_{n\in\mathbb{Z}_+}(-1, 1/n)$. From this clearly $\mathbf{x} \notin U$ so that $\mathbb{R}^\infty \cap B_d(\mathbf{0}, \epsilon)$ cannot be a subset of U. Since the ball $B_d(\mathbf{0}, \epsilon)$ was arbitrary, this shows that U is not open in the inherited ℓ^2-topology as desired.

Next we show that there is an open set in the inherited ℓ^2-topology that is not open in the inherited uniform topology. So consider the set $U = \mathbb{R}^\infty \cap B_d(\mathbf{0}, 1)$, which is clearly open in the inherited ℓ^2-topology. We show that U is not open in the inherited uniform topology, again by the contrapositive of Corollary 20.8.3. Consider the point $\mathbf{0}$, clearly in U, and any $\epsilon > 0$ so that $B_{\bar{\rho}}(\mathbf{0}, \epsilon)$ is an arbitrary ball in the uniform topology on $\mathbb{R}^\omega$. Now clearly there is an $N \in \mathbb{Z}_+$ large enough that $N \geq (2/\epsilon)^2$. It then follows from Corollary 20.1.2 that $\sqrt{N} \geq 2/\epsilon$.

Now define the sequence $\mathbf{x}$ by

$$x_n = \begin{cases} \epsilon/2 & n \leq N \\ 0 & n > N \end{cases}$$

for $n \in \mathbb{Z}_+$, so that clearly $\mathbf{x} \in \mathbb{R}^\infty$. Then clearly we have

$$\bar{p}(x_n, 0) \leq p(x_n, 0) = |\epsilon/2 - 0| = |\epsilon/2| = \epsilon/2$$

for any $n \leq N$, where again p and $\bar{p}$ are the standard and standard bounded metrics on $\mathbb{R}$, respectively. If $n > N$ clearly

$$\bar{p}(x_n, 0) \leq p(x_n, 0) = |0 - 0| = |0| = 0 \leq \epsilon/2\,.$$

Hence it follows that

$$\bar{\rho}(\mathbf{x}, \mathbf{0}) = \sup\{\bar{\rho}(x_n, 0) \mid n \in \mathbb{Z}_+\} \leq \epsilon/2 < \epsilon$$

so that $\mathbf{x} \in B_{\bar{\rho}}(\mathbf{0}, \epsilon)$. Therefore of course $\mathbf{x} \in \mathbb{R}^\infty \cap B_{\bar{\rho}}(\mathbf{0}, \epsilon)$. However, we also have

$$\begin{aligned} d(\mathbf{x}, 0) &= \left[\sum_{i=1}^{\infty}(x_i - 0)^2\right]^{1/2} = \left[\sum_{i=1}^{\infty} x_i^2\right]^{1/2} = \left[\sum_{i=1}^{N}\left(\frac{\epsilon}{2}\right)^2\right]^{1/2} \\ &= \left[N\left(\frac{\epsilon}{2}\right)^2\right]^{1/2} = \sqrt{N}\frac{\epsilon}{2} \\ &\geq \frac{2}{\epsilon}\left(\frac{\epsilon}{2}\right) = 1 \end{aligned}$$

since $\sqrt{N} \geq 2/\epsilon$. Therefore clearly $\mathbf{x} \notin B_d(\mathbf{0}, 1)$ so that $\mathbf{x} \notin U$. It follows that $\mathbb{R}^\infty \cap B_{\bar{\rho}}(\mathbf{0}, \epsilon)$ cannot be a subset of U. Since $B_{\bar{\rho}}(\mathbf{0}, \epsilon)$ was an arbitrary ball, this shows that U is not open in the uniform topology as desired.

Lastly we show that there is an open set in the inherited uniform topology that is not open in the inherited product topology. So let $U = \mathbb{R}^\infty \cap B_{\bar{\rho}}(\mathbf{0}, 1)$, which is clearly open in the inherited uniform topology. We show that U is not open in the inherited product topology using the definition of a basis. Consider any basis element B of the inherited product topology that contains $\mathbf{0}$ so that $B = \mathbb{R}^\infty \cap \prod_{n \in \mathbb{Z}_+} B_n$, where each B_n is open in $\mathbb{R}$ and $B_n \neq \mathbb{R}$ for only finitely many $n \in \mathbb{Z}_+$. Then clearly there is an $N \in \mathbb{Z}_+$ where $B_N = \mathbb{R}$, and clearly we have $0 \in B_n$ for all $n \in \mathbb{Z}_+$.

So define the sequence $\mathbf{x}$ by

$$x_n = \begin{cases} 0 & n \neq N \\ 1 & n = N \end{cases}$$

for $n \in \mathbb{Z}_+$. Clearly $\mathbf{x} \in \mathbb{R}^\infty$ since $x_n = 0$ for all $n \geq N + 1$. For any $n \in \mathbb{Z}_+$ we have that $x_n = 0 \in B_n$ if $n \neq N$, and $x_n = 1 \in \mathbb{R} = B_n$ for $n = N$. Hence clearly $\mathbf{x} \in \prod B_n$ so that $\mathbf{x} \in \mathbb{R}^\infty \cap \prod B_n = B$ as well. However, we clearly have that

$$p(x_N, 0) = |x_N - 0| = |1 - 0| = |1| = 1 \geq 1$$

so that $\bar{p}(x_N, 0) = \min\{p(x_N, 0), 1\} = 1 \geq 1$. Then it has to be that

$$\bar{\rho}(\mathbf{x}, \mathbf{0}) = \sup\{\bar{p}(x_n, 0) \mid n \in \mathbb{Z}_+\} \geq 1$$

so that $\mathbf{x} \notin B_{\bar{\rho}}(\mathbf{0}, 1)$ and hence $\mathbf{x} \notin U$. This shows that B is not a subset of U, which shows that U is not open in the inherited product topology since B was an arbitrary basis element. □

(c) First, we note that H is contained in X by the comparison test since the series $\sum(1/n)^2$ converges. Then, again since relative topological "fineness" is preserved when inherited by subspace topologies, we know that

$$\text{box topology} \supset \ell^2\text{-topology} \supset \text{uniform topology} \supset \text{product topology}$$

on H. We claim, however, that the inherited box topology is distinct from the other three, which are all the same.

Proof. First we show that the inherited box topology has an open set that is not open in the inherited ℓ^2-topology, in a very similar way to how this was shown in part (b). Consider the set $U = H \cap \prod_{n \in \mathbb{Z}_+}(-1, 1/n)$, which is clearly a basis element and open set in the inherited box topology.

We show that U is not open in the inherited ℓ^2-topology using the contrapositive of Corollary 20.8.3. So consider the point $\mathbf{0}$, which is clearly contained in U and any $\epsilon > 0$ and the arbitrary ball $B_d(\mathbf{0}, \epsilon)$ of X. Of course, there is positive integer N large enough that $N \geq 2/\epsilon$, and hence $1/N \leq \epsilon/2$.

Now, consider the sequence $\mathbf{x}$ defined by

$$x_n = \begin{cases} 0 & n \neq N \\ 1/N & n = N \end{cases}$$

for $n \in \mathbb{Z}_+$. Clearly $\mathbf{x} \in H$ since each $x_n \in [0, 1/n]$. We also clearly have

$$d(\mathbf{x}, \mathbf{0}) = \left[\sum_{i=1}^{\infty}(x_i - 0)^2\right]^{1/2} = \left[\sum_{i=1}^{\infty} x_i^2\right]^{1/2} = \left[x_N^2\right]^{1/2} = x_N = 1/N \leq \epsilon/2 < \epsilon$$

so that $\mathbf{x} \in B_d(\mathbf{0}, \epsilon)$. Therefore $\mathbf{x} \in H \cap B_d(\mathbf{0}, \epsilon)$. However, we have that $x_N = 1/N$ so that $x_N \notin (-1, 1/N)$, and hence $\mathbf{x} \notin \prod_{n \in \mathbb{Z}_+}(-1, 1/n)$. From this clearly $\mathbf{x} \notin U$ so that $H \cap B_d(\mathbf{0}, \epsilon)$ cannot be a subset of U. Since the ball $B_d(\mathbf{0}, \epsilon)$ was arbitrary, this shows that U is not open in the inherited ℓ^2-topology as desired.

To show that the other three topologies are the same on H, it suffices to show that the inherited product topology is finer than the inherited ℓ^2-topology, which we do using Lemma 13.3. So consider any $\mathbf{x} \in H$ and any basis element B of the inherited ℓ^2-topology that contains $\mathbf{x}$. It then follows from Corollary 20.8.3 that there is an $\epsilon > 0$ where $B' = H \cap B_d(\mathbf{x}, \epsilon) \subset B$ since of course B is open. Now, by Lemma 20.8.5 the sequence of partial series $p_n = \sum_{i=n}^{\infty}(1/i)^2$ converges to zero so that there is an $N \in \mathbb{Z}_+$ where $p_n = |p_n| = |p_n - 0| < \epsilon^2/2$ for all $n \geq N$ since each p_n is non-negative (also by Lemma 20.8.5). In particular $p_{N+1} = \sum_{i=N+1}^{\infty}(1/i)^2 < \epsilon^2/2$. So define the following sets:

$$C_n = \begin{cases} B_p(x_n, \epsilon/\sqrt{2N}) & n \leq N \\ \mathbb{R} & n > 0 \end{cases}$$

for $n \in \mathbb{Z}_+$, where again p is the usual metric on $\mathbb{R}$. Clearly $C = H \cap \prod C_n$ is a basis element in the inherited product topology that contains $\mathbf{x}$. We now claim that $C \subset B' \subset B$, which shows that the inherited product topology is finer by Lemma 13.3 since B was an arbitrary basis element.

To see this, consider any $\mathbf{y} \in C$ so that $\mathbf{y} \in H$ and $\mathbf{y} \in \prod C_n$. Now, for any $n \leq N$, we have that $y_n \in C_n = B_p(x_n, \epsilon/\sqrt{2N})$ so that $|y_n - x_n| < \epsilon/\sqrt{2N}$. It then follows from Lemma 20.1.1 that

$$(y_n - x_n)^2 = |y_n - x_n|^2 < \left(\frac{\epsilon}{\sqrt{2N}}\right)^2 = \frac{\epsilon^2}{2N}.$$

Since this is true of each $n \leq N$, we clearly have that

$$\sum_{i=1}^{N}(y_i - x_i)^2 < \sum_{i=1}^{N}\frac{\epsilon^2}{2N} = N\frac{\epsilon^2}{2N} = \frac{\epsilon^2}{2}.$$

Now, for any $n > N$ we have that of course that $\mathbf{y} \in H = \prod[0, 1/n]$ so that $y_n \in [0, 1/n]$. Similarly $\mathbf{x} \in H$ so that $x_n \in [0, 1/n]$. Then both $0 \leq y_n \leq 1/n$ and $0 \leq x_n \leq 1/n$ so that

$$\begin{gathered} 0 \leq x_n \leq 1/n \\ 0 \geq -x_n \geq -1/n \\ y_n \geq y_n - x_n \geq y_n - 1/n \\ 1/n \geq y_n \geq y_n - x_n \geq y_n - 1/n \geq 0 - 1/n = -1/n. \end{gathered}$$

Hence $|y_n - x_n| \leq 1/n$, from which it follows that $(y_n - x_n)^2 = |y_n - x_n|^2 \leq (1/n)^2$ by Lemma 20.1.1. Since this it true for any $n > N$, it follow from either Lemma 20.8.1 or Corollary 20.8.2 that

$$\sum_{i=N+1}^{\infty} (y_i - x_i)^2 \leq \sum_{i=N+1}^{\infty} \left(\frac{1}{i}\right)^2 < \frac{\epsilon^2}{2}.$$

We then have, from the definition of partial series (Definition 20.8.4), that

$$\sum_{i=1}^{\infty}(y_i - x_i)^2 = \sum_{i=1}^{N}(y_i - x_i)^2 + \sum_{i=N+1}^{\infty}(y_i - x_i)^2 < \frac{\epsilon^2}{2} + \frac{\epsilon^2}{2} = \epsilon^2$$

Then Corollary 20.1.2 means that

$$d(\mathbf{y}, \mathbf{x}) = \left[\sum_{i=1}^{\infty}(y_i - x_i)^2\right]^{1/2} < \left(\epsilon^2\right)^{1/2} = \epsilon$$

so that $\mathbf{y} \in B_d(\mathbf{x}, \epsilon)$, and hence clearly $\mathbf{y} \in H \cap B_d(\mathbf{x}, \epsilon) = B'$. Since $\mathbf{y}$ was arbitrary, this shows that $C \subset B' \subset B$ as desired. □

Exercise 20.9

Show that the euclidean metric d on $\mathbb{R}^n$ is a metric, as follows: If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $c \in \mathbb{R}$, define

$$\begin{aligned}
\mathbf{x} + \mathbf{y} &= (x_1 + y_1, \ldots, x_n + y_n), \\
c\mathbf{x} &= (cx_1, \ldots, cx_n), \\
\mathbf{x} \cdot \mathbf{y} &= x_1 y_1 + \cdots + x_n y_n.
\end{aligned}$$

(a) Show that $\mathbf{x} \cdot (\mathbf{y} + \mathbf{z}) = (\mathbf{x} \cdot \mathbf{y}) + (\mathbf{x} \cdot \mathbf{z})$.

(b) Show that $|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\| \|\mathbf{y}\|$. [Hint: If $\mathbf{x}, \mathbf{y} \neq 0$, let $a = 1/\|\mathbf{x}\|$ and $b = 1/\|\mathbf{y}\|$, and use the fact that $\|a\mathbf{x} \pm b\mathbf{y}\| \geq 0$.]

(c) Show that $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$. [Hint: Compute $(\mathbf{x} + \mathbf{y}) \cdot (\mathbf{x} + \mathbf{y})$ and apply (b).]

(d) Verify that d is a metric.

Solution:

First we show some basic properties of these operations that will be useful:

Lemma 20.9.1. *For any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $a \in \mathbb{R}$, we assert the following:*

(1) The dot product is commutative, that is $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$.

(2) $\mathbf{0} \cdot \mathbf{x} = 0$.

(3) $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = \mathbf{0}$.

(4) $\|\mathbf{x}\| \geq 0$

(5) $\mathbf{x} \cdot \mathbf{x} = \|\mathbf{x}\|^2$

(6) $\|a\mathbf{x}\| = |a| \|\mathbf{x}\|$

Proof. For assertion (1) clearly

$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{n} x_i y_i = \sum_{i=1}^{n} y_i x_i = \mathbf{y} \cdot \mathbf{x}$$

by the definition of the dot product. Regarding (2), we clearly have

$$\mathbf{0} \cdot \mathbf{x} = \sum_{i=1}^{n} 0 \cdot x_i = \sum_{i=1}^{n} 0 = 0\,.$$

For (3) first suppose that $\mathbf{x} \neq \mathbf{0}$ so that that there is an $i \in \{1, \ldots, n\}$ where $x_k \neq 0$ so that clearly $x_k^2 > 0$. Then we have

$$\|\mathbf{x}\| = \sum_{i=1}^{n} x_i^2 = \sum_{i=k} x_i^2 + \sum_{i \neq k} x_i^2 = x_k^2 + \sum_{i \neq k} x_i^2 \geq x_k^2 + 0 = x_k^2 > 0$$

since each term in the sum $\sum_{i \neq k} x_i^2$ is non-negative so that the overall sum is as well. Thus of course $\|\mathbf{x}\| \neq 0$. This shows the forward implication by contrapositive. For the reverse direction, suppose that $\mathbf{x} = \mathbf{0}$ so that

$$\|\mathbf{x}\| = \|\mathbf{0}\| = \left[\sum_{i=1}^{n} 0^2\right]^{1/2} = \left[\sum_{i=1}^{n} 0\right]^{1/2} = [0]^{1/2} = 0\,.$$

Assertion (4) is fairly obvious from the definition. Clearly each $x_i^2 \geq 0$ since it is a square, so that $\sum_{i=1}^{n} x_i^2 \geq 0$ as well. It then follows from Corollary 20.1.2 that

$$\|\mathbf{x}\| = \left[\sum_{i=1}^{n} x_i^2\right]^{1/2} \geq 0^{1/2} = 0$$

as desired. Assertion (5) is also easy to show:

$$\mathbf{x} \cdot \mathbf{x} = \sum_{i=1}^{n} x_i x_i = \sum_{i=1}^{n} x_i^2 = \left(\left[\sum_{i=1}^{n} x_i^2\right]^{1/2}\right)^2 = \|\mathbf{x}\|^2$$

by definition.

For part (6) we have by definition that

$$\|a\mathbf{x}\| = \left[\sum_{i=1}^{n} (a x_i)^2\right]^{1/2} = \left[\sum_{i=1}^{n} a^2 x_i^2\right]^{1/2} = \left[a^2 \sum_{i=1}^{n} x_i^2\right]^{1/2} = \left[a^2\right]^{1/2} \left[\sum_{i=1}^{n} x_i^2\right]^{1/2} = |a|\,\|\mathbf{x}\|$$

as desired. □

Main Problem.

(a)

Proof. By definition we have that

$$\mathbf{y} + \mathbf{z} = (y_1 + z_1, \ldots, y_n + y_n)$$

so that clearly

$$\mathbf{x}\cdot(\mathbf{y}+\mathbf{z}) = \sum_{i=1}^{n} x_i(y_i+z_i) = \sum_{i=1}^{n}(x_iy_i + x_iz_i) = \sum_{i=1}^{n} x_iy_i + \sum_{i=1}^{n} x_iz_i = (\mathbf{x}\cdot\mathbf{y}) + (\mathbf{x}\cdot\mathbf{z})$$

as desired. □

(b)

Proof. First, if $\mathbf{x} = \mathbf{0}$ then clearly by Lemma 20.9.1 parts (2) and (3)

$$|\mathbf{x}\cdot\mathbf{y}| = |\mathbf{0}\cdot\mathbf{y}| = |0| = 0 \le 0 = 0\,\|\mathbf{y}\| = \|\mathbf{0}\|\,\|\mathbf{y}\| = \|\mathbf{x}\|\,\|\mathbf{y}\|\ .$$

Similarly if $\mathbf{y} = \mathbf{0}$ then by all parts of Lemma 20.9.1

$$|\mathbf{x}\cdot\mathbf{y}| = |\mathbf{x}\cdot\mathbf{0}| = |\mathbf{0}\cdot\mathbf{x}| = |0| = 0 \le 0 = \|\mathbf{x}\|\,0 = \|\mathbf{x}\|\,\|\mathbf{0}\| = \|\mathbf{x}\|\,\|\mathbf{y}\|\ .$$

So in what follows assume that $\mathbf{x},\mathbf{y} \ne \mathbf{0}$. Then by Lemma 20.9.1 part (3) we have that $\|\mathbf{x}\|$ and $\|\mathbf{y}\|$ are both nonzero so that $a = 1/\|\mathbf{x}\|$ and $b = 1/\|\mathbf{y}\|$ are defined. Then, by Lemma 20.9.1 part (4), we of course have

$$\|a\mathbf{x} \pm b\mathbf{y}\| \ge 0\,.$$

Since clearly

$$a\mathbf{x} \pm b\mathbf{y} = (ax_1 \pm by_1, \ldots, ax_n \pm by_n)$$

by the definition of the operations, we have

$$\left[\sum_{i=1}^{n}(ax_i \pm by_i)^2\right]^{1/2} = \|a\mathbf{x} \pm b\mathbf{y}\| \ge 0\,.$$

It then follows from Lemma 20.1.1 that

$$\begin{aligned}
\sum_{i=1}^{n}(ax_i \pm by_i)^2 &\ge 0^2 = 0 \\
\sum_{i=1}^{n}(a^2x_i^2 \pm 2abx_iy_i + b^2y_i^2) &\ge 0 \\
a^2\sum_{i=1}^{n} x_i^2 \pm 2ab\sum_{i=1}^{n} x_iy_i + b^2\sum_{i=1}^{n} y_i^2 &\ge 0 \\
a^2\|\mathbf{x}\|^2 \pm 2ab(\mathbf{x}\cdot\mathbf{y}) + b^2\|\mathbf{y}\|^2 &\ge 0 \\
\pm 2ab(\mathbf{x}\cdot\mathbf{y}) &\ge -a^2\|\mathbf{x}\|^2 - b^2\|\mathbf{y}\|^2 \\
\pm 2\frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{x}\|\,\|\mathbf{y}\|} &\ge -\frac{\|\mathbf{x}\|^2}{\|\mathbf{x}\|^2} - \frac{\|\mathbf{y}\|^2}{\|\mathbf{y}\|^2} \\
\pm 2\frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{x}\|\,\|\mathbf{y}\|} &\ge -1 - 1 \\
\pm 2\frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{x}\|\,\|\mathbf{y}\|} &\ge -2 \\
\mp\mathbf{x}\cdot\mathbf{y} &\le \|\mathbf{x}\|\,\|\mathbf{y}\|\ .
\end{aligned}$$

Hence we have that both $\mathbf{x}\cdot\mathbf{y} \le \|\mathbf{x}\|\,\|\mathbf{y}\|$ and $-\mathbf{x}\cdot\mathbf{y} \le \|\mathbf{x}\|\,\|\mathbf{y}\|$ so that $\mathbf{x}\cdot\mathbf{y} \ge -\|\mathbf{x}\|\,\|\mathbf{y}\|$. Hence

$$-\|\mathbf{x}\|\,\|\mathbf{y}\| \le \mathbf{x}\cdot\mathbf{y} \le \|\mathbf{x}\|\,\|\mathbf{y}\|$$

so we can conclude that $|\mathbf{x}\cdot\mathbf{y}| \le \|\mathbf{x}\|\,\|\mathbf{y}\|$ as desired. □

(c)

Proof. We have

$$\begin{aligned}
\|\mathbf{x}+\mathbf{y}\|^2 &= (\mathbf{x}+\mathbf{y})\cdot(\mathbf{x}+\mathbf{y}) && \text{(by Lemma 20.9.1 part (5))} \\
&= (\mathbf{x}+\mathbf{y})\cdot\mathbf{x} + (\mathbf{x}+\mathbf{y})\cdot\mathbf{y} && \text{(by part (a))} \\
&= \mathbf{x}\cdot(\mathbf{x}+\mathbf{y}) + \mathbf{y}\cdot(\mathbf{x}+\mathbf{y}) && \text{(by Lemma 20.9.1 part (1))} \\
&= \mathbf{x}\cdot\mathbf{x} + \mathbf{x}\cdot\mathbf{y} + \mathbf{y}\cdot\mathbf{x} + \mathbf{y}\cdot\mathbf{y} && \text{(by part (a))} \\
&= \|\mathbf{x}\|^2 + \mathbf{x}\cdot\mathbf{y} + \mathbf{y}\cdot\mathbf{x} + \|\mathbf{y}\|^2 && \text{(by Lemma 20.9.1 part (5))} \\
&= \|\mathbf{x}\|^2 + \mathbf{x}\cdot\mathbf{y} + \mathbf{x}\cdot\mathbf{y} + \|\mathbf{y}\|^2 && \text{(by Lemma 20.9.1 part (1))} \\
&\le \|\mathbf{x}\|^2 + \|\mathbf{x}\|\,\|\mathbf{y}\| + \|\mathbf{x}\|\,\|\mathbf{y}\| + \|\mathbf{y}\|^2 && \text{(by part (b))} \\
&= \|\mathbf{x}\|^2 + 2\,\|\mathbf{x}\|\,\|\mathbf{y}\| + \|\mathbf{y}\|^2 \\
&= (\|\mathbf{x}\| + \|\mathbf{y}\|)^2\,.
\end{aligned}$$

The desired result that $\|\mathbf{x}+\mathbf{y}\| \le \|\mathbf{x}\| + \|\mathbf{y}\|$ then follows from Corollary 20.1.2. □

(d)

Proof. First recall that the euclidean metric on $\mathbb{R}^n$ is defined as

$$d(\mathbf{x},\mathbf{y}) = \|\mathbf{x}-\mathbf{y}\|\ .$$

Then part (1) of the definition of a metric follows directly from Lemma 20.9.1 since of course

$$d(\mathbf{x},\mathbf{y}) = \|\mathbf{x}-\mathbf{y}\| \ge 0\,.$$

We also have that $\mathbf{x}=\mathbf{y}$ if and only if $\mathbf{x}-\mathbf{y}=0$, which is true if and only if

$$d(\mathbf{x},\mathbf{y}) = \|\mathbf{x}-\mathbf{y}\| = 0$$

by Lemma 20.9.1 part (3). Similarly part (2) of the definition follows from Lemma 20.9.1 part (6) as follows:

$$d(\mathbf{x},\mathbf{y}) = \|\mathbf{x}-\mathbf{y}\| = \|(-1)(\mathbf{y}-\mathbf{x})\| = |-1|\,\|\mathbf{y}-\mathbf{x}\| = \|\mathbf{y}-\mathbf{x}\| = d(\mathbf{y},\mathbf{x})\,.$$

Lastly, for part (3) of the definition, we have that

$$\begin{aligned}
d(\mathbf{x},\mathbf{z}) = \|\mathbf{x}-\mathbf{y}\| &= \|\mathbf{x}-\mathbf{z}+\mathbf{y}-\mathbf{y}\| = \|(\mathbf{x}-\mathbf{y})+(\mathbf{y}-\mathbf{z})\| \\
&\le \|\mathbf{x}-\mathbf{y}\| + \|\mathbf{x}-\mathbf{z}\| = d(\mathbf{x},\mathbf{y}) + d(\mathbf{y},\mathbf{z})\,,
\end{aligned}$$

where we have used what was shown in part (c). This shows that d has all the properties required to be a metric. □

Exercise 20.10

Let X denote the subset of $\mathbb{R}^\omega$ consisting of all sequences $(x_1, x_2, \ldots)$ such that $\sum x_i^2$ converges. (You may assume standard facts about infinite series. In case they are not familiar to you, we shall give them in Exercise 11 of the next section.)

(a) Show that if $\mathbf{x},\mathbf{y} \in X$, then $\sum |x_i y_i|$ converges. [Hint: Use (b) of Exercise 9 to show that the partial sums are bounded.]

(b) Let $c \in \mathbb{R}$. Show that if $\mathbf{x},\mathbf{y} \in X$, then so are $\mathbf{x}+\mathbf{y}$ and $c\mathbf{x}$.

(c) Show that

$$d(\mathbf{x}, \mathbf{y}) = \left[\sum_{i=1}^{\infty} (x_i - y_i)^2 \right]^{1/2}$$

is a well-defined metric on X.

Solution:

(a)

Proof. First, denote the partial sums of the series $\sum |x_i y_i|$ by

$$s_n = \sum_{i=1}^{n} |x_i y_i|$$

for any $n \in \mathbb{Z}_+$. Clearly then each term in the sum is non-negative so that the sequence of partial sums is non-decreasing, which is to say that $s_{n+1} \geq s_n$ for every $n \in \mathbb{Z}_+$. Then, to show that the series converges, it suffices to show that the partial sums are bounded, since convergence then follows from what will be shown in Exercise 21.11 part (a). To this end we show that the sequence $(s_1, s_2, \ldots)$ is a bounded above.

So, first let us define the real numbers

$$\|\mathbf{x}\| = \left[\sum_{i=1}^{\infty} x_i^2 \right]^{1/2} \qquad \|\mathbf{y}\| = \left[\sum_{i=1}^{\infty} y_i^2 \right]^{1/2},$$

which we know are well defined since $\mathbf{x}, \mathbf{y} \in X$. Also, for any $n \in \mathbb{Z}_+$, define the truncated sequences $\mathbf{x}_n = (x_1, \ldots, x_n)$ and $\mathbf{y}_n = (y_1, \ldots, y_n)$. We know from Lemma 20.8.5 that the partial series $\sum_{i=n}^{\infty} x_i^2$ and $\sum_{i=n}^{\infty} y_i^2$ are non-negative for any $n \in \mathbb{Z}_+$. From this and Definition 20.8.4 we have that

$$\sum_{i=1}^{n} x_i^2 \leq \sum_{i=1}^{n} x_i^2 + \sum_{i=n+1}^{\infty} x_i^2 = \sum_{i=1}^{\infty} x_i^2$$

for any $n \in \mathbb{Z}_+$. It then follows from Corollary 20.1.2 that

$$\|\mathbf{x}_n\| = \left[\sum_{i=1}^{n} x_i^2 \right]^{1/2} \leq \left[\sum_{i=1}^{\infty} x_i^2 \right]^{1/2} = \|\mathbf{x}\|,$$

and similarly $\|\mathbf{y}_n\| \leq \|\mathbf{y}\|$ for any $n \in \mathbb{Z}_+$. Hence, clearly $\|\mathbf{x}_n\| \|\mathbf{y}_n\| \leq \|\mathbf{x}\| \|\mathbf{y}\|$ since norms are always non-negative by Lemma 20.9.1 part (4).

Lastly, we define $\overline{\mathbf{x}}_n = (|x_1|, \ldots, |x_n|)$ and $\overline{\mathbf{y}}_n = (|y_1|, \ldots, |y_n|)$ for any $n \in \mathbb{Z}_+$. From this definition it follows that

$$\|\overline{\mathbf{x}}_n\| = \left[\sum_{i=1}^{n} |x_i|^2 \right]^{1/2} = \left[\sum_{i=1}^{n} x_i^2 \right]^{1/2} = \|\mathbf{x}_n\|,$$

and similarly that $\|\overline{\mathbf{y}}_n\| = \|\mathbf{y}_n\|$.

Putting all of this together we have by Exercise 20.9 part (b) that

$$s_n = \sum_{i=1}^{n} |x_i y_i| = \sum_{i=1}^{n} |x_i| \, |y_i| = \overline{\mathbf{x}}_n \cdot \overline{\mathbf{y}}_n \leq \|\overline{\mathbf{x}}_n\| \|\overline{\mathbf{y}}_n\| = \|\mathbf{x}_n\| \|\mathbf{y}_n\| \leq \|\mathbf{x}\| \|\mathbf{y}\|$$

for any $n \in \mathbb{Z}_+$. This shows that the sequence of partial sums is bounded by $\|\mathbf{x}\| \|\mathbf{y}\|$, which shows the desired result as previously discussed. $\square$

(b)

Proof. First, it is a well known fact that if a series converges absolutely, then it converges. Hence, since it was shown in part (a) that $\sum |x_i y_i|$ converges, we have that $\sum x_i y_i$ also converges. We of course also know that $\sum x_i^2$ and $\sum y_i^2$ converge since $\mathbf{x}, \mathbf{y} \in X$. It then follows from Exercise 21.11 part (b) that

$$\sum_{i=1}^{\infty}(x_i + y_i)^2 = \sum_{i=1}^{\infty}(x_i^2 + 2x_i y_i + y_i^2)$$

converges to $\sum x_i^2 + 2\sum x_i y_i + \sum y_i^2$. Since of course

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, \ldots)$$

we have that $\mathbf{x} + \mathbf{y} \in X$ as desired. We also have that $c\mathbf{x} = (cx_1, cx_2, \ldots)$ by definition, and it again follows from Exercise 21.11 part (b) that

$$\sum_{i=1}^{\infty}(cx_i)^2 = \sum_{i=1}^{\infty} c^2 x_i^2$$

converges to $c^2 \sum x_i^2$. Hence $c\mathbf{x} \in X$ as desired. $\square$

(c)

Proof. Suppose that $\mathbf{x}, \mathbf{y} \in X$. It then follows from part (b) that $-\mathbf{y} = (-1)\mathbf{y} \in X$. Then also $\mathbf{x} - \mathbf{y} = \mathbf{x} + (-\mathbf{y}) \in X$ as well, again by what was shown in part (b). Since we clearly have

$$\mathbf{x} - \mathbf{y} = (x_1 - y_1, x_2 - y_2, \ldots)\,,$$

it therefore follows that $\sum (x_i - y_i)^2$ converges since $\mathbf{x} - \mathbf{y} \in X$. Hence the function

$$d(\mathbf{x}, \mathbf{y}) = \left[\sum_{i=1}^{\infty}(x_i - y_i)^2\right]^{1/2}$$

is well defined.

To show that d is a metric, first it is obvious that $d(\mathbf{x}, \mathbf{y}) \geq 0$ for all $\mathbf{x}, \mathbf{y} \in X$ since each term is non-negative so that the sequence of partial sums is non-negative and non-decreasing. Also, clearly if $\mathbf{y} = \mathbf{x}$ then

$$d(\mathbf{x}, \mathbf{y}) = d(\mathbf{x}, \mathbf{x}) = \left[\sum_{i=1}^{\infty}(x_i - x_i)^2\right]^{1/2} = \left[\sum_{i=1}^{\infty} 0^2\right]^{1/2} = \left[\sum_{i=1}^{\infty} 0\right]^{1/2} = 0^{1/2} = 0\,.$$

To show the converse, suppose that $\mathbf{x} \neq \mathbf{y}$ so that there is an $n \in \mathbb{Z}_+$ where $x_n \neq y_n$, and hence $x_n - y_n \neq 0$ and $(x_n - y_n)^2 > 0$. Referencing Definition 20.8.4, we then have

$$\begin{aligned} 0 < (x_n - y_n)^2 &= \sum_{i=n}^{n}(x_i - y_i)^2 \leq \sum_{i=1}^{n-1}(x_i - y_i)^2 + \sum_{i=n}^{n}(x_i - y_i)^2 + \sum_{i=n+1}^{\infty}(x_i - y_i)^2 \\ &= \sum_{i=1}^{n}(x_i - y_i)^2 + \sum_{i=n+1}^{\infty}(x_i - y_i)^2 = \sum_{i=1}^{\infty}(x_i - y_i)^2 \end{aligned}$$

Thus by Corollary 20.1.2 we have

$$d(\mathbf{x},\mathbf{y}) = \left[\sum_{i=1}^{\infty}(x_i - y_i)^2\right]^{1/2} > 0^{1/2} = 0$$

so that of course $d(\mathbf{x},\mathbf{y}) \neq 0$ as desired. This shows part (1) of the definition of a metric.

Showing part (2) of the definition is even easier:

$$d(\mathbf{x},\mathbf{y}) = \left[\sum_{i=1}^{\infty}(x_i - y_i)^2\right]^{1/2} = \left[\sum_{i=1}^{\infty}(y_i - x_i)^2\right]^{1/2} = d(\mathbf{y},\mathbf{x})\,.$$

For part (3) consider any $\mathbf{x},\mathbf{y},\mathbf{z} \in X$. Denote the truncated sequences by $\mathbf{x}_n = (x_1,\ldots,x_n)$ for any $n \in \mathbb{Z}_+$, and similarly for $\mathbf{y}_n$ and $\mathbf{z}_n$. Then we have

$$\sum_{i=1}^{\infty}(x_i - y_i)^2 = \lim_{n\to\infty}\sum_{i=1}^{n}(x_i - y_i)^2 = \lim_{n\to\infty}\|\mathbf{x}_n - \mathbf{y}_n\|^2\,.$$

Since the function $f(x) = x^{1/2}$ is continuous on the domain $\{x \in \mathbb{R} \mid x \geq 0\}$ by elementary calculus, it follows from Theorem 21.3 in the next section that

$$d(\mathbf{x},\mathbf{y}) = \left[\sum_{i=1}^{\infty}(x_i - y_i)^2\right]^{1/2} = \left[\lim_{n\to\infty}\|\mathbf{x}_n - \mathbf{y}_n\|^2\right]^{1/2} = \lim_{n\to\infty}\left[\|\mathbf{x}_n - \mathbf{y}_n\|^2\right]^{1/2} = \lim_{n\to\infty}\|\mathbf{x}_n - \mathbf{y}_n\|$$

Since $\mathbf{x}$ and $\mathbf{y}$ are arbitrary, we of course also have that

$$d(\mathbf{x},\mathbf{z}) = \lim_{n\to\infty}\|\mathbf{x}_n - \mathbf{z}_n\| \qquad d(\mathbf{y},\mathbf{z}) = \lim_{n\to\infty}\|\mathbf{y}_n - \mathbf{z}_n\|\,.$$

Now, it was shown in Exercise 20.9 part (d) that

$$\|\mathbf{x}_n - \mathbf{z}_n\| \leq \|\mathbf{x}_n - \mathbf{y}_n\| + \|\mathbf{y}_n - \mathbf{z}_n\|$$

for any $n \in \mathbb{Z}_+$. It then follows from Lemma 20.8.1 that

$$\begin{aligned} d(\mathbf{x},\mathbf{y}) = \lim_{n\to\infty}\|\mathbf{x}_n - \mathbf{z}_n\| &\leq \lim_{n\to\infty}\left(\|\mathbf{x}_n - \mathbf{y}_n\| + \|\mathbf{y}_n - \mathbf{z}_n\|\right) \\ &= \lim_{n\to\infty}\|\mathbf{x}_n - \mathbf{y}_n\| + \lim_{n\to\infty}\|\mathbf{y}_n - \mathbf{z}_n\| \\ &= d(\mathbf{x},\mathbf{y}) + d(\mathbf{y},\mathbf{z}) \end{aligned}$$

as desired. Note that we have used the well known property of sequences that, if $(a_1, a_2, \ldots)$ and $(b_1, b_2, \ldots)$ are real sequences that converge to a and b, respectively, then $\lim_{n\to\infty}(a_n + b_n) = a + b$. (This is shown later in Exercise 21.5.) This shows that d has all three of the properties required to be a metric. □

Exercise 20.11

Show that if d is a metric for X, then

$$d'(x,y) = d(x,y)/(1 + d(x,y))$$

is a bounded metric that gives the same topology of X. [Hint: If $f(x) = x/(1+x)$ for $x > 0$, use the mean value theorem to show that $f(a+b) - f(b) \leq f(a)$.]

Solution:

Proof. First we show that d' is a valid metric. Since d is a metric, we have that

$$d(x,y) \geq 0$$
$$1 + d(x,y) \geq 1 > 0\,.$$

Hence clearly $d'(x,y) = d(x,y)/(1+d(x,y)) \geq 0$ as well. If $x = y$ then $d(x,y) = 0$ so that

$$d'(x,y) = \frac{d(x,y)}{1+d(x,y)} = \frac{0}{1+0} = 0\,.$$

Conversely, if $d'(x,y) = 0$ then it has to be that $d(x,y) = 0$ as well since $1 + d(x,y) > 0$. It then must be that $x = y$ since d is a metric. This shows part (1) of the definition of a metric.

part (2) is easy to show:

$$d'(x,y) = \frac{d(x,y)}{1+d(x,y)} = \frac{d(y,x)}{1+d(y,x)} = d'(y,x)$$

since again d is a metric so that $d(x,y) = d(y,x)$.

For part (3) define the function $f(x) = x/(1+x)$ over the domain $\{x \in \mathbb{R} \mid x \geq 0\}$ so that clearly $d'(x,y) = f(d(x,y))$. We first show that f is monotonically increasing. The easiest way to do this is to show that its derivative is always positive, from which monotonicity follows from elementary calculus. Using the quotient rule, we have

$$f'(x) = \frac{1\cdot(1+x) - x\cdot 1}{(1+x)^2} = \frac{1+x-x}{(1+x)^2} = \frac{1}{(1+x)^2}\,.$$

Clearly we have $x \geq 0$ so that $1 + x \geq 1 > 0$, and hence $(1+x)^2 > 0^2 = 0$ by Lemma 20.1.1. It then follows that $f'(x) = 1/(1+x)^2 > 0$, and thus f is monotonically increasing.

Now, for any $a, b \geq 0$ we clearly have

$$1 + a \leq 1 + a + b + b + ab + b^2$$
$$1 + a \leq 1\cdot(1+a+b) + b(1+a+b)$$
$$1 + a \leq (1+a+b)(1+b)$$
$$\frac{1}{(1+a+b)(1+b)} \leq \frac{1}{1+a}$$

since $1 + a \geq 0$ and $1 + a + b$ and $1 + b$ are both non-negative so that their product is as well. It then follows that

$$\begin{aligned} f(a+b) - f(b) &= \frac{a+b}{1+a+b} - \frac{b}{1+b} = \frac{(a+b)(1+b) - b(1+a+b)}{(1+a+b)(1+b)} \\ &= \frac{a + ab + b + b^2 - b - ab - b^2}{(1+a+b)(1+b)} = \frac{a}{(1+a+b)(1+b)} \\ &\leq \frac{a}{1+a} = f(a) \end{aligned}$$

by what was just shown before since $a \geq 0$. Hence we have $f(a+b) \leq f(a) + f(b)$. Since d is a metric, we then of course have

$$d(x,z) \leq d(x,y) + d(y,z)$$

$$
\begin{aligned}
f(d(x,z)) &\le f(d(x,y)+d(y,z)) && \text{(since } f \text{ is monotonically increasing)} \\
f(d(x,z)) &\le f(d(x,y)+d(y,z)) \\
&\le f(d(x,y)) + f(d(y,z)) && \text{(by what was just shown)} \\
d'(x,z) &\le d'(x,y) + d'(y,z)\,,
\end{aligned}
$$

for any $x, y, z \in X$, which of course shows part (3) of the definition. This completes the proof that d' is a valid metric.

It is easy to see that d' is bounded by 1:

$$
\begin{gathered}
0 < 1 \\
d(x,y) < 1 + d(x,y) \\
\frac{d(x,y)}{1+d(x,y)} < 1 \\
d'(x,y) < 1
\end{gathered}
$$

so that d' is a bounded metric.

Now we must show that the metric topologies induced by d and d' are the same. First we show that the topology induced by d is finer than that induced by d' using Lemma 20.2. So consider any $x \in X$ and any $\epsilon > 0$. If $\epsilon \ge 1$ set $\delta = 999$, otherwise set $\delta = \epsilon/(1-\epsilon)$ noting that in this case $\epsilon < 1$ so that $0 < 1 - \epsilon$ and hence $\delta > 0$. Now consider any $y \in B_d(x,\delta)$ so that $d(y,x) < \delta$. If $\epsilon \ge 1$ then of course $d'(y,x) < 1 \le \epsilon$ since we have previously shown the d' is bounded by 1. On the other hand, if $\epsilon < 1$, then we have

$$
\begin{gathered}
d(y,x) < \delta = \frac{\epsilon}{1-\epsilon} \\
d(y,x)(1-\epsilon) < \epsilon \\
d(y,x) - \epsilon d(y,x) < \epsilon \\
d(y,x) < \epsilon + \epsilon d(y,x) = \epsilon(1 + d(y,x)) \\
\frac{d(y,x)}{1+d(y,x)} < \epsilon \\
d'(x,y) < \epsilon\,.
\end{gathered}
$$

Thus either was we have $d'(y,x) < \epsilon$ so that $y \in B_{d'}(x,\epsilon)$, which of course shows that $B_d(x,\delta) \subset B_{d'}(x,\epsilon)$ since y was arbitrary. Since ϵ was arbitrary this shows the desired result that the topology induced by d is finer than that induced by d'.

Now we show the other direction, i.e. that the d' topology is also finer than the d topology, again using Lemma 20.2. So consider any $x \in X$ and $\epsilon > 0$ again. This time set $\delta = \epsilon/(1+\epsilon)$ noting that $\delta > 0$ follows trivially from the fact that $\epsilon > 0$. Then, for any $y \in B_{d'}(x,\delta)$, we have that

$$
\begin{gathered}
d'(y,x) < \delta = \frac{\epsilon}{1+\epsilon} \\
\frac{d(y,x)}{1+d(y,x)} < \frac{\epsilon}{1+\epsilon} \\
d(y,x)(1+\epsilon) < \epsilon(1+d(y,x)) \\
d(y,x) + \epsilon d(y,x) < \epsilon + \epsilon d(y,x) \\
d(y,x) < \epsilon
\end{gathered}
$$

so that $y \in B_d(x,\epsilon)$. This of course shows that $B_{d'}(x,\delta) \subset B_d(x,\epsilon)$, which in turn shows that the d' topology is finer than the d topology as desired. Since each topology is finer than the other, of course they must be the same as desired. □

§21 The Metric Topology (continued)

Exercise 21.1

Let $A \subset X$. If d is a metric for the topology of X, show that $d \restriction A \times A$ is a metric for the subspace topology on A.

Solution:

Proof. Let us denote the restricted function $d \restriction A \times A$ by d'. Clearly d' is a metric on A, since for $x, y \in A$ we have $d'(x, y) = d(x, y)$ and d has all the properties of a metric. We show that the metric topology induced by d' is the same as the subspace topology using Lemma 13.3 in both directions.

First we show that $B_{d'}(x, \epsilon) = A \cap B_d(x, \epsilon)$ for any $x \in A$ and $\epsilon > 0$. For any $y \in B_{d'}(x, \epsilon)$ we have that clearly $y \in A$ since the metric $d'(y, x)$ must be defined, and $d(y, x) = d'(y, x) < \epsilon$ so that also $y \in B_d(x, \epsilon)$. Thus $y \in A \cap B_d(x, \epsilon)$, and hence $B_{d'}(x, \epsilon) \subset A \cap B_d(x, \epsilon)$. For the other direction suppose that $y \in A \cap B_d(x, \epsilon)$ so that $y \in A$ and $y \in B_d(x, \epsilon)$. Thus y and x are in A so that $d'(y, x)$ is defined and $d'(y, x) = d(y, x) < \epsilon$, and hence $y \in B_{d'}(x, \epsilon)$. This shows that $B_{d'}(x, \epsilon) \supset A \cap B_d(x, \epsilon)$, which in turn shows that the two sets are the same.

Now, clearly each basis element of the metric topology $B_{d'}(x, \epsilon) = A \cap B_d(x, \epsilon)$ is also a basis element of the subspace topology by Lemma 16.1. Hence for any $x \in A$ we have that $B_m = B_{d'}(x, \epsilon)$ is a basis element of the metric topology and $B_s = A \cap B_d(x, \epsilon)$ is a basis element of the subspace topology. But $B_m = B_s$ so that $x \in B_s \subset B_m$ and $x \in B_m \subset B_s$ so that each topology is finer than the other by Lemma 13.3. Thus they are the same topologies. □

Exercise 21.2

Let X and Y be metric spaces with metrics d_X and d_Y, respectively. Let $f : X \to Y$ have the property that for every pair of points x_1, x_2 of X,

$$d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2)\,.$$

Show that f is an imbedding. It is called an ***isometric imbedding*** of X in Y.

Solution:

First, let $Z = f(X)$ and let f' denote the function f with the range restricted to Z so that clearly f' is surjective. First we must show that f and therefore also f' is injective, from which it clearly follows that f' is a bijection. So suppose that $x_1, x_2 \in X$ where $f(x_1) = f(x_2)$. Then we have that

$$d_X(x_1, x_2) = d_Y(f(x_1), f(x_2)) = d_Y(f(x_1), f(x_1)) = 0$$

by property (1) of the metric d_Y, so that it must be that $x_1 = x_2$ since d_X is a metric and $d_X(x_1, x_2) = 0$. This of course means that f and f' are injective and hence f' is bijective.

We show that f' and f'^{-1} are both continuous using Theorem 21.1 since X and Z are both metric spaces, noting that $Z \subset Y$ is a metric space with metric $d_Z = d_Y \restriction Z \times Z$ by the previous exercise. So consider any $x \in X$ and any $\epsilon > 0$, and let $\delta = \epsilon$. Then for any $x_1, x_2 \in X$ where $d_X(x_1, x_2) < \delta$ we have

$$d_Z(f'(x_1), f'(x_2)) = d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2) < \delta = \epsilon\,,$$

which suffices to show that f' is continuous. A similar argument shows that $f'^{-1}: Z \to X$ is continuous. For $y_1, y_2 \in Z$ and $\epsilon > 0$, again let $\delta = \epsilon$ and suppose that $d_Z(y_1, y_2) < \delta$. Let $x_1 = f'^{-1}(y_1)$ and $x_2 = f'^{-1}(y_2)$. Then we have

$$\begin{aligned} d_X(f'^{-1}(y_1), f'^{-1}(y_2)) &= d_X(x_1, x_2) = d_Y(f(x_1), f(x_2)) = d_Y(f'(x_1), f'(x_2)) \\ &= d_Y(f'(f'^{-1}(y_1)), f'(f'^{-1}(y_2))) = d_Y(y_1, y_2) \\ &= d_Z(y_1, y_2) < \delta = \epsilon, \end{aligned}$$

which of course shows that f'^{-1} is also continuous. This shows that f' is a homeomorphism so that f is an imbedding as desired.

Exercise 21.3

Let X_n be a metric space with metric d_n, for $n \in \mathbb{Z}_+$.

(a) Show that

$$\rho(x, y) = \max\{d_1(x_1, y_1), \ldots, d_n(x_n, y_n)\}$$

is a metric for the product space $X_1 \times \cdots \times X_n$.

(b) Let $\bar{d}_i = \min\{d_i, 1\}$. Show that

$$D(x, y) = \sup\{\bar{d}_i(x_i, y_i)/i\}$$

is a metric for the product space $\prod X_i$.

Solution:

Lemma 21.3.1. *Suppose that $(x_\alpha)_{\alpha \in J}$ and $(y_\alpha)_{\alpha \in J}$ are sequences of real numbers indexed by J and that $\{x_\alpha\}$ and $\{y_\alpha\}$ are bounded above so that $\sup\{x_\alpha\}$ and $\sup\{y_\alpha\}$ exist. We assert the following facts:*

(1) If $x_\alpha \le y_\alpha$ for each $\alpha \in J$, then $\sup\{x_\alpha\} \le \sup\{y_\alpha\}$.

(2) $\sup\{x_\alpha + y_\alpha\}$ exists and $\sup\{x_\alpha + y_\alpha\} \le \sup\{x_\alpha\} + \sup\{y_\alpha\}$.

Proof. The proofs of both parts are quite simple. Regarding part (1) we have that $x_\beta \le y_\beta \le \sup\{y_\alpha\}$ for any $\beta \in J$ so that clearly $\sup\{y_\alpha\}$ is an upper bound of the set $\{x_\alpha\}$. It then follows by the definition of the supremum as the least upper bound that $\sup\{x_\alpha\} \le \sup\{y_\alpha\}$ as desired.

For part (2) consider any $\beta \in J$. Clearly $x_\beta \le \sup\{x_\alpha\}$ and $y_\beta \le \sup\{y_\alpha\}$ so that

$$x_\beta + y_\beta \le \sup\{x_\alpha\} + \sup\{y_\alpha\} .$$

Since β was arbitrary, this shows that $\sup\{x_\alpha\} + \sup\{y_\alpha\}$ is an upper bound for the set $\{x_\alpha + y_\alpha\}$. Therefore $\sup\{x_\alpha + y_\alpha\} \le \sup\{x_\alpha\} + \sup\{y_\alpha\}$ as desired by the definition of the supremum as the least upper bound. □

Main Problem.

(a)

Proof. First we must show that ρ is a metric at all. In what follows suppose that $x, y, z \in X_1 \times \cdots \times X_n$ and that k and m are elements of $\{1, \ldots, n\}$ such that

$$\rho(x, y) = \max\{d_1(x_1, y_1), \ldots, d_n(x_n, y_n)\} = d_k(x_k, y_k)$$

and

$$\rho(x, z) = \max\{d_1(x_1, z_1), \ldots, d_n(x_n, z_n)\} = d_m(x_m, z_m) .$$

First, we have $\rho(x, y) = d_k(x_k, y_k) \geq 0$ since d_k is a metric. If $x = y$ then of course $x_k = y_k$ so that $\rho(x, y) = d_k(x_k, y_k) = 0$. Now suppose that $\rho(x, y) = 0$ and consider any $l \in \{1, \ldots, n\}$. Then we have that $d_l(x_l, y_l) \geq 0$ and that $d_l(x_l, y_l) \leq \rho(x, y) = 0$, and hence it must be that $d_l(x_l, y_l) = 0$ so that $x_l = y_l$ since d_l is a metric. Since l was arbitrary, this shows that $x = y$, which shows that ρ satisfies part (1) of the definition of a metric.

As usual, part (2) of the definition is the easiest to show since

$$\begin{aligned}\rho(x, y) &= \max\{d_1(x_1, y_1), \ldots, d_n(x_n, y_n)\} = \\ &= \max\{d_1(y_1, x_1), \ldots, d_n(y_n, x_n)\} = \\ &= \rho(y, x) ,\end{aligned}$$

as we have that each $d_l(x_l, y_l) = d_l(y_l, x_l)$ since d_l is a metric. Lastly, for part (3) we have

$$\rho(x, z) = d_m(x_m, z_m) \leq d_m(x_m, y_m) + d_m(y_m, z_m) \leq \rho(x, y) + \rho(y, z)$$

since of course d_m is a metric. This completes the proof that ρ is a proper metric.

Now we show that both topologies are the same using Lemma 13.3. So suppose that $x \in X_1 \times \cdots \times X_n$ and $B_\rho(x, \epsilon)$ is any basis element of the metric topology and let $B = \prod_{i=1}^n B_{d_i}(x_i, \epsilon)$, which is clearly a basis element of the product topology that contains x since each $B_{d_i}(x_i, \epsilon)$ is open in the metric space X_i. Now suppose that $y \in B$ so that each $y_i \in B_{d_i}(x_i, \epsilon)$. So, for every $i \in \{1, \ldots, n\}$, we have $d_i(y_i, x_i) < \epsilon$ so that clearly

$$\rho(y, x) = \max\{d_1(y_1, x_1), \ldots, d_n(y_n, x_n)\} < \epsilon ,$$

which shows that $y \in B_\rho(x, \epsilon)$. This shows that $x \in B \subset B_\rho(x, \epsilon)$ so that $B_\rho(x, \epsilon)$ the product topology is finer than the metric topology by Lemma 13.3.

Now consider again any $x \in X_1 \times \cdots \times X_n$ and any basis element $B = \prod_{i=1}^n U_i$ of the product topology. Then of course each U_i is open in X_i and $x_i \in U_i$ so that there is a ball $B_{d_i}(x_i, \epsilon_i)$ such that $B_{d_i}(x_i, \epsilon_i) \subset U_i$ by Lemma 20.4.1. Let $\epsilon = \min\{\epsilon_1, \ldots, \epsilon_n\}$ and consider the basis element $B_\rho(x, \epsilon)$ in the metric space induced by ρ. Clearly $x \in B_\rho(x, \epsilon)$ so consider any $y \in B_\rho(x, \epsilon)$ so that

$$\rho(y, x) = \max\{d_1(y_1, x_1), \ldots, d_n(y_n, x_n)\} < \epsilon .$$

Then, for any $i \in \{1, \ldots, n\}$, we have

$$d_i(y_i, x_i) \leq \rho(x, \epsilon) < \epsilon \leq \epsilon_i$$

so that $y_i \in B_{d_i}(x_i, \epsilon_i) \subset U_i$. Since this is true of any $i \in \{1, \ldots, n\}$, it follows that $y \in \prod_{i=1}^n U_i = B$. We can then conclude that $x \in B_\rho(x, \epsilon) \subset B$ since y was arbitrary. This shows that the ρ-metric topology is finer than the product topology, again by Lemma 13.3. Hence the two topologies are the same as desired. $\square$

(b)

Proof. First we show that the metric D is well-defined and is in fact a metric. In what follows suppose that $x, y, z \in \prod X_i$. For any $j \in \mathbb{Z}_+$ we have that $0 < 1 \leq j$ so that $1/j \leq 1$. We also have that $\bar{d}_j(x_j, y_j) = \min\{d_j(x_j, y_j), 1\} \leq 1$ so that $\bar{d}_j(x_j, y_j)/j \leq 1/j \leq 1$. Hence 1 is an upper bound for the set $\{\bar{d}_i(x_i, y_i)/i\}$ since j was arbitrary, so that

$$D(x, y) = \sup\{\bar{d}_i(x_i, y_i)/i\}$$

exists and hence D is well defined.

To show part (1) of the definition of a metric, pick any $j \in \mathbb{Z}_+$ so that clearly $\bar{d}_j(x_j, y_j) \geq 0$ and hence $\bar{d}_j(x_j, y_j)/j \geq 0$ also since $j \geq 1 > 0$. Thus we have

$$D(x, y) = \sup\{\bar{d}_i(x_i, y_i)/i\} \geq \bar{d}_j(x_j, y_j)/j \geq 0\,.$$

If $x = y$ then we have that $x_j = y_j$ for any $j \in \mathbb{Z}_+$. Thus $\bar{d}_j(x_j, y_j) = 0$ since $\bar{d}_j$ is a standard bounded metric, and therefore

$$\frac{\bar{d}_j(x_j, y_j)}{j} = \frac{0}{j} = 0\,.$$

Since j was arbitrary, this shows that

$$D(x, y) = \sup\left\{\frac{\bar{d}_i(x_i, y_i)}{j}\right\} = \sup\{0\} = 0\,.$$

On the other hand, if $D(x, y) = 0$ then, for any $j \in \mathbb{Z}_+$, we have

$$0 \leq \frac{\bar{d}_j(x_j, y_j)}{j} \leq \sup\left\{\frac{\bar{d}_i(x_i, y_i)}{i}\right\} = D(x, y) = 0\,.$$

This shows that $\bar{d}_j(x_j, y_j)/j = 0$ so that clearly $\bar{d}_j(x_j, y_j) = 0$ as well, from which it follows that $x_j = y_j$ since $\bar{d}_j$ is a standard bounded metric. Since j was arbitrary, this shows that $x = y$, which completes the proof of part (1).

Showing part (2) is quite easy:

$$D(x, y) = \sup\left\{\frac{\bar{d}_i(x_i, y_i)}{i}\right\} = \sup\left\{\frac{\bar{d}_i(y_i, x_i)}{i}\right\} = D(x, y)$$

since $\bar{d}_i$ is a standard bounded metric for every $i \in \mathbb{Z}_+$.

For part (3) consider any $j \in \mathbb{Z}_+$. Then

$$\bar{d}_j(x_j, z_j) \leq \bar{d}_j(x_j, y_j) + \bar{d}_j(y_j, z_j)$$

since $\bar{d}_j$ is a standard bounded metric. Then of course

$$\frac{\bar{d}_j(x_j, z_j)}{j} \leq \frac{\bar{d}_j(x_j, y_j) + \bar{d}_j(y_j, z_j)}{j} = \frac{\bar{d}_j(x_j, y_j)}{j} + \frac{\bar{d}_j(y_j, z_j)}{j}$$

since $j \geq 1 > 0$. Then, since j was arbitrary this shows that

$$\begin{aligned} D(x, z) = \sup\left\{\frac{\bar{d}_i(x_i, z_i)}{i}\right\} &\leq \sup\left\{\frac{\bar{d}_i(x_i, y_i)}{i} + \frac{\bar{d}_i(y_i, z_i)}{i}\right\} \\ &\leq \sup\left\{\frac{\bar{d}_i(x_i, y_i)}{i}\right\} + \sup\left\{\frac{\bar{d}_i(y_i, z_i)}{i}\right\} \end{aligned}$$

$$= D(x,y) + D(y,z)$$

by both parts of Lemma 21.3.1. This of course completes the proof that D is a well defined metric.

Now we show that the metric topology induced by D is the same as the product topology on $\prod X_i$, which we do using Lemma 13.3. So consider any $x \in \prod X_i$ and any basis element of the metric topology $B_D(x,\epsilon)$ centered at x. Clearly there is a positive integer N large enough such that $N > 2/\epsilon$, and so $1/N < \epsilon/2$. Now define the sets

$$U_i = \begin{cases} B_{d_i}(x_i, \epsilon/2) & i < N \\ \mathbb{R} & i \geq N \end{cases}$$

for $i \in \mathbb{Z}_+$, and the set $B = \prod U_i$. Clearly B is a basis element of the product topology since each U_i is open in X_i and $U_i \neq \mathbb{R}$ for only finitely many i, namely when $i < N$. Clearly also $x \in B$ since each $x_i \in U_i$. Now suppose that $y \in B$ and consider any $j \in \mathbb{Z}_+$. If $j < N$ then we of course have that $y_j \in U_j = B_{d_j}(x_j, \epsilon/2)$ so that

$$\frac{\bar{d}_j(y_j, x_j)}{j} \leq \bar{d}_j(y_j, x_j) \leq d_j(y_j, x_j) < \epsilon/2$$

since $j \geq 1$. If $j \geq N$ then we have $1/j \leq 1/N$ so that

$$\frac{\bar{d}_j(y_i, x_i)}{j} \leq \frac{1}{j} \leq \frac{1}{N} < \frac{\epsilon}{2}$$

since $\bar{d}_j(y_j, x_j) \leq 1$. Since j was arbitrary, this shows that

$$D(y,x) = \sup\left\{\frac{\bar{d}_i(y_i, x_i)}{i}\right\} \leq \frac{\epsilon}{2} < \epsilon$$

so that $y \in B_D(x,\epsilon)$. Therefore $x \in B \subset B_D(x,\epsilon)$ since y was arbitrary. Hence the product topology is finer than the metric topology by Lemma 13.3.

Now again consider any $x \in \prod X_i$ and any basis element $B = \prod U_i$ of the product topology where $x \in B$. Then of course each U_i is open in X_i and there is a finite subset $J \subset \mathbb{Z}_+$ where $U_i = \mathbb{R}$ for every $i \notin J$. Of course also $x \in U_i$ for each $i \in \mathbb{Z}_+$. For any $j \in J$ we have that $x \in U_j$ and U_j is open in X_i so that there is a basis element $B_{d_j}(x_j, \epsilon_j)$ such that $B_{d_j}(x_j, \epsilon_j) \subset U_j$. So let $\epsilon = \min(\{\epsilon_j \mid j \in J\} \cup \{1\})$ and $k = \max\{j \mid j \in J\}$, which both exists since J is finite, and also clearly $\epsilon > 0$.

Now consider the set $B_D(x, \epsilon/k)$, which is a basis element of the metric topology that clearly contains x. Suppose that $y \in B_D(x, \epsilon/k)$ so that $D(y,x) < \epsilon/k$. Then, for any $j \in \mathbb{Z}_+$, clearly $y_i \in \mathbb{R} = U_i$ if $j \neq J$. On the other hand, if $j \in J$ then we have that $k \geq j$ so that $1/k \leq 1/j$. We also have

$$\frac{\bar{d}(y_j, x_j)}{j} \leq \sup\left\{\frac{\bar{d}_i(y_i, x_i)}{i}\right\} = D(y,x) < \frac{\epsilon}{k} \leq \frac{\epsilon}{j} \leq \frac{1}{j}$$

since $1/k \leq 1/j$ and $\epsilon \leq 1$. Therefore $\bar{d}_j(y_j, x_j) < 1$ so that it must be that $\bar{d}_j(y_j, x_j) = d_j(y_j, x_j)$. Hence we have $d_j(y_j, x_j) = \bar{d}_j(y_j, x_j) < \epsilon \leq \epsilon_j$ so that $y_j \in B_{d_j}(x_j, \epsilon_j) \subset U_j$. Therefore $y_i \in U_i$ for all $i \in \mathbb{Z}_+$ so that $y \in \prod U_i = B$. Since y was arbitrary, this shows that $x \in B_D(x,\epsilon) \subset B$, which in turn proves that the metric topology is also finer than the product topology by Lemma 13.3. Thus the two topologies must be the same. □

Exercise 21.4

Show that $\mathbb{R}_\ell$ and the ordered square satisfy the first countability axiom. (This result does not, of course, imply that they are metrizable.)

Solution:

Proof. Suppose that x is any element of $\mathbb{R}_\ell$. Define the sets $U_n = [x, x+1/n)$ for $n \in \mathbb{Z}_+$. Clearly this is a countable collection of neighborhoods of x since each U_n is a basis element of $\mathbb{R}_\ell$ and $x \in U_n$. Now consider any other neighborhood U of x so that there is a basis element $B = [a, b)$ that contains x and $B \subset U$. Thus of course $a \leq x < b$. There is clearly a positive integer N large enough that $N > 1/(b-x)$, noting that $b - x > 0$ since $x < b$. Now consider any $y \in U_N = [x, x+1/N)$ so that $x \leq y < x + 1/N$. Then we have

$$\begin{aligned} \frac{1}{b-x} &< N \\ \frac{1}{N} &< b - x && \text{(since both } N \geq 1 > 0 \text{ and } b - x > 0\text{)} \\ x + \frac{1}{N} &< b \end{aligned}$$

so that $y \leq x + 1/N < b$. As we also clearly have $a \leq x \leq y$ it follows that $y \in [a, b) = B \subset U$. Thus $U_N \subset U$ since y was arbitrary. This shows that $\mathbb{R}_\ell$ satisfies the first countability axiom since U and x were arbitrary.

Now recall that the ordered square is the set $I \times I$ where $I = [0, 1]$ with the dictionary order topology. In what follows let $\prec$ denote the dictionary order on $I \times I$. So suppose that $\mathbf{x} = x_1 \times x_2 \in I \times I$ so that of course $0 \leq x_1 \leq 1$ and $0 \leq x_2 \leq 1$. Now define the following

$$a_{n,1} = \begin{cases} \max\{x_1 - 1/n, 0\} & x_2 = 0 \\ x_1 & x_2 > 0 \end{cases} \qquad b_{n,1} = \begin{cases} \min\{x_1 + 1/n, 1\} & x_2 = 1 \\ x_1 & x_2 < 1 \end{cases}$$

$$a_{n,2} = \max\{x_2 - 1/n, 0\} \qquad b_{n,2} = \min\{x_2 + 1/n, 1\}$$

for $n \in \mathbb{Z}_+$, which are all well defined since it is never the case that $x_2 < 0$ or $x_2 > 1$. Also define $\mathbf{a}_n = a_{n,1} \times a_{n,2}$ and $\mathbf{b}_n = b_{n,1} \times b_{n,2}$ for $n \in \mathbb{Z}_+$. Lastly, define the sets

$$U_n = \begin{cases} [\mathbf{a}_n, \mathbf{b}_n) & x_1 = x_2 = 0 \\ (\mathbf{a}_n, \mathbf{b}_n] & x_1 = x_2 = 1 \\ (\mathbf{a}_n, \mathbf{b}_n) & \text{otherwise} \end{cases}$$

for $n \in \mathbb{Z}_+$, noting that the intervals are in the dictionary order so that these are basis elements of the dictionary order topology and so are open.

As it is not obvious with all the different cases going on, we now show that $\mathbf{x} \in U_n$ for every $n \in \mathbb{Z}_+$. So consider any $n \in \mathbb{Z}_+$.

Case: $x_2 = 0$. Then we have $b_{n,1} = x_1$ and $x_2 < \min\{x_2 + 1/n, 1\} = b_{n,2}$ since $x_2 = 0 < 1$ and clearly $x_2 < x_2 + 1/n$. This shows that $\mathbf{x} \prec \mathbf{b}_n$.

> Case: $x_1 = 0$. Then we clearly have that $x_1 - 1/n < x_1 = 0$, and hence $a_{n,1} = \max\{x_1 - 1/n, 0\} = 0 = x_1$. Also clearly $a_{n,1} = 0 = x_2$ since $x_2 = 0$. Hence $\mathbf{a}_n = 0 \times 0 = \mathbf{x}$ so that $\mathbf{a}_n \preccurlyeq \mathbf{x}$ is true. This shows that $\mathbf{x} \in [\mathbf{a}_n, \mathbf{b}_n) = U_n$.
>
> Case: $x_1 > 0$. Then we have $x_1 - 1/n < x_1$ and $0 < x_1$ so that $a_{n,1} = \max\{x_1 - 1/n, 0\} < x_1$. Therefore $\mathbf{a}_n \prec \mathbf{x}$ so that $\mathbf{x} \in (\mathbf{a}_n, \mathbf{b}_n) = U_n$.

Case: $x_2 = 1$. Then we have $a_{n,1} = x_1$ and $a_{n,2} = \max\{x_2 - 1/n, 0\} < x_2$ since $x_2 = 1 > 0$ and clearly $x_2 > x_2 - 1/n$. This shows that $\mathbf{a}_n \prec \mathbf{x}$.

Case: $x_1 = 1$. Then $x_1 + 1/n > x_1 = 1$ so that $b_{n,1} = \min\{x_1 + 1/n, 1\} = 1 = x_1$. Likewise we have that $x_2 + 1/n > x_2 = 1$ so that $b_{n,2} = \min\{x_2 + 1/n, 1\} = 1$. Thus $\mathbf{b}_n = 1 \times 1 = \mathbf{x}$ and hence $\mathbf{x} \preccurlyeq \mathbf{b}_n$ is true. Therefore $\mathbf{x} \in (\mathbf{a}_n, \mathbf{b}_n] = U_n$.

Case: $x_1 < 1$. Then we have $x_1 + 1/n > x_1$ and $1 > x_1$ so that $b_{n,1} = \min\{x_1 + 1/n, 1\} > x_1$. Therefore $\mathbf{x} \prec \mathbf{b}_n$ so that $\mathbf{x} \in (\mathbf{a}_n, \mathbf{b}_n) = U_n$.

Case: $0 < x_2 < 1$. Then $a_{n,1} = x_1$, and $x_2 - 1/n < x_2$ and $0 < x_2$ so that $a_{n,2} = \max\{x_2 - 1/n, 0\} < x_2$. This shows that $\mathbf{a}_n \prec \mathbf{x}$. Similarly $b_{n,1} = x_1$, and $x_2 + 1/n > x_2$ and $1 > x_2$ so that $b_{n,2} = \min\{x_2 + 1/n, 1\} > x_2$. This shows that $\mathbf{x} \prec \mathbf{b}_n$. Therefore we have $\mathbf{x} \in (\mathbf{a}_n, \mathbf{b}_n) = U_n$.

Hence $\mathbf{x} \in U_n$ in all of the exhaustive cases so that $\{U_n\}_{n \in \mathbb{Z}_+}$ is a countable collection of neighborhoods of $\mathbf{x}$.

Lastly consider any other neighborhood of U of $\mathbf{x}$ in the dictionary order topology. Then there is a basis element B of the dictionary order topology such that $\mathbf{x} \in B \subset U$. Then we have that either $B = [\mathbf{c}, \mathbf{d})$ where $\mathbf{c} = \mathbf{0}$, $B = (\mathbf{c}, \mathbf{d})$, or $B = (\mathbf{c}, \mathbf{d}]$ where $\mathbf{d} = \mathbf{1}$ (where we denote $\mathbf{1} = 1 \times 1$). Now we set $N_a, N_b \in \mathbb{Z}_+$ based on the different cases we might have. First, we know that no matter what we have $\mathbf{c} \preccurlyeq \mathbf{x}$ since $\mathbf{x} \in B$. We therefore have:

Case: $c_1 < x_1$. If $x_2 > 0$ then set $N_a = 1$, and otherwise $x_2 = 0$ and there is an N_a large enough such that $N_a > 1/(x_1 - c_1)$, noting that this is defined since $x_1 - c_1 > 0$.

Case: $c_1 = x_1$. Then it has to be that $c_2 \leq x_2$ since $\mathbf{c} \preccurlyeq \mathbf{x}$. If $c_2 = x_2$ then it must be that $B = [\mathbf{c}, \mathbf{d})$ and $\mathbf{c} = \mathbf{x} = \mathbf{0}$, so just set $N_a = 1$. On the other hand, if $c_2 < x_2$, then there must be an N_a large enough such that $N_a > 1/(x_2 - c_2)$, noting that $x_2 - c_2 > 0$.

Now we set N_b in an analogous way, noting that we have $\mathbf{x} \preccurlyeq \mathbf{d}$ no matter what since $\mathbf{x} \in B$:

Case: $x_1 < d_1$. If $x_2 < 1$ then simply set $N_b = 1$, and otherwise $x_2 = 1$ and there is an N_b large enough such that $N_b > 1/(d_1 - x_1)$, noting that $d_1 - x_1 > 0$.

Case: $x_1 = d_1$. Then it must be that $x_2 \leq d_2$ since $\mathbf{x} \preccurlyeq \mathbf{d}$. If $x_2 = d_2$ then it has to be that $B = (\mathbf{c}, \mathbf{d}]$ and $\mathbf{x} = \mathbf{d} = \mathbf{1}$, so just set $N_b = 1$. On the other hand, if $x_2 < d_2$, then there is an N_b large enough such that $N_b > 1/(d_2 - x_2)$, noting that $d_2 - x_2 > 0$.

Now let $N = \max\{N_a, N_b\}$, and we claim that $U_N \subset B$ in every case. We have again know that $\mathbf{c} \preccurlyeq \mathbf{x}$ so that we have:

Case: $c_1 < x_1$. If $x_2 > 0$ then $c_1 < x_1 = a_{N,1}$. If $x_2 = 0$ then we have $N \geq N_a > 1/(x_1 - c_1)$, from which it readily follows that $c_1 < x_1 - 1/n \leq a_{N,1}$. Thus either way we have $c_1 < a_{N,1}$ so that $\mathbf{c} \prec \mathbf{a}_N$.

Case: $c_1 = x_1$. If also $c_2 = x_2$ then again it has to be that $B = [\mathbf{c}, \mathbf{d})$ and $\mathbf{c} = \mathbf{x} = \mathbf{0}$. In this case it was established above that $U_N = [\mathbf{a}_N, \mathbf{b}_N)$ and $\mathbf{a}_N = \mathbf{0}$. So we have here that $\mathbf{c} = \mathbf{0} \preccurlyeq \mathbf{0} = \mathbf{a}_N$. On the other hand if $c_2 < x_2$ then $N \geq N_a > 1/(x_2 - c_2)$, from which it follows that $c_2 < x_2 - 1/n \leq a_{N,2}$. Also $c_1 = x_1 = a_{N,1}$ so that $\mathbf{c} \prec \mathbf{a}_N$ since $0 \leq c_2 < x_2$.

We also of course again have that $\mathbf{x} \preccurlyeq \mathbf{d}$ so that

Case: $x_1 < d_1$. If $x_2 < 1$ then $b_{N,1} = x_1 < d_1$. If $x_2 = 1$ then we have $N \geq N_b > 1/(d_1 - x_1)$, from which it follows that $b_{N,1} \leq x_1 + 1/N < d_1$. Hence either was $b_{N,1} < d_1$ so that $\mathbf{b}_N \prec \mathbf{d}$.

Case: $x_1 = d_1$. If also $x_2 = d_2$ then again it has to be that $B = (\mathbf{c}, \mathbf{d}]$ and $\mathbf{d} = \mathbf{x} = \mathbf{1}$. In this case it was established above that $U_N = (\mathbf{a}_N, \mathbf{b}_N]$ and $\mathbf{b}_N = \mathbf{1}$. So we have here that $\mathbf{b}_N = \mathbf{1} \preccurlyeq \mathbf{1} = \mathbf{d}$. On the other hand if $x_2 < d_2$ then we have $N \geq N_b > 1/(d_2 - x_2)$, from which it readily follows that $b_{N,2} \leq x_2 + 1/N < d_2$. We also have $b_{N,1} = x_1 = d_1$ so that $\mathbf{b}_N \prec \mathbf{d}$ since $x_2 < d_2 \leq 1$.

Therefore in every case we have that $\mathbf{c} \prec \mathbf{a}_N$ except when $\mathbf{c} = \mathbf{x} = \mathbf{a}_N = \mathbf{0}$ so that $U_N = [\mathbf{0}, \mathbf{b}_N)$ and $B = [\mathbf{0}, \mathbf{d})$. Similarly we always have $\mathbf{b}_N \prec \mathbf{d}$ except when $\mathbf{b}_N = \mathbf{x} = \mathbf{d} = \mathbf{1}$ so that $U_N = (\mathbf{a}_N, \mathbf{1}]$ and $B = (\mathbf{c}, \mathbf{1}]$. When $\mathbf{c} = \mathbf{x} = \mathbf{a}_N = \mathbf{0}$ we cannot have $\mathbf{x} = \mathbf{1}$, so that $\mathbf{b}_N \prec \mathbf{d}$. Analogously, when $\mathbf{b}_N = \mathbf{x} = \mathbf{d} = \mathbf{1}$ it cannot be that $\mathbf{x} = \mathbf{0}$, and hence $\mathbf{c} \prec \mathbf{a}_N$. Otherwise we have $U_N = (\mathbf{a}_N, \mathbf{b}_N)$,

$\mathbf{c} \prec \mathbf{a}_N$, and $\mathbf{b}_N \prec \mathbf{d}$ so that in every case $\mathbf{x} \in U_N \subset B \subset U$. Since U was an arbitrary neighborhood and $\mathbf{x}$ was also arbitrary, this shows that $I \times I$ satisfies the first countability axiom as desired. $\square$

Exercise 21.5

Theorem. *Let $x_n \to x$ and $y_n \to y$ in the space $\mathbb{R}$. Then*

$$\begin{aligned} x_n + y_n &\to x + y \\ x_n - y_n &\to x - y \\ x_n y_n &\to xy\,, \end{aligned}$$

and provided that each $y_n \neq 0$ and $y \neq 0$,

$$x_n / y_n \to x/y\,.$$

[Hint: Apply Lemma 21.4; recall from the exercises of §19 that if $x_n \to x$ and $y_n \to y$, then $x_n \times y_n \to x \times y$.]

Solution:

Proof. First, we have that the sequence $x_n \times y_n$ converges to $x \times y$ in the product space $\mathbb{R} \times \mathbb{R}$ by Exercise 19.6 since both $x_n \to x$ and $y_n \to y$ in $\mathbb{R}$. Now suppose that $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous. Then we have that

$$\lim_{n\to\infty} f(x_n \times y_n) = f\left(\lim_{n\to\infty} x_n \times y_n\right) = f(x \times y)$$

by Theorem 21.3. Now, we have that addition, subtraction, and multiplication are all continuous functions from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$ by Lemma 21.4. It then follows that for the continuous function $+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ we have

$$\lim_{n\to\infty} (x_n + y_n) = \lim_{n\to\infty} +(x_n \times y_n) = +(x \times y) = x + y\,.$$

It similarly follows that $x_n - y_n \to x - y$ and $x_n y_n \to xy$ as desired.

Regarding the quotient, we note that (y_n) is a sequence in the subspace topology $\mathbb{R} - \{0\}$ since each $y_n \neq 0$. We also note that $y \neq 0$ and hence also $y \in \mathbb{R} - \{0\}$ so that the sequence (y_n) converges to a point still within the space $\mathbb{R} - \{0\}$. It then again follows that $x_n \times y_n \to x \times y$ in the product space $\mathbb{R} \times (\mathbb{R} - \{0\})$ by Exercise 19.6. Since the quotient function is continuous from $\mathbb{R} \times (\mathbb{R} - \{0\})$ to $\mathbb{R}$ by Lemma 21.4, it then follows as before that $x_n / y_n \to x/y$ by Theorem 21.3 just as we would like. $\square$

Exercise 21.6

Define $f_n : [0, 1] \to \mathbb{R}$ by the equation $f_n(x) = x^n$. Show that the sequence $(f_n(x))$ converges for each $x \in [0, 1]$, but that the sequence (f_n) does not converge uniformly.

Solution:

First we build up a little more theory

Definition 21.6.1. *Let $f_n : X \to Y$ be a sequence of functions from a set X to topological space Y. We say that the sequence of functions converges* pointwise *to a function $f : X \to Y$ if the sequence $(f_n(x))$ converges to $f(x)$ for every $x \in X$.*

Lemma 21.6.2. *Let $f_n : X \to Y$ be a sequence of functions from a set X to topological space Y. If Y is a Hausdorff space and (f_n) converges pointwise to f, then f is unique.*

Proof. Suppose that (f_n) converges pointwise to two distinct functions f and g. Since they are distinct, there is an $x_0 \in X$ where $f(x_0) \neq g(x_0)$. But then the sequence $(f_n(x_0))$ converges to both of the distinct points $f(x_0)$ and $g(x_0)$ since (f_n) converges pointwise to both f and g. As Y is Hausdorff, this violates Theorem 17.10 so that (f_n) can only converge pointwise to a unique function. □

Lemma 21.6.3. *Let $f_n : X \to Y$ be a sequence of functions from a set X to metric space Y with metric d. If (f_n) converges uniformly to a function f, then it also converges pointwise to f.*

Proof. Suppose that (f_n) converges uniformly to f and consider any $x_0 \in X$ and $\epsilon > 0$. Then there is an $N \in \mathbb{Z}_+$ where $d(f_n(x), f(x)) < \epsilon$ for all $n > N$ and $x \in X$. Then, since $x_0 \in X$, we have $d(f_n(x_0), f(x_0)) < \epsilon$ for all $n \geq N + 1$ since then $n > N$. This shows that the sequence $(f_n(x_0))$ converges to $f(x_0)$ since ϵ was arbitrary. Since x_0 was arbitrary, this shows that (f_n) converges pointwise to f as desired. □

Main Problem.

Proof. First we show that the sequence (f_n) converges pointwise to the function $g : [0, 1] \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} 0 & x < 1 \\ 1 & x = 1 \end{cases}$$

for $x \in [0, 1]$. This of course shows that $(f_n(x))$ converges for each $x \in [0, 1]$ by Definition 21.6.1.

So consider any such $x \in [0, 1]$ so that we have the following exhaustive cases:

Case: $x = 0$. Then $f_n(x) = x^n = 0^n = 0$ for any $n \in \mathbb{Z}_+$. Thus clearly the sequence $(f_n(x)) = (0, 0, \ldots)$ converges to 0.

Case: $x = 1$. Somewhat similarly we have $f_n(x) = x^n = 1^n = 1$ for every $n \in \mathbb{Z}_+$ so that clearly the sequence $(f_n(x)) = (1, 1, \ldots)$ converges to 1.

Case: $0 < x < 1$. Here we show that, for any $x \in (0, 1)$, that the sequence $(f_n(x))$ is strictly decreasing and that it is bounded below by 0. We prove this using induction on n. So first for $n = 1$ we have

$$\begin{aligned} 0 &< x < 1 \\ 0 &< x^2 < x && \text{(since } x > 0\text{)} \\ 0 &< f_2(x) < f_1(x) \\ 0 &< f_{n+1}(x) < f_n(x) \,. \end{aligned}$$

For the inductive step suppose that $f_n(x) > 0$ and $f_{n+1}(x) < f_n(x)$. Then we of course have $0 < f_n(x) = x^n$ so that clearly $0 = x \cdot 0 < x \cdot x^n = x^{n+1} = f_{n+1}(x)$ as well since $x > 0$. We also clearly have

$$\begin{aligned} f_{n+1} &< f_n(x) \\ x^{n+1} &< x^n \end{aligned}$$

$$
\begin{aligned}
x \cdot x^{n+1} &< x \cdot x^n && \text{(since } x > 0\text{)} \\
x^{n+2} &< x^{n+1} \\
f_{n+2}(x) &< f_{n+1}(x)\,,
\end{aligned}
$$

which completes the inductive step.

Hence the sequence is strictly decreasing (and therefore of course non-increasing) and bounded below by zero, from which it follows that it converges by a theorem analogous in an obvious way to that shown in Exercise 21.11 part (a). So suppose that the sequence converges to the value a. Then of course also the sequence $(f_{n+1}(x))$ must also converge to a. Thus we have

$$a = \lim_{n\to\infty} f_{n+1}(x) = \lim_{n\to\infty} x^{n+1} = \lim_{n\to\infty} x \cdot x^n = \lim_{n\to\infty} x f_n(x) = x \lim_{n\to\infty} f_n(x) = xa\,,$$

where we note that clearly the function $h(y) = xy$ for $y \in \mathbb{R}$ is clearly continuous so that

$$\lim_{n\to\infty} x f_n(x) = \lim_{n\to\infty} h(f_n(x)) = h\left(\lim_{n\to\infty} f_n(x)\right) = x \lim_{n\to\infty} f_n(x)$$

by Theorem 21.3. Therefore we have $a = xa$ so that it would be that $1 = x$ if a were any nonzero value. However, we know that $x < 1$ so it must be that $a = 0$. This completes the proof that (f_n) converges pointwise to g.

It then becomes fairly easy to show that $(f_n(x))$ does not converge uniformly. Consider any function $f : [0,1] \to \mathbb{R}$. If $f \neq g$ then (f_n) cannot converge pointwise to f since it was shown to converge pointwise to g above and g is unique by Lemma 21.6.2 since $\mathbb{R}$ is Hausdorff. Hence (f_n) cannot converge uniformly to f by the converse of Lemma 21.6.3 since it does even not converge pointwise. On the other hand if $f = g$ then clearly $f = g$ has a discontinuity at 1 so that it is not continuous. However each $f_n(x) = x^n$ is continuous on $[0,1]$ by elementary calculus. This shows that (f_n) cannot converge uniformly to $f = g$ since this would violate Theorem 21.6. Therefore (f_n) does converge uniformly to any function since f was arbitrary. □

Exercise 21.7

Let X be a set, and let $f_n : X \to \mathbb{R}$ be a sequence of functions. Let $\bar{\rho}$ be the uniform metric on the space $\mathbb{R}^X$. Show that the sequence (f_n) converges uniformly to the function $f : X \to \mathbb{R}$ if any only if the sequence converges to f as elements of the metric space $(\mathbb{R}^X, \bar{\rho})$.

Solution:

Proof. (⇒) First suppose that (f_n) converges uniformly to f in the functional sense. Consider any $\epsilon > 0$. Then, by the definition of uniform convergence there is an $N \in \mathbb{Z}_+$ where

$$d(f_n(x), f(x)) < \epsilon/2$$

for all $n > N$ and $x \in X$, noting that d of course denotes the usual metric on $\mathbb{R}$. Now consider any $x_0 \in X$ and an $n \geq N+1$ so that $n > N$. Then of course we have

$$\bar{d}(f_n(x_0), f(x_0)) = \min\{d(f_n(x_0), f(x_0)), 1\} \leq d(f_n(x_0), f(x_0)) < \epsilon/2\,,$$

from which it follows that

$$\bar{\rho}(f_n, f) = \sup\left\{\bar{d}(f_n(x), f(x)) \mid x \in X\right\} \leq \epsilon/2 < \epsilon$$

since x_0 was arbitrary. This shows that $f_n \in B_{\bar{\rho}}(f, \epsilon)$. Since $n \geq N + 1$ and $\epsilon > 0$ were both arbitrary, this shows that the sequence (f_n) converges to f as elements of the metric space $\mathbb{R}^X$.

($\Leftarrow$) Now suppose that (f_n) converges to f in the uniform metric space $\mathbb{R}^X$. Consider any $\epsilon > 0$ and let $\delta = \min\{\epsilon, 1\}$. Then, since $B_{\bar{\rho}}(f, \delta)$ is a neighborhood of f in the uniform metric space, there is an $N \in \mathbb{Z}_+$ where $f_n \in B_{\bar{\rho}}(f, \delta)$ for all $n \geq N$. So consider any $n > N$ and $x_0 \in X$. Then clearly $f_n \in B_{\bar{\rho}}(f, \delta)$ so that $\bar{\rho}(f_n, f) < \delta$. Hence

$$\bar{d}(f_n(x_0), f(x_0)) \leq \sup\left\{\bar{d}(f_n(x), f(x)) \mid x \in X\right\} = \bar{\rho}(f_n, f) < \delta \leq 1$$

so it has to be that $d(f_n(x_0), f(x_0)) = \bar{d}(f_n(x_0), f(x_0))$ since

$$\bar{d}(f_n(x_0), f(x_0)) = \min\{d(f_n(x_0), f(x_0)), 1\} < 1\,.$$

Therefore we have

$$d(f_n(x_0), f(x_0)) = \bar{d}(f_n(x_0), f(x_0)) \leq \bar{\rho}(f_n, f) < \delta \leq \epsilon\,.$$

Since $n > N$, $x_0 \in X$, and $\epsilon > 0$ were arbitrary, this shows that (f_n) converges to f uniformly in the functional sense. □

Exercise 21.8

Let X be a topological space and let Y be a metric space. Let $f_n : X \to Y$ be a sequence of continuous functions. Let x_n be a sequence of points of X converging to x. Show that if the sequence (f_n) converges uniformly to f, then $(f_n(x_n))$ converges to $f(x)$.

Solution:

Proof. Suppose that (f_n) converges to f uniformly and let d denote the metric for Y. Consider any $\epsilon > 0$. Since (f_n) converges to f uniformly there is an $N_1 \in \mathbb{Z}_+$ where $d(f_n(x'), f(x')) < \epsilon/2$ for any $n > N_1$ and $x' \in X$. We also know from the uniform limit theorem (Theorem 21.6) that f is continuous since each f_n is continuous. It then follows from Theorem 21.3 that the sequence $(f(x_n))$ converges to $f(x)$ since $x_n \to x$. Hence there is an $N_2 \in \mathbb{Z}_+$ such that $d(f(x_n), f(x)) < \epsilon/2$ for all $n \geq N_2$.

So set $N = \max\{N_1 + 1, N_2\}$ and consider any $n \geq N$. Then of course $n \geq N \geq N_1 + 1 > N_1$ and $x_n \in X$ so that

$$d(f_n(x_n), f(x_n)) < \epsilon/2\,.$$

We also have that $n \geq N \geq N_2$ so that

$$d(f(x_n), f(x)) < \epsilon/2\,.$$

We then have

$$d(f_n(x_n), f(x)) \leq d(f_n(x_n), f(x_n)) + d(f(x_n), f(x)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

since d is a metric, and so $f_n(x_n) \in B_d(f(x), \epsilon)$. Since $n \geq N$ and $\epsilon > 0$ were arbitrary, we have shown that the sequence $(f_n(x_n))$ converges to $f(x)$ in the metric space Y by definition. □

Exercise 21.9

Let $f_n : \mathbb{R} \to \mathbb{R}$ be the function

$$f_n(x) = \frac{1}{n^3[x - (1/n)]^2 + 1} .$$

See Figure 21.1. Let $f : \mathbb{R} \to \mathbb{R}$ be the zero function.

(a) Show that $f_n(x) \to f(x)$ for each $x \in \mathbb{R}$.

(b) Show that f_n does not converge uniformly to f. (This shows that the converse of Theorem 21.6 does not hold; the limit function f may be continuous even though the convergence is not uniform.)

Solution:

(a)

Proof. This is easy to show by evaluating the limit using techniques from elementary calculus. Fix $x \in \mathbb{R}$ and first suppose that $x \neq 0$. Clearly $1/n \to 0$ as $n \to \infty$ so that $[x - (1/n)]^2 \to x^2$. Since $x^2 > 0$ it follows that $n^3[x - (1/n)]^2 \to n^3 x^2 \to \infty$ as $n \to \infty$. Hence the overall function

$$f_n(x) = \frac{1}{n^3[x - (1/n)]^2 + 1} \to \frac{1}{\infty + 1} \to 0 = f(x)$$

as $n \to \infty$. Of course this is a little informal, but it can be justified rigorously using nothing more than Exercise 21.5. If $x = 0$ then we clearly have

$$f_n(x) = f_n(0) = \frac{1}{n^3[-(1/n)]^2 + 1} = \frac{1}{n^3/n^2 + 1} = \frac{1}{n + 1} \to 0 = f(x)$$

as $n \to \infty$. □

(b)

Proof. Let $\epsilon = 1/2$ and consider any $N \in \mathbb{Z}_+$ and let $n = N + 1$ so of course $n > N$. Also set $x = 1/n$. Then we have

$$f_n(x) = \frac{1}{n^3[x - (1/n)]^2 + 1} = \frac{1}{n^3[(1/n) - (1/n)]^2 + 1} = \frac{1}{n^3 \cdot 0 + 1} = \frac{1}{1} = 1$$

whereas of course $f(x) = 0$. We therefore have

$$d(f_n(x), f(x)) = d(1, 0) = |1 - 0| = 1 \geq 1/2 = \epsilon .$$

This shows the negation of the definition of uniform convergence so that (f_n) does not converge uniformly to f as desired. □

Note that this also shows that (f_n) does not uniformly converge to any function at all since, if it did, it can only converge uniformly to f since this is the only function to which it converges pointwise. This follows from Lemmas 21.6.2 and 21.6.3 as in Exercise 21.6.

Exercise 21.10

Using the closed set formulation of continuity (Theorem 18.1), show that the following are closed subsets of $\mathbb{R}^2$:

$$A = \{x \times y \mid xy = 1\} ,$$

$$S^1 = \{x \times y \mid x^2 + y^2 = 1\} ,$$
$$B^2 = \{x \times y \mid x^2 + y^2 \leq 1\} .$$

The set B^2 is called the (closed) ***unit ball*** in $\mathbb{R}^2$.

Solution:

Proof. Regarding the set A, the function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $f(x \times y) = xy$ is simply the multiplication function, which is continuous by Lemma 21.4. Clearly the set $A = f^{-1}(\{1\})$ by definition. We also have that $\{1\}$ is closed in $\mathbb{R}$ by Theorem 17.8 since $\mathbb{R}$ is Hausdorff. Thus $A = f^{-1}(\{1\})$ is closed in $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ by Theorem 18.1 since f is continuous.

For S^1, let d denote the usual euclidean metric on $\mathbb{R} \times \mathbb{R}$, which we know induces the same topology as the product topology. We also know from Exercise 20.3 that $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous. The function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$f(x \times y) = d(x, y) \cdot d(x, y) = [d(x, y)]^2 = \left[\sqrt{x^2 + y^2}\right]^2 = x^2 + y^2 .$$

is also then continuous by Theorem 21.5 being the product of two continuous functions. As previously mentioned, the finite set $\{1\}$ is closed in $\mathbb{R}$. We also have that $S^1 = f^{-1}(\{1\})$ by definition, which then also must be closed in $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ again by Theorem 18.1 since f continuous.

Regarding the closed unit ball B^2, define $f = d \cdot d$ as above for S^1, which is of course still continuous. Define the subset $C = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$ of $\mathbb{R}$. We claim that $B^2 = f^{-1}(C)$. This is pretty obvious since clearly $0 \leq f(x \times y) = x^2 + y^2 \leq 1$ for any $x \times y \in B^2$ by definition so that $f(x \times y) \in C$ and hence $x \times y \in f^{-1}(C)$. Conversely, for any $x \times y \in f^{-1}(C)$, we have that $f(x \times y) \in C$ so that $0 \leq f(x \times y) = x^2 + y^2 \leq 1$ so that $x \times y \in B^2$ by definition. This shows the desired equality of B^2 and $f^{-1}(C)$. As it is trivial to show that C is closed in $\mathbb{R}$, and f is continuous, it follows that $B^2 = f^{-1}(C)$ is also closed in $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$, again by Theorem 18.1. □

Exercise 21.11

Prove the following standard facts about infinite series:

(a) Show that if (s_n) is a bounded sequence of real numbers and $s_n \leq s_{n+1}$ for each n, then (s_n) converges.

(b) Let (a_n) be a sequence of real numbers; define

$$s_n = \sum_{i=1}^{n} a_i .$$

If $s_n \to s$, we say that the ***infinite series***

$$\sum_{i=1}^{\infty} a_i$$

converges to s also. Show that if $\sum a_i$ converges to s and $\sum b_i$ converges to t, then $\sum(ca_i + b_i)$ converges to $cs + t$.

(c) Prove the ***comparison test*** for infinite series: If $|a_i| \leq b_i$ for each i, and if the series $\sum b_i$ converges, then the series $\sum a_i$ converges. [Hint: Show that the series $\sum |a_i|$ and $\sum c_i$ converge, where $c_i = |a_i| + a_i$.]

(d) Given a sequence of functions $f_n : X \to \mathbb{R}$, let

$$s_n(x) = \sum_{i=1}^{n} f_i(x)\,.$$

Prove the ***Weierstrass M-test*** for uniform convergence: If $|f_i(x)| \leq M_i$ for all $x \in X$ and all i, and if the series $\sum M_i$ converges, then the sequence (s_n) converges uniformly to a function s. [Hint: Let $r_n = \sum_{i=n+1}^{\infty} M_i$. Show that if $k > n$, then $|s_k(x) - s_n(x)| \leq r_n$; conclude that $|s(x) - s_n(x)| \leq r_n$.]

Solution:

(a)

Proof. Clearly image of the sequence $S = \{s_n \mid n \in \mathbb{Z}_+\}$ is a set of real numbers that is bounded above since the sequence is bounded above. Therefore $s = \sup S$ exists, noting that of course $s_n \leq s$ for any $n \in \mathbb{Z}_+$ since $s_n \in S$. We claim that $s_n \to s$. So consider any $\epsilon > 0$ so that clearly $s - \epsilon < s$, and hence $s - \epsilon$ cannot be an upper bound of S (since s is the least upper bound). Thus there is an $N \in \mathbb{Z}_+$ where $s - \epsilon < s_N$. Then, for any $n \geq N$ we have that $s - \epsilon < S_N \leq s_n \leq s$ since the sequence is non-decreasing. Thus we have

$$\begin{aligned}
s - \epsilon &< s_n \\
s - s_n &< \epsilon \\
|s - s_n| &< \epsilon \\
d(s_n, s) &< \epsilon
\end{aligned}$$

since $s \geq s_n$, where d denotes the usual metric on $\mathbb{R}$. Hence $s_n \in B_d(s, \epsilon)$, which shows that the sequence converges to s since $n \geq N$ and ϵ were arbitrary. □

(b)

Proof. Define the partials sums

$$s_n = \sum_{i=1}^{n} a_i \qquad\qquad t_n = \sum_{i=1}^{n} b_i$$

of $\sum a_i$ and $\sum b_i$ so that $s_n \to s$ and $t_n \to t$ by the definition of infinite series. Also define the constant sequence $c_n = c$ for real c, so that of course $c_n \to c$. For any $n \in \mathbb{Z}_+$, we of course have

$$c_n s_n + t_n = c\sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i = \sum_{i=1}^{n} ca_i + \sum_{i=1}^{n} b_i = \sum_{i=1}^{n} (ca_i + b_i)$$

since these are just finite sums. It then follows from what was shown in Exercise 21.5 that

$$\sum_{i=1}^{\infty} (ca_i + b_i) = \lim_{n\to\infty} \sum_{i=1}^{n} (ca_i + b_i) = \lim_{n\to\infty} (c_n s_n + t_n) = cs + t$$

as desired. □

(c)

Proof. Denote the partial sums

$$s_n = \sum_{i=1}^{n} |a_i| \qquad\qquad t_n = \sum_{i=1}^{n} b_i \,.$$

We know that (t_n) converges by the definition of an infinite series, since $\sum b_i$ converges. So suppose that $t_n \to t$. We also clearly have that $0 \leq |a_i| \leq b_i$ for all $i \in \mathbb{Z}_+$, so that each term in both sums is always non-negative. It then follows that the sequence of partial sums (s_n) and (t_n) are non-decreasing, and moreover that $t_n \leq t$ for all $n \in \mathbb{Z}_+$. Lastly, since each $|a_i| \leq b_i$, it follows from a simple inductive argument that each $s_n \leq t_n \leq t$. Hence (s_n) is a non-decreasing sequence that is also bounded (by t), and so it converges by part (a). Therefore the infinite series $\sum |a_i|$ converges by definition. Denote its convergence value by u.

Now, we also have that clearly

$$\begin{gathered} -|a_i| \leq a_i \leq |a_i| \\ 0 \leq |a_i| + a_i \leq 2\,|a_i| \end{gathered}$$

for every $i \in \mathbb{Z}_+$, and that $\sum 2\,|a_i|$ converges to $2u$ by what was shown in part (b). Therefore, by the same argument as above, the sequence of partial sums of $\sum(|a_i| + a_i)$ is non-decreasing and bounded by $2u$ so that the series converges, say the the value v. Then, again by part (b), we have that the series

$$\sum_{i=1}^{\infty} a_i = \sum_{i=1}^{\infty}(|a_i| + a_i - |a_i|) = \sum_{i=1}^{\infty}\left[(|a_i| + a_i) + (-1)\,|a_i|\right]$$

converges to $v - u$ since both $v = \sum(|a_i| + a_i)$ and $u = \sum |a_i|$ have been shown to converge. This shows the desired result. □

(d)

Proof. First, since $|f_i(x)| \leq M_i$ for all $x \in X$ and $\sum M_i$ converges, it follows from part (c) that the series $\sum f_i(x)$ converges (as does the sequence $(s_n(x))$) for each $x \in X$. So set the function $s : X \to \mathbb{R}$ to $s(x) = \lim_{n\to\infty} s_n(x)$ for each $x \in X$. Thus (s_n) converges pointwise to s by Definition 21.6.1, and s can be thought of as the (pointwise) infinite sum of the functions f_i.

To show that $s_n \to s$ uniformly, first define

$$r_n = \sum_{i=n+1}^{\infty} M_i$$

for $n \in \mathbb{Z}_+$ as the partial series of $\sum M_i$ as in Definition 20.8.4. Now consider any $x \in X$, any $n \in \mathbb{Z}_+$, and any $k > n$. Then we have that

$$|s_k(x) - s_n(x)| = \left|\sum_{i=1}^{k} f_i(x) - \sum_{i=1}^{n} f_i(x)\right| = \left|\sum_{i=n+1}^{k} f_i(x)\right| \leq \sum_{i=n+1}^{k} |f_i(x)| \leq \sum_{i=n+1}^{k} M_i \leq \sum_{i=n+1}^{\infty} M_i = r_n\,,$$

where we note that $\sum_{i=n+1}^{k} M_i \leq \sum_{i=n+1}^{\infty} M_i$ since each M_i is non-negative. Since the absolute value function is continuous on $\mathbb{R}$, it follows from Theorem 21.3 and Exercise 21.5 that the sequence $|s_k(x) - s_n(x)|$ converges as $k \to \infty$, and moreover converges to $|s(x) - s_n(x)|$ since $s_k(x) \to s(x)$ as $k \to \infty$ as shown above. Then, since the above inequality holds for any $k > n$, it follows from Lemma 20.8.1 that

$$|s(x) - s_n(x)| = \lim_{k\to\infty} |s_k(x) - s_n(x)| \leq \lim_{k\to\infty} r_n = r_n\,.$$

Now consider any $\epsilon > 0$. We know from Lemma 20.8.5 that the sequence of partial series (r_n) converges to zero since each M_i is non-negative. Hence there is an $N \in \mathbb{Z}_+$ where $|r_n - 0| < \epsilon$ for all $n \geq N$. Moreover, Lemma 20.8.5 asserts that each $r_n \geq 0$ so that $|r_n - 0| = |r_n| = r_n < \epsilon$ for all $n \geq N$. Thus, for any $n > N$ and any $x \in X$, we have

$$|s(x) - s_n(x)| \leq r_n < \epsilon\,.$$

This suffices to show that (s_n) uniformly converges to s since ϵ was arbitrary. □

Exercise 21.12

Prove continuity of the algebraic operations on $\mathbb{R}$, as follows: Use the metric $d(a, b) = |a - b|$ on $\mathbb{R}$ and the metric on $\mathbb{R}^2$ given by the equation

$$\rho((x, y), (x_0, y_0)) = \max\{|x - x_0|, |y - y_0|\}\;.$$

(a) Show that addition is continuous. [Hint: Given ϵ, let $\delta = \epsilon/2$ and note that

$$d(x + y, x_0 + y_0) \leq |x - x_0| + |y - y_0|\;.]$$

(b) Show that multiplication is continuous. [Hint: Given (x_0, y_0) and $0 < \epsilon < 1$, let

$$3\delta = \epsilon/(|x_0| + |y_0| + 1)$$

and note that

$$d(xy, x_0y_0) \leq |x_0|\,|y - y_0| + |y_0|\,|x - x_0| + |x - x_0|\,|y - y_0|\;.]$$

(c) Show that the operation of taking reciprocals is a continuous map from $\mathbb{R} - \{0\}$ to $\mathbb{R}$. [Hint: Show the inverse image of the interval (a, b) is open. Consider five cases, according as a and b are positive, negative, or zero.]

(d) Show that the subtraction and quotient operations are continuous.

Solution:

First we note that this exercise proves Lemma 21.4 in the text. We also note that the square metric ρ as defined above induces the usual product topology on $\mathbb{R}^2$, which was shown in Theorem 20.3.

(a)

Proof. We show the continuity of $+ : \mathbb{R}^2 \to \mathbb{R}$ using Theorem 21.1. So consider any $x_0 \times y_0 \in \mathbb{R}^2$ and any $\epsilon > 0$. Following the hint, let $\delta = \epsilon/2$ and consider any $x \times y \in \mathbb{R}^2$ where $\rho(x \times y, x_0 \times y_0) < \delta$. Then we have that

$$\rho(x \times y, x_0 \times y_0) = \max\{|x - x_0|, |y - y_0|\} < \delta = \epsilon/2\,,$$

from which it clearly follows that both $|x - x_0| < \epsilon/2$ and $|y - y_0| < \epsilon/2$. Then we have

$$\begin{aligned} d(+(x \times y), +(x_0 \times y_0)) &= d(x + y, x_0 + y_0) = |x + y - (x_0 + y_0)| \\ &= |(x - x_0) + (y - y_0)| \leq |x - x_0| + |y - y_0| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\,. \end{aligned}$$

This suffices to show that $+$ is continuous by Theorem 21.1 since ϵ was arbitrary. □

(b)

Proof. We again use Theorem 21.1 so show that the multiplication function $\times : \mathbb{R}^2 \to \mathbb{R}$ is continuous. So consider any $x \times y \in \mathbb{R}^2$ and any $\epsilon > 0$. Let

$$\delta = \min \left\{ \frac{\epsilon}{|x_0| + |y_0| + 1}, 1 \right\}$$

so that of course $\delta > 0$. Now consider any $x \times y \in \mathbb{R}^2$ where $\rho(x \times y, x_0 \times y_0) < \delta$. Then of course

$$\rho(x \times y, x_0 \times y_0) = \max \{|x - x_0|, |y - y_0|\} < \delta$$

so that both $|x - x_0| < \delta$ and $|y - y_0| < \delta$. We then have

$$\begin{aligned} d(\times(x \times y), \times(x_0 \times y_0)) &= d(xy, x_0y_0) = |xy - x_0y_0| \\ &= |xy - x_0y_0 + (x_0y - x_0y) + (xy_0 - xy_0) + (x_0y_0 - x_0y_0)| \\ &= |(x_0y - x_0y_0) + (xy_0 - x_0y_0) + (xy - xy_0 - x_0y + x_0y_0)| \\ &= |x_0(y - y_0) + y_0(x - x_0) + (x - x_0)(y - y_0)| \\ &\le |x_0(y - y_0)| + |y_0(x - x_0)| + |(x - x_0)(y - y_0)| \\ &= |x_0| |y - y_0| + |y_0| |x - x_0| + |x - x_0| |y - y_0| \\ &< |x_0| \delta + |y_0| \delta + \delta^2 \\ &\le |x_0| \delta + |y_0| \delta + \delta \\ &= \delta(|x_0| + |y_0| + 1) \\ &\le \frac{\epsilon}{|x_0| + |y_0| + 1}(|x_0| + |y_0| + 1) \\ &= \epsilon, \end{aligned}$$

where we have used the fact that $0 < \delta \le 1$, from which it follows that $\delta^2 \le d$. This suffices to show that $\times$ is continuous by Theorem 21.1 since ϵ was arbitrary. □

(c)

Proof. Define the reciprocal function $f : \mathbb{R} - \{0\} \to \mathbb{R}$ by $f(x) = \frac{1}{x}$. To show that f is continuous, it suffices to show that $f^{-1}(B)$ is open in $\mathbb{R} - \{0\}$ for every basis element B of $\mathbb{R}$, which was shown back in §18. So let $B = (a, b)$ be any basis element of $\mathbb{R}$.

Case: $a > 0$. Then it has to be that $0 < a < b$. We claim that $f^{-1}(B) = (\frac{1}{b}, \frac{1}{a})$, noting that $0 < \frac{1}{b} < \frac{1}{a}$ follows readily from the fact that $0 < a < b$. We have the following

$$\begin{aligned} x \in (\tfrac{1}{b}, \tfrac{1}{a}) &\Leftrightarrow \tfrac{1}{b} < x < \tfrac{1}{a} \Leftrightarrow \tfrac{1}{b} < x \wedge x < \tfrac{1}{a} \\ &\Leftrightarrow 1 < bx \wedge ax < 1 \Leftrightarrow \tfrac{1}{x} < b \wedge a < \tfrac{1}{x} \\ &\Leftrightarrow a < \tfrac{1}{x} < b \Leftrightarrow f(x) = \tfrac{1}{x} \in (a, b) = B \\ &\Leftrightarrow x \in f^{-1}(B), \end{aligned}$$

noting that $x > a > 0$ and $\frac{1}{x} > \frac{1}{b} > 0$. This of course shows that $f^{-1}(B) = (\frac{1}{b}, \frac{1}{a})$ as desired.

Case: $a = 0$. Then we have $0 = a < b$ and hence also $0 < \frac{1}{b}$. We claim that $f^{-1}(B) = (\frac{1}{b}, \infty)$. So first suppose that $x \in f^{-1}(B)$ so that $f(x) \in B = (a, b) = (0, b)$, and hence $0 < x < b$. Then $0 < 1 < b/x$, from which follows $0 < \frac{1}{b} < x$ so that $x \in (\frac{1}{b}, \infty)$. Now consider any $x \in (\frac{1}{b}, \infty)$ so that $0 < \frac{1}{b} < x$. Then $0 < 1 < bx$ so that $0 < \frac{1}{x} < b$, and hence $f(x) = \frac{1}{x} \in (0, b) = (a, b) = B$. Therefore $x \in f^{-1}(B)$, from which we conclude that $f^{-1}(B) = (\frac{1}{b}, \infty)$ as desired.

Case: $a < 0$. Then also $\frac{1}{a} < 0$ and we have the following sub-cases:

Case: $b > 0$. Then also $\frac{1}{b} > 0$. Here we claim that $f^{-1}(B) = (-\infty, \frac{1}{a}) \cup (\frac{1}{b}, \infty)$. So suppose that $x \in f^{-1}(B)$ so that $f(x) = \frac{1}{x} \in B = (a, b)$, and hence $a < \frac{1}{x} < b$. We know that $x \neq 0$ since the domain of f is $\mathbb{R} - \{0\}$. If $x < 0$ then we have $a < \frac{1}{x}$ so that $ax > 1$, and hence $x < \frac{1}{a}$ since $a < 0$. Therefore $x \in (\infty, \frac{1}{a})$. On the other hand, if $x > 0$ then we have $\frac{1}{x} < b$ so that $1 < bx$, and hence $\frac{1}{b} < x$ since $b > 0$. Therefore $x \in (\frac{1}{b}, \infty)$. So either way $x \in (-\infty, \frac{1}{a}) \cup (\frac{1}{b}, \infty)$ so that $f^{-1}(B) \subset (-\infty, \frac{1}{a}) \cup (\frac{1}{b}, \infty)$.

Now consider $x \in (-\infty, \frac{1}{a}) \cup (\frac{1}{b}, \infty)$. If $x \in (-\infty . \frac{1}{a})$ then $x < \frac{1}{a} < 0$ so that $ax > 1$, and hence $a < \frac{1}{x} < 0 < b$. If $x \in (\frac{1}{b}, \infty)$ then $0 < \frac{1}{b} < x$ so that $1 < bx$, and hence $a < 0 < \frac{1}{x} < b$. Thus either way we have $a < \frac{1}{x} = f(x) < b$ so that $f(x) \in (a, b) = B$ and $x \in f^{-1}(B)$. Hence $(-\infty, \frac{1}{a}) \cup (\frac{1}{b}, \infty) \subset f^{-1}(B)$, which completes the proof that $f^{-1}(B) = (-\infty, \frac{1}{a}) \cup (\frac{1}{b}, \infty)$ as desired.

Case: $b = 0$. Then we have $a < b = 0$. An argument analogous to the previous case when $a = 0 < b$ shows that $f^{-1}(B) = (-\infty, \frac{1}{a})$.

Case: $b < 0$. Then we have $a < b < 0$. An argument analogous to the previous case when $0 < a < b$ again shows that $f^{-1}(B) = (\frac{1}{b}, \frac{1}{a})$.

Thus in all cases and sub-cases clearly $f^{-1}(B)$ is an open set of $\mathbb{R} - \{0\}$, which shows that f is continuous as previously discussed since $B = (a, b)$ was an arbitrary basis element. □

(d)

Proof. Regarding the subtraction function $- : \mathbb{R}^2 \to \mathbb{R}$, first define the function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = -x$, which is clearly continuous by elementary calculus. We also know that the coordinate functions $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$ and $\pi_2 : \mathbb{R}^2 \to \mathbb{R}$ are also continuous. Therefore the composition $f \circ \pi_2 : \mathbb{R}^2 \to \mathbb{R}$ is also continuous by Theorem 18.2 part (c). We then have that the function $g : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $g(x \times y) = \pi_1(x \times y) \times (f \circ \pi_2)(x \times y)$ is continuous by Theorem 18.4. Then the composition $+ \circ g : \mathbb{R}^2 \to \mathbb{R}$ is continuous, again by Theorem 18.2 part (c), since it was shown in part (a) that the addition function $+ : \mathbb{R}^2 \to \mathbb{R}$ is continuous.

However, for any $x \times y \in \mathbb{R}^2$, we have

$$\begin{aligned}(+ \circ g)(x \times y) &= +(g(x \times y)) = +(\pi_1(x \times y) \times (f \circ \pi_2)(x \times y)) \\ &= \pi_1(x \times y) + (f \circ \pi_2)(x \times y) = x + f(\pi_1(x \times y)) \\ &= x + f(y) = x - y \\ &= -(x \times y)\end{aligned}$$

so that $- = + \circ g$ is continuous just as we would like.

Regarding the quotient function $/ : \mathbb{R} \times (\mathbb{R} - \{0\}) \to \mathbb{R}$, let $f : \mathbb{R} - \{0\} \to \mathbb{R}$ be the reciprocal function, which we know is continuous by part (c). Again the coordinate functions $\pi_1 : \mathbb{R} \times (\mathbb{R} - \{0\}) \to \mathbb{R}$ and $\pi_2 : \mathbb{R} \times (\mathbb{R} - \{0\}) \to \mathbb{R} - \{0\}$ are continuous so that the composition $f \circ \pi_2 : \mathbb{R} \times (\mathbb{R} - \{0\}) \to \mathbb{R}$ is also continuous by Theorem 18.2 part (c). Then we have that the function $g : \mathbb{R} \times (\mathbb{R} - \{0\}) \to \mathbb{R}^2$ defined by $g(x \times y) = \pi_1(x \times y) \times (f \circ \pi_2)(x \times y)$ is continuous by Theorem 18.4. Thus the composition $\times \circ g : \mathbb{R} \times (\mathbb{R} - \{0\}) \to \mathbb{R}$ is continuous again by Theorem 18.2 part (c) since it was shown in part (b) that the multiplication function $\times : \mathbb{R}^2 \to \mathbb{R}$ is continuous.

Now, for any $x \times y \in \mathbb{R} \times (\mathbb{R} - \{0\})$, we have that

$$\begin{aligned}(\times \circ g)(x \times y) &= \times(g(x \times y)) = \times(\pi_1(x \times y) \times (f \circ \pi_2)(x \times y)) \\ &= \pi_1(x \times y) \cdot (f \circ \pi_2)(x \times y) = x \cdot f(\pi_2(x \times y)) \\ &= x \cdot f(y) = x \cdot \tfrac{1}{y} = \tfrac{x}{y} \\ &= /(x \times y)\end{aligned}$$

so that $/ = \times \circ g$ is continuous as desired. □

§22 The Quotient Topology

Exercise 22.1

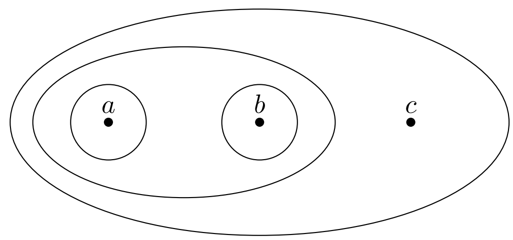
Check the details of Example 3.

Solution:

Recall that Example 22.3 includes a function $p : \mathbb{R} \to A$ where $A = \{a, b, c\}$ is a three-point set. The function is defined by

$$p(x) = \begin{cases} a & x > 0 \\ b & x < 0 \\ c & x = 0 \,. \end{cases}$$

We are then asked to verify that the quotient topology on A induced by p is that indicated by the following diagram:



Proof. Clearly the diagram illustrates the topology $\mathcal{T} = \{\varnothing, \{a\}, \{b\}, \{a, b\}, A\}$ on A. We also note that p is surjective, and that $\mathcal{T}$ is the unique topology such that p is a quotient map. First, obviously $\varnothing$ and A must be open in the quotient topology $\mathcal{T}$ since it is a topology. Then, clearly the following sets

$$p^{-1}(\{a\}) = (0, \infty)$$
$$p^{-1}(\{b\}) = (-\infty, 0)$$
$$p^{-1}(\{a, b\}) = (0, \infty) \cup (-\infty, 0)$$

are all open in $\mathbb{R}$ so that $\{a\}$, $\{b\}$, and $\{a, b\}$ should be open in the quotient topology since p is a quotient map. On the contrary, the sets

$$p^{-1}(\{c\}) = \{0\}$$
$$p^{-1}(\{a, c\}) = [0, \infty)$$
$$p^{-1}(\{b, c\}) = (-\infty, 0]$$

are all clearly not open in $\mathbb{R}$ (in fact they are all closed) so that $\{c\}$, $\{a, c\}$ and $\{b, c\}$ should not be open in the quotient topology. As we have considered all eight of the possible subsets of A, this shows the desired result. □

Exercise 22.2

(a) Let $p : X \to Y$ be a continuous map. Show that if there is a continuous map $f : Y \to X$ such that $p \circ f$ equals the identity map of Y, then p is a quotient map.

(b) If $A \subset X$, a ***retraction*** of X onto A is a continuous map $r : X \to A$ such that $r(a) = a$ for each $a \in A$. Show that a retraction is a quotient map.

Solution:

(a)

Proof. First, f is a right inverse for p by definition so that p is surjective by Exercise 2.5 part (a). Suppose that U is a subset of Y. If U is open in Y then $p^{-1}(U)$ is open in X since p is continuous. So suppose that $V = p^{-1}(U)$ is open in X. Since we have $p \circ f = i_Y$ is bijective and $i_Y = i_Y^{-1}$, it follows from Exercise 2.4 part (a) that

$$U = i_Y(U) = i_Y^{-1}(U) = (p \circ f)^{-1}(U) = f^{-1}(p^{-1}(U)) = f^{-1}(V).$$

Then, since V is open in X, we have that $f^{-1}(V) = U$ is open in Y since f is continuous. This shows that p is a quotient map by definition. □

(b)

Proof. Suppose X is a topological space, $A \subset X$, and $r : X \to A$ is a retraction. Let $f : A \to X$ be defined by $f(a) = a$ for all $a \in A$, i.e. f is the identity function on A with the range expanded to X. Now, i_A is continuous (in fact it is a homeomorphism) by Exercise 18.3 so that f is also continuous by Theorem 18.2 part (e) since it is just i_A with an expanded range. Then for any $a \in A$ we have that

$$(p \circ f)(a) = p(f(a)) = p(a) = a$$

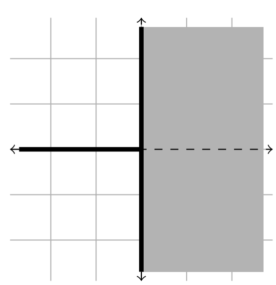
since p is a retraction. Thus $p \circ f = i_A$, which shows that p is a quotient map by what was shown in part (a). □

Exercise 22.3

Let $\pi_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be projection on the first coordinate. Let A be the subspace of $\mathbb{R} \times \mathbb{R}$ consisting of all points $x \times y$ for which either $x \geq 0$ or $y = 0$ (or both); let $q : A \to \mathbb{R}$ be obtained by restricting π_1. Show that q is a quotient map that is neither open nor closed.

Solution:

Proof. An illustration of the subspace $A \subset \mathbb{R} \times \mathbb{R}$ is shown below:



First, we know that π_1 is continuous from §18. It then follows that the restriction q is a continuous map as well by Theorem 18.2 part (d).

Now define a map $f : \mathbb{R} \to A$ by $f(x) = x \times 0$ for any $x \in \mathbb{R}$, noting that clearly $f(x) \in A$. We show that f is continuous by considering any basis element $U \times V$ of $\mathbb{R} \times \mathbb{R}$ so that U and V are open in $\mathbb{R}$. Now, if either of U or V are empty, then of course $U \times V$ is empty as well so that $f^{-1}(U \times V) = f^{-1}(\varnothing) = \varnothing$ is open in $\mathbb{R}$. Otherwise if $0 \notin V$ then also $f^{-1}(U \times V) = \varnothing$ is open in $\mathbb{R}$. If $0 \in V$ then we claim that $f^{-1}(U \times V) = U$, which is of course open in $\mathbb{R}$. This is easy to show:

$$\begin{aligned} x \in f^{-1}(U \times V) &\Leftrightarrow f(x) \in U \times V \\ &\Leftrightarrow x \times 0 \in U \times V \\ &\Leftrightarrow x \in U \end{aligned}$$

since we know that $0 \in V$. Thus in all cases $f^{-1}(U \times V)$ is open in $\mathbb{R}$, which suffices to show that f is continuous.

Now consider any $x \in \mathbb{R}$ so that we have

$$(q \circ f)(x) = q(f(x)) = q(x \times 0) = \pi_1(x \times 0) = x\,,$$

which shows that $q \circ f = i_{\mathbb{R}}$. Since $q : A \to \mathbb{R}$ and $f : \mathbb{R} \to A$ have both been shown to be continuous, it follows from Exercise 22.1 part (a) that q is a quotient map as desired.

To show that q is not an open map, consider that subset $U = [0, 1) \times (1, 2) \subset A$, which is open in the subspace A since $U = A \cap [(-1, 1) \times (1, 2)]$ and clearly $(-1, 1) \times (1, 2)$ is a basis element of $\mathbb{R} \times \mathbb{R}$ and so is open. However, clearly the set $q(U) = \pi_1(U) = [0, 1)$ is not open in $\mathbb{R}$. To show that q is not a closed map, consider the set $C = \{x \times (1/x) \mid x > 0\}$. It is easy to see and not difficult to show that C is a closed subset of the subspace A because no point of $A - C$ is a limit point of C. Also clearly $q(C) = \pi_1(C) = \mathbb{R}_+$, which is not closed in $\mathbb{R}$ since its complement $\{x \mid x \leq 0\}$ is not open. Thus q is not a closed map either. □

Exercise 22.4

(a) Define an equivalence relation on the plane $X = \mathbb{R}^2$ as follows:

$$x_0 \times y_0 \sim x_1 \times y_1 \qquad \text{if } x_0 + y_0^2 = x_1 + y_1^2.$$

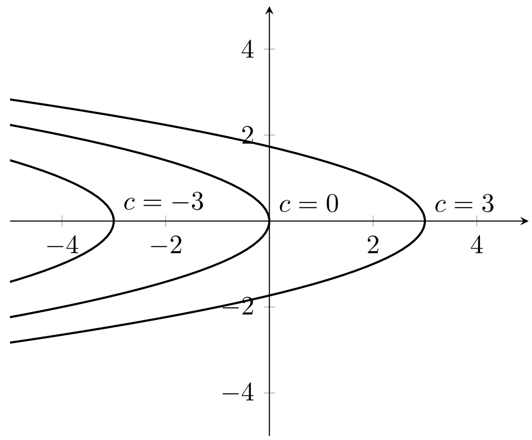
Let X^* be the corresponding quotient space. It is homeomorphic to a familiar space; what is it? [Hint: Set $g(x \times y) = x + y^2$.]

(b) Repeat (a) for the equivalence relation

$$x_0 \times y_0 \sim x_1 \times y_1 \qquad \text{if } x_0^2 + y_0^2 = x_1^2 + y_1^2.$$

Solution:

(a) Two points in the plane are in the same equivalence class if $x + y^2$ have the same value, say c. Then we have that $x + y^2 = c$ and hence $x = c - y^2$, which is the equation for horizontally-oriented parabola opening to the left and shifted in x by c. A few of these parabolic equivalence classes are shown below for various values of c:

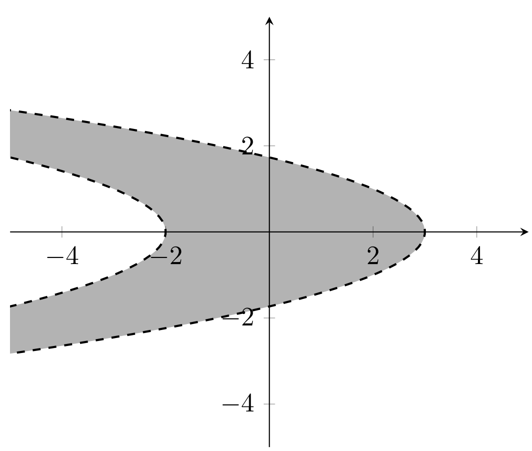


Then, if X^* is the quotient space induced, by this equivalence relation, then each element of X^* is one of these equivalence classes. We claim that this space is homeomorphic to $\mathbb{R}$.

Proof. So define $f : X^* \to \mathbb{R}$ as follows: if U is an equivalence class (so that $U \in X^*$) containing $x \times y$ then set $f(U) = x + y^2$, noting that clearly $f(U) \in \mathbb{R}$. We also note that if $x_0 \times y_0$ and $x_1 \times y_1$ are both in the equivalence class X then $x_0 + y_0^2 = x_1 + y_1^2$ so that $f(X) = x_0 + y_0^2 = x_1 + y_1^2$ is well defined.

Now suppose that U_0 and U_1 are two distinct equivalence classes and that $x_0 \times y_0 \in U_0$ and $x_1 \times y_1 \in U_1$. It then follows that it is not true that $x_0 \times y_0 \sim x_1 \times y_1$ so that $x_0 + y_0^2 \neq x_1 + y_1^2$. Thus we have $f(U_0) = x_0 + y_0^2 \neq x_1 + y_1^2 = f(U_1)$, which shows that f is injective. Next consider any real c and, the point $c \times 0$ in the plane, and let U be the equivalence class containing $c \times 0$, which must exist since the equivalence classes form a partition of the plane. Then we have $f(U) = c + 0^2 = c$, which shows that f is surjective since c was arbitrary. This completes the proof that f is a bijection, noting that of course this means that f^{-1} is also a bijective function.

Now, the standard topology of $\mathbb{R}$ of course can have different bases, but we will concern ourselves with the order topology basis first. So consider any basis element $B = (a, b)$ of $\mathbb{R}$ in the order topology. Clearly then the inverse image of B is the collection $\mathcal{U}$ of equivalence classes U where $a < f(U) < b$. Then the union of all the sets in $\mathcal{U}$ is the set of points in the plane $x \times y$ where $a < x + y^2 < b$. An illustration of such a subset of the plane is illustrated below for $a = -2$ and $b = 3$:



It is easy to see that such a set is open in $\mathbb{R}^2$ for any a and b, which we shall not show formally. Thus $\mathcal{U}$ is an open subset of X^*, which suffices to show that f is continuous.

Now, to show that f^{-1} is continuous we utilize metric topology bases for both $\mathbb{R}^2$ and $\mathbb{R}$, which we know induce the standard topologies on those sets. In particular, we use the standard metric d on $\mathbb{R}$ but use the square metric ρ on $\mathbb{R}^2$, which induces the standard topology on $\mathbb{R}^2$ by Theorem 20.3. Recall that the square metric is defined by

$$\rho(x_0 \times y_0, x_1 \times y_1) = \max\{|x_0 - x_1|, |y_0 - y_1|\} .$$

So consider any $x \in \mathbb{R}$, let U be the equivalence class in X^* such that $U = f^{-1}(x)$ so that $f(U) = x$, and let $\mathcal{U}$ be a neighborhood of $U = f^{-1}(x)$ in the quotient topology on X^*. Then we have that $\mathcal{U}$ is a collection of equivalence classes and that $x \times 0$ must be in U since $x + 0^2 = x = f(U)$. Clearly also the union $A = \bigcup_{U' \in \mathcal{U}} U'$ is then open in $\mathbb{R}^2$ since $\mathcal{U}$ is open in the quotient space, and $x \times 0 \in A$ since $x \times 0 \in U$ and $U \in \mathcal{U}$. Thus by Lemma 20.4.1 there is a $\delta > 0$ where $B_\rho(x \times 0, \delta) \subset A$.

Now, consider the set $B_d(x, \delta)$, which is clearly a neighborhood of x in $\mathbb{R}$. Consider any $V \in f^{-1}(B_d(x, \delta))$ and let $v = f(V)$ so that it must be that $v \times 0 \in V$ since $v + 0^2 = v = f(V)$. Then $v = f(V) \in B_d(x, \delta)$ so that $d(v, x) = |v - x| < \delta$. Since we have $|v - x| \geq 0 = |0 - 0|$, it follows that

$$\rho(v \times 0, x \times 0) = \max\{|v - x|, |0 - 0|\} = |v - x| < \delta ,$$

and hence $v \times 0 \in B_\rho(x \times 0, \delta) \subset A = \bigcup_{U' \in \mathcal{U}} U'$. Therefore there is an equivalence class W in $\mathcal{U}$ such that $v \times 0 \in W$ and $W \in \mathcal{U}$. However, since the equivalence classes are all disjoint, it has to be that $W = V$ since $v \times 0 \in V$ as well. Thus $V = W \in \mathcal{U}$ so that $f^{-1}(B_d(x, \delta)) \subset \mathcal{U}$ since V was arbitrary. This suffices to show that f^{-1} is continuous by Theorem 18.1 since $\mathcal{U}$ was an arbitrary neighborhood of $U = f^{-1}(x)$. This completes the proof that f is a homeomorphism so that X^* is homeomorphic to $\mathbb{R}$ as desired. □

(b) Here two points are in the same equivalence class if $x^2 + y^2$ have the same value, say c. Then we have that the equivalence class is all the points in the plane such that $x^2 + y^2 = c$, which is clearly a circle in the plane with radius $\sqrt{c}$ centered at the origin, noting that of course always $c = x^2 + y^2 \geq 0$. We also note that the only point such that $x^2 + y^2 = 0$ is 0×0 itself, so this the only point in its equivalence class. We claim that the quotient topology on the set of equivalence classes X^* is homeomorphic to the subspace topology of nonnegative reals, i.e. the subspace $A = \{x \in \mathbb{R} \mid x \geq 0\}$ of $\mathbb{R}$.

Proof. Taking a similar approach to that in part (a), define $f : X^* \to A$ by $f(U) = \|x \times y\| = \sqrt{x^2 + y^2}$ if $x \times y$ is a point in the equivalence class U. Clearly we have $f(U) \geq 0$ so that $f(U) \in A$. It also follows that f is well-defined since, if $x_0 \times y_0$ and $x_1 \times y_1$ are two points in the same equivalence class U, then $f(U) = \sqrt{x_0^2 + y_0^2} = \sqrt{x_1^2 + y_1^2}$ since $x_0^2 + y_0^2 = x_1^2 + y_1^2$.

Now, if U_0 and U_1 are two distinct equivalence classes and $x_0 \times y_0 \in U_0$ and $x_1 \times y_1 \in U_1$ then it is not true that $x_0 \times y_0 \sim x_1 \times y_1$ so that $f(U_0) = \sqrt{x_0^2 + y_0^2} \neq \sqrt{x_1^2 + y_1^2} = f(U_1)$ since $x_0^2 + y_0^2 \neq x_1^2 + y_1^2$ and the square root function is injective on the nonnegative reals. This shows that f is injective. Also, if c is any element of A then let U be the equivalence class containing $c \times 0$, which exists since the classes form a partition on the plane. Then we have $f(U) = \sqrt{c^2 + 0^2} = |c| = c$ since $c \geq 0$, which shows that f is surjective since c was arbitrary. Hence f is a bijection so that of course f^{-1} is also a bijective function.

Next, consider the order topology basis of $\mathbb{R}$ and any corresponding basis element B of the subspace A. Then clearly either $B = [0, b)$ (since then, for example, $B = A \cap (-1, b)$ and $(-1, b)$ is clearly a basis element of $\mathbb{R}$) or $B = (a, b)$ for some $0 \leq a < b$. In the former case clearly $f^{-1}(B)$ is the collection $\mathcal{U}$ of equivalence classes U such that $0 \leq f(U) < b$, the union of which is clearly the set of points $x \times y$ in the plane such that $\sqrt{x^2+y^2} < b$, which is an open filled circle of radius b centered at the origin. This is obviously an open set of $\mathbb{R}^2$. In the later case clearly $f^{-1}(B)$ is the collection $\mathcal{U}$ of equivalence classes U such that $a < f(U) < b$, the union of which is the set of points in the plane $x \times y$ such that $a < \sqrt{x^2+y^2} < b$. As this is clearly an open annular ring centered at the origin, it is open in $\mathbb{R}^2$. Hence in either case the union of the collection $\mathcal{U}$ is open in $\mathbb{R}^2$ so that $\mathcal{U} = f^{-1}(B)$ is an open subset of X^*. Since B was an arbitrary basis element of A, this shows that f is continuous.

Similar to what was done in part (a), we show that f^{-1} is also continuous by utilizing the euclidean metric d on $\mathbb{R}^2$ and the standard metric d' on A, i.e. d' is the standard metric d on $\mathbb{R}$ restricted to $A \times A$, which is a metric for the subspace A by the remarks at the beginning of §21. So consider any $x \in A$ so that $x \geq 0$, let U be the equivalence class in X^* such that $U = f^{-1}(x)$ so that $f(U) = x$, and let $\mathcal{U}$ be a neighborhood of $U = f^{-1}(x)$ in the quotient topology on X^*. Then we have that $\mathcal{U}$ is a collection of equivalence classes and that $x \times 0$ must be in U since $\sqrt{x^2+0^2} = |x| = x = f(U)$. Clearly also the union $C = \bigcup_{U' \in \mathcal{U}} U'$ is then open in $\mathbb{R}^2$ since $\mathcal{U}$ is open in the quotient space, and $x \times 0 \in C$ since $x \times 0 \in U$ and $U \in \mathcal{U}$. Thus by Lemma 20.4.1 there is a $\delta > 0$ where $B_d(x \times 0, \delta) \subset C$.

Now, consider the set $B_{d'}(x, \delta)$, which is clearly a neighborhood of x in A. Consider any $V \in f^{-1}(B_{d'}(x,\delta))$ and let $v = f(V) \in A$ so that it must be that $v \times 0 \in V$ since $\sqrt{v^2+0^2} = |v| = v = f(V)$. Then $v = f(V) \in B_{d'}(x,\delta)$ so that $d'(v,x) = d(v,x) = |v-x| < \delta$. Then also

$$d(v \times 0, x \times 0) = \sqrt{(z-x)^2 + (0-0)^2} = \sqrt{(z-x)^2} = |v-x| < \delta\,,$$

and hence $v \times 0 \in B_d(x \times 0, \delta) \subset C = \bigcup_{U' \in \mathcal{U}} U'$. Therefore there is an equivalence class W in $\mathcal{U}$ such that $v \times 0 \in W$ and $W \in \mathcal{U}$. However, since the equivalence classes are all disjoint, it has to be that $W = V$ since $v \times 0 \in V$ as well. Thus $V = W \in \mathcal{U}$ so that $f^{-1}(B_{d'}(x,\delta)) \subset \mathcal{U}$ since V was arbitrary. This suffices to show that f^{-1} is continuous by Theorem 18.1 since $\mathcal{U}$ was an arbitrary neighborhood of $U = f^{-1}(x)$. This completes the proof that f is a homeomorphism so that X^* is homeomorphic to A as desired. □

Exercise 22.5

Let $p : X \to Y$ be an open map. Show that if A is open in X, then the map $q : A \to p(A)$ obtained by restricting p is an open map.

Solution:

Consider any open set U in the subspace A so that $U = A \cap V$ for some open set V in X by the definition of a subspace. Since A is also open in X we have that $A \cap V = U$ is also open in X by the definition of a topology. Then $q(U) = p(U)$ is open in Y since p is an open map. Since $U \subset A$, it follows that $p(U) \subset p(A)$ by Exercise 2.2 part (e) so that $p(U) \cap p(A) = p(U) = q(U)$. This shows that $q(U)$ is open in the subspace $p(A)$ since we have shown that $p(U)$ is open in Y. Therefore q is an open map since U was an arbitrary open set of A.

Exercise 22.6

Recall that $\mathbb{R}_K$ denotes the real line in the K-topology. (See §13.) Let Y be the quotient space obtained from $\mathbb{R}_K$ by collapsing the set K to a point: let $p : \mathbb{R}_K \to Y$ be the quotient map.

(a) Show that Y satisfies the T_1 axiom, but is not Hausdorff.

(b) Show that $p \times p : \mathbb{R}_K \times \mathbb{R}_K \to Y \times Y$ is not a quotient map. [Hint: The diagonal is not closed in $Y \times Y$, but its inverse image is closed in $\mathbb{R}_K \times \mathbb{R}_K$.]

Solution:

In what follows let

$$Y = \{K\} \cup \{\{x\} \mid x \in \mathbb{R} - K\} .$$

It is easy to see and trivial to show that Y is a partition of $\mathbb{R}_K$, and that K becomes a single point in the quotient space Y.

(a)

Proof. First, consider a single point U in the collection Y. If $U = K$ then clearly $p^{-1}(\{U\}) = K$, which we know is closed in $\mathbb{R}_K$ as was shown in Exercise 17.16 part (b). If $U \neq K$ then $U = \{x\}$ for some $x \in \mathbb{R} - K$ so that of course $p^{-1}(\{U\}) = \{x\}$ is closed in $\mathbb{R}_K$ since $\mathbb{R}_K$ was shown to satisfy the T_1 axiom, again in Exercise 17.16 part (b). Hence either way $p^{-1}(\{U\})$ is closed in $\mathbb{R}_K$ so that $\{U\}$ must be closed in Y since p is a quotient map. This suffices to show that Y satisfies the T_1 axiom since U was arbitrary.

To show that Y is *not* Hausdorff consider any neighborhood V_0 in Y of the point $\{0\}$ and any neighborhood V_K in Y of the point K, noting that clearly $\{0\}$ and K are distinct points in Y. Then set $U_K = p^{-1}(V_K) = \bigcup_{V \in V_K} V$ so that $K \subset U_K$ since $K \in V_K$. Similarly, set $U_0 = p^{-1}(V_0) = \bigcup_{V \in V_0} V$ so that $\{0\} \subset U_0$ since $\{0\} \in V_0$, and hence $0 \in U_0$. Since V_0 is open in Y, it follows that $U_0 = p^{-1}(V_0)$ must be open in $\mathbb{R}_K$ since p is a quotient map. It then follows that there is a basis element B_0 in $\mathbb{R}_K$ such that $0 \in B_0 \subset U_0$. Hence $B_0 = (a, b)$ or $B_0 = (a, b) - K$ for some $a < 0 < b$. Now, since we have $0 < b$, clearly there is an $n \in \mathbb{Z}_+$ large enough that $0 < 1/n < b$, and by definition $1/n \in K \subset U_K$. Then, since U_K is open in $\mathbb{R}_K$, there must be a basis element B_K of $\mathbb{R}_K$ such that $1/n \in B_K \subset U_K$. Then it must be that $B_K = (c, d)$ for some $c < 1/n < d$ since $1/n \in K$.

Next, set $e = \max\{1/(n+1), c\}$ and $x = (e + 1/n)/2$. Then we then have that $1/(n+1) \le e < x < 1/n$ so that $x \notin K$. We also have $a < 0 < 1/(n+1) \le e < x < 1/n < b$ so that $x \in B_0 \subset U_0$ regardless of whether or not K is included in B_0 or not. Lastly, we have that $c \le e < x < 1/n < d$ so that $x \in (c, d) = B_K \subset U_K$. Therefore $x \in U_0$ and $x \notin K$ so that $p(x) = \{x\} \in V_0$ since $U_0 = p^{-1}(V_0)$. Likewise we have $x \in U_K$ and $U_K = p^{-1}(V_K)$ so that $p(x) = \{x\} \in V_K$ as well. Hence $\{x\} = p(x) \in V_0 \cap V_K$ so that these neighborhoods intersect. Since V_0 and V_K were arbitrary neighborhoods of $\{0\}$ and K, respectively, in Y, this shows that Y fails to be Hausdorff. □

(b) In what follows we define the map $p \times p : \mathbb{R}_K \times \mathbb{R}_K \to Y \times Y$ by

$$(p \times p)(x \times y) = p(x) \times p(y)$$

for $x \times y \in \mathbb{R}_K \times \mathbb{R}_K$.

Proof. Define the set $\Delta_Y = \{(U, U) \mid U \in Y\} \subset Y \times Y$ to be the diagonal of $Y \times Y$. Since it was shown in part (a) that Y is not Hausdorff, it follows that Δ_Y is not closed in $Y \times Y$ by Exercise 17.13. Now set $\Delta = (p \times p)^{-1}(\Delta_Y) \subset \mathbb{R}_K \times \mathbb{R}_K$, and we claim that $\Delta = \Delta_{\mathbb{R}} \cup (K \times K)$, where of course $\Delta_{\mathbb{R}} = \{(x, x) \mid x \in \mathbb{R}\}$ is the diagonal of $\mathbb{R}_K$.

To show this, first consider $x \times y \in \Delta = (p \times p)^{-1}(\Delta_Y)$ so that $(p \times p)(x \times y) = p(x) \times p(y) \in D_Y$. Hence $p(x) = p(y) \in Y$ so that either $x, y \in K$ so that $p(x) = K = p(y)$, or $x, y \notin K$ so that $p(x) = \{x\} = \{y\} = p(y)$ and $x = y$. Clearly in the former case we have $x \times y \in K \times K$, and in the latter case $x \times y = x \times x \in \Delta_{\mathbb{R}}$. Thus either way we have $x \times y \in \Delta_{\mathbb{R}} \cup (K \times K)$ so that $\Delta \subset \Delta_{\mathbb{R}} \cup (K \times K)$. Now consider any $x \times y \in \Delta_{\mathbb{R}} \cup (K \times K)$. If $x \times y \in \Delta_{\mathbb{R}}$ then $x = y$ so that of course $p(x) = p(y)$ since p is a function. On the other hand, if $x \times y \in K \times K$ then we again have $p(x) = K = p(y)$. Therefore either way we have $p(x) = p(y)$ so that $(p \times p)(x \times y) = p(x) \times p(y) = p(x) \times p(x) \in \Delta_Y$, and thus $x \times y \in (p \times p)^{-1}(\Delta_Y) = \Delta$. Hence $\Delta \supset \Delta_{\mathbb{R}} \cup (K \times K)$ as well so that equality has been shown.

Now, since $\mathbb{R}_K$ is Hausdorff (again, as shown in Exercise 17.16), it follows from Exercise 17.13 that $\Delta_{\mathbb{R}}$ is closed in $\mathbb{R}_K \times \mathbb{R}_K$, and therefore $\overline{\Delta_{\mathbb{R}}} = \Delta_{\mathbb{R}}$. Since K is also closed in $\mathbb{R}_K$ (also shown in Exercise 17.16), we have that $K \times K$ is closed in $\mathbb{R}_K \times \mathbb{R}_K$ per Exercise 17.3, and so $\overline{K \times K} = K \times K$. By Exercise 17.6 part (b), we then have that

$$\overline{\Delta} = \overline{\Delta_{\mathbb{R}} \cup (K \times K)} = \overline{\Delta_{\mathbb{R}}} \cup \overline{(K \times K)} = \Delta_{\mathbb{R}} \cup (K \times K) = \Delta \,,$$

which shows that $\Delta = (p \times p)^{-1}(\Delta_Y)$ is in fact closed in $\mathbb{R}_K \times \mathbb{R}_K$. This suffices to show that $p \times p$ is *not* a quotient map as desired since Δ_Y is *not* closed in $Y \times Y$. □

§TG Supplementary Exercises: Topological Groups

For this section, recall that a group is a set G together with some operation $\cdot$ that satisfies the following properties, called the group axioms:

(1) *(Closure)* For all $a, b \in G$, the result $a \cdot b$ is also in G.

(2) *(Associativity)* $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in G$.

(3) *(Identity Element)* There is an element $e \in G$ such that $a \cdot e = e \cdot a = a$ for all $a \in G$, which is called an identity element.

(4) *(Inverse Element)* For every $a \in G$, there is an element $b \in G$ such that $a \cdot b = b \cdot a = e$, where e is the identity element. This b is called an inverse element of a.

It is easy to show directly from these axioms that the identity element of a group is unique, and so we refer to *the* identity element. Similarly, if $a \in G$, then its inverse element is also unique, and is usually denoted by a^{-1}.

Exercise TG.1

Let H denote a group that is also a topological space satisfying the T_1 axiom. Show that H is a topological group if and only if the map of $H \times H$ into H sending $x \times y$ to $x \cdot y^{-1}$ is continuous.

Solution:

Lemma TG.1.1. *In any group G the inverse element of an inverse element is the element itself, i.e. $(x^{-1})^{-1} = x$ for any $x \in G$.*

Proof. Consider any x in the group G with operation $\cdot$ and identity element e, and let $y = (x^{-1})^{-1}$. Then we have of course that $y \cdot x^{-1} = e$ since y is the inverse of x^{-1}. Since we of course also have $x \cdot x^{-1} = e$ since x^{-1} is the inverse of x, it has to be that $y = x$ because the inverse of x^{-1} must be unique. Therefore of course $(x^{-1})^{-1} = y = x$ as desired. □

Main Problem.

Proof. (⇒) First suppose that H is a topological group. Then $f : H \to H$ defined by $f(x) = x^{-1}$ and $g : H \times H \to H$ defined by $g(x \times y) = x \cdot y$ are both continuous. Define $h : H \times H \to H$ by

$$h = g \circ (i_H \times f) \,,$$

where of course i_H is the identity function on H, and we have defined the function $i_H \times f : H \times H \to H \times H$ as in Exercise 18.10.

Now, we know that both i_H and f are continuous so that $i_H \times f$ is continuous by Exercise 18.10. It then follows that $g \circ (i_H \times f) = h$ is continuous by Theorem 18.2 part (c) since g is also continuous. Now, for any $x \times y \in H \times H$, we have

$$\begin{aligned} h(x \times y) &= (g \circ (i_H \times f))\,(x \times y) \\ &= g((i_H \times f)(x \times y)) \\ &= g(i_H(x) \times f(y)) \\ &= g(x \times y^{-1}) \\ &= x \cdot y^{-1} . \end{aligned}$$

Since we have shown that h is continuous, this shows the desired result.

($\Leftarrow$) Now again define $h : H \times H \to H$ by $h(x \times y) = x \cdot y^{-1}$, and suppose that h is continuous. Then h is continuous in each variable separately by Exercise 18.11. So, if we let e be the unique identity element of H, then we have that

$$h(e \times x) = e \cdot x^{-1} = x^{-1}$$

is continuous for any x. Similarly, for any $x, y \in H$, we have that $\left(y^{-1}\right)^{-1} = y$ by Lemma TG.1.1 so that

$$h(x \times y^{-1}) = x \cdot \left(y^{-1}\right)^{-1} = x \cdot y$$

must also be continuous. □

Exercise TG.2

Show that the following are topological groups:

(a) $(\mathbb{Z}, +)$

(b) $(\mathbb{R}, +)$

(c) $(\mathbb{R}_+, \cdot)$

(d) $(S^1, \cdot)$, where we take S^1 to be the space of all complex numbers z for which $|z| = 1$.

(e) The *general linear group* $\mathrm{GL}(n)$, under the operation of matrix multiplication. ($\mathrm{GL}(n)$ is the set of all nonsingular n by n matrices, topologized by considering it as a subset of euclidean space of dimension n^2 in the obvious way.)

Solution:

Lemma TG.2.1. *Any discrete topology satisfies the T_1 axiom.*

Proof. Suppose that X is a set with the discrete topology and C is a finite point set. Then $X - C$ is clearly still a subset of X and so is open since X is discrete. This shows by definition that C is closed. In fact by this same argument *any* subset of X is both open and closed. □

Lemma TG.2.2. *If Y and Y are sets both with discrete topologies, then $X \times Y$ is also the discrete topology.*

Proof. It suffices to show that the subset of $X \times Y$ containing a single arbitrary element is open, since clearly any other subset is the union of such single-element open subsets and is therefore also open by the definition of a topology. So consider any $(x, y) \in X \times Y$ and the subset $\{(x, y)\} \subset X \times Y$. Then clearly $\{(x, y)\} = \{x\} \times \{y\}$, which is a basis element of $X \times Y$ and therefore open by the definition of a product topology since both $\{x\}$ and $\{y\}$ are open in X and Y, respectively, since they are discrete. □

Lemma TG.2.3. *If X and Y are topological spaces and X has the discrete topology then any function $f : X \to Y$ is continuous.*

Proof. This is fairly obvious since, for any open subset V of Y, of course $f^{-1}(V)$ is a subset of X and so is open since X is discrete. □

Main Problem.

(a)

Proof. First we must show that $(\mathbb{Z}, +)$ is even a group. Clearly $a + b$ is an integer when a and b are so that the closure axiom is satisfied. Also, we know that integer addition is associative. We clearly have that $0 \in \mathbb{Z}$ and that $a + 0 = a$ for any $a \in \mathbb{Z}$ so that 0 is the identity element of $(\mathbb{Z}, +)$. Lastly, for any $a \in \mathbb{Z}$, we have that $-a \in \mathbb{Z}$ and that $a + (-a) = a - a = 0$ so that clearly $-a$ is the inverse of a. This shows that $(\mathbb{Z}, +)$ is in fact a group.

To show that it is a topological group, we first note that $\mathbb{Z}$ clearly has the discrete topology when considered both an order topology or as a subspace of $\mathbb{R}$, for similar reason as discussed in Example 3 of §14. Thus $\mathbb{Z}$ satisfies the T_1 axiom by Lemma TG.2.1 since it is discrete. Also $X \times X$ is the discrete topology by Lemma TG.2.2. Thus the function f defined by $f(x \times y) = x + y^{-1} = x + (-y) = x - y$ is a function from $X \times X$ to X, so that it follows that f is continuous by Lemma TG.2.3 since $X \times X$ is discrete. Hence $(\mathbb{Z}, +)$ is a topological group by Exercise TG.1. □

(b)

Proof. Similarly to part (a), clearly $(\mathbb{R}, +)$ is a group with identity element 0 and inverse element $-x$ for any $x \in \mathbb{R}$. However, this time the topology is no longer discrete. Of course we know that $\mathbb{R}$ satisfies the T_1 axiom. Now consider the function $f(x \times y) = x + y^{-1} = x + (-y) = x - y$ for any $x, y \in \mathbb{R}$. Consider also any basis element $B = (a, b) \in \mathbb{R}$, where here we are of course using the order topology basis. Then we clearly have

$$\begin{aligned} f^{-1}(B) &= \{x \times y \mid f(x \times y) \in (a, b)\} = \{x \times y \mid a < f(x \times y) < b\} \\ &= \{x \times y \mid a < x - y < b\} = \{x \times y \mid a - x < -y < b - x\} \\ &= \{x \times y \mid x - a > y > x - b\} \ . \end{aligned}$$

Clearly this is the region in $\mathbb{R}^2$ between the lines $y = x - b$ and $y = x - a$, which is obviously an open set in $\mathbb{R}^2$. This shows that f is continuous since B was an arbitrary basis element, so that $(\mathbb{R}, +)$ is a topological group by Exercise TG.1. □

(c)

Proof. First, clearly $\mathbb{R}_+$ satisfies the T_1 axiom since $\mathbb{R}$ does. Next we note that for any $x, y \in \mathbb{R}_+$ we have that $x \cdot y$ is also positive so that $x \cdot y \in \mathbb{R}_+$ as well, which shows the closure property of a group. Also, clearly $1 \in \mathbb{R}_+$ is the identity element of multiplication, and the inverse element is $x^{-1} = 1/x$ for any $x \in \mathbb{R}_+$, noting that this is defined since $x > 0$, and that $1/x > 0$ so that $x^{-1} = 1/x \in \mathbb{R}_+$. Lastly, we know that multiplication is associative on the reals (and therefore also on $\mathbb{R}_+$), which completes the check that $(\mathbb{R}_+, \cdot)$ is in fact a group.

As before, define the function $f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ by $f(x \times y) = x \cdot y^{-1} = x \cdot 1/y = x/y$. Consider the order topology basis of $\mathbb{R}$ and consider any basis element B of the subspace $\mathbb{R}_+$ so that $B = \mathbb{R}_+ \cap (a, b)$ for some $a, b \in \mathbb{R}$ where $a < b$ by Lemma 16.1. Now, if $a \leq 0$ then clearly $B = (0, b)$, and we have that

$$\begin{aligned} f^{-1}(B) &= \{x \times y \mid f(x \times y) \in B\} = \{x \times y \mid 0 < f(x \times y) < b\} \\ &= \{x \times y \mid 0 < x/y < b\} = \{x \times y \mid 0 < x < by\} \\ &= \{x \times y \mid 0 < x/b < y\} \ , \end{aligned}$$

noting that $0 < b$ so that x/b is defined. Obviously this is the region in $\mathbb{R}_+ \times \mathbb{R}_+$ ($\mathbb{R}_+ \times \mathbb{R}_+$ being the upper right quadrant of $\mathbb{R}^2$ that does not include either axis) above the line $y = x/b$, which is easy to show is open in $\mathbb{R}_+ \times \mathbb{R}_+$.

On the other hand, if $a > 0$ than $B = (a, b)$ so that

$$\begin{aligned} f^{-1}(B) &= \{x \times y \mid f(x \times y) \in B\} = \{x \times y \mid a < f(x \times y) < b\} \\ &= \{x \times y \mid a < x/y < b\} = \{x \times y \mid ay < x < by\} \\ &= \{x \times y \mid ay < x \wedge x < by\} = \{x \times y \mid y < x/a \wedge x/b < y\} \\ &= \{x \times y \mid x/b < y < x/a\} \ , \end{aligned}$$

which is clearly the region of $\mathbb{R}_+ \times \mathbb{R}_+$ between the lines $y = x/b$ and $y = x/a$. It is easy to see that again this is an open subset of $\mathbb{R}_+ \times \mathbb{R}_+$, which shows that f is continuous either way. This in turn proves that $(\mathbb{R}_+, \cdot)$ is a topological space, again by Exercise TG.1. □

(d)

Proof. Topologies on the complex plane $\mathbb{C}$ have not really been discussed, but $\mathbb{C}$ is usually defined as $\mathbb{R} \times \mathbb{R}$ having the usual product topology. Then of course S^1 is the unit circle in $\mathbb{C}$. We know that $\mathbb{R}$ is Hausdorff so that $\mathbb{C} = \mathbb{R} \times \mathbb{R}$ is as well by Theorem 17.11. Then, again by Theorem 17.11, S^1 is Hausdorff since it is a subspace of $\mathbb{C}$, and so it also satisfies the T_1 axiom.

While perhaps not immediately obvious, it is easy to show that S^1 is closed under multiplication. If $z, w \in S^1$ then $|z| = |w| = 1$ so that $|z \cdot w| = |z| \cdot |w| = 1 \cdot 1 = 1$ by familiar rules of complex analysis so that $z \cdot w \in S^1$ as well. Clearly $1 \in S^1$ is the identity element where the inverse element of $z \in S^1$ is $1/z$, noting that $|1/z| = 1/|z| = 1/1 = 1$ since $z \in S^1$, and so $1/z \in S^1$. We also note that $|0| = 0$, and hence $0 \notin S^1$ so that the inverse $1/z$ is always defined. Lastly, we know that multiplication is associative within $\mathbb{C}$ and therefore also within S^1. This shows that $(S^1, \cdot)$ satisfies all of the group axioms.

To rigorously show that S^1 is a topological group is actually quite tedious so we shall omit some details. Suppose that U is open in S^1 and that $z \times w \in f^{-1}(U)$ so that $f(z \times w) \in U$. Now, clearly the unit circle in $\mathbb{C} = \mathbb{R} \times \mathbb{R}$ is the set $S^1 = \left\{e^{i\theta} \mid \theta \in \mathbb{R}\right\}$ so that we can express $z = e^{i\theta}$ and $w = e^{i\phi}$ for some $\theta, \phi \in \mathbb{R}$. We then have that

$$f(z \times w) = z/w = e^{i\theta}/e^{i\phi} = e^{i\theta}e^{-i\phi} = e^{i(\theta-\phi)} \in U \, .$$

While tedious to show rigorously, it follows from the fact that U is open in S^1 that there is an $\epsilon > 0$ where $f(z \times w) \in A_{\theta-\phi,\epsilon} \subset U$, where we define

$$A_{\alpha,\epsilon} = \left\{e^{i\gamma} \mid \alpha - \epsilon < \gamma < \alpha + \epsilon\right\} \ ,$$

noting that of course $A_{\alpha,\epsilon} \subset S^1$. Now consider $A_{\theta,\epsilon/2}$ and $A_{\phi,\epsilon/2}$, which are both clearly open in S^1 and noting that clearly $z \in A_{\theta,\epsilon/2}$ and $w \in A_{\phi,\epsilon/2}$. For any $z' = e^{i\theta'} \in A_{\theta,\epsilon/2}$ and $w' = e^{i\phi'} \in A_{\phi,\epsilon/2}$ we then have that

$$\theta - \epsilon/2 < \theta' < \theta + \epsilon/2 \qquad\qquad \phi - \epsilon/2 < \phi' < \phi + \epsilon/2 \, .$$

Hence

$$\begin{gathered}-\phi+\epsilon/2 > -\phi' > -\phi-\epsilon/2 \\ \theta'-\phi+\epsilon/2 > \theta'-\phi' > \theta'-\phi-\epsilon/2 \\ \theta+\epsilon/2-\phi+\epsilon/2 > \theta'-\phi+\epsilon/2 > \theta'-\phi' > \theta'-\phi-\epsilon/2 > \theta-\epsilon/2-\phi-\epsilon/2 \\ (\theta-\phi)+\epsilon > \theta'-\phi' > (\theta-\phi)-\epsilon\end{gathered}$$

so that $f(z'\times w') = e^{i(\theta'-\phi')} \in A_{\theta-\phi,\epsilon} \subset U$. Thus $z'\times w' \in f^{-1}(U)$ so that $z\times w \in A_{\theta,\epsilon/2}\times A_{\phi,\epsilon/2} \subset f^{-1}(U)$ since z' and w' were arbitrary. We also have that $A_{\theta,\epsilon/2}\times A_{\phi,\epsilon/2}$ is open in $S^1\times S^1$ since both $A_{\theta,\epsilon/2}$ and $A_{\phi,\epsilon/2}$ are open in S^1. Since $z\times w$ was an arbitrary element of $f^{-1}(U)$, this shows that $f^{-1}(U)$ is open in $S^1\times S^1$, which in turn shows that f is continuous by definition. Thus by Exercise TG.1 we have that S^1 is a topological group. □

(e)

Proof. First, from linear algebra we know that the matrix product of two nonsingular n by n matrices is another nonsingular n by n matrix, so that $\mathrm{GL}(n)$ is closed under matrix multiplication. Clearly the identity matrix is the identity element of $\mathrm{GL}(n)$, while the inverse matrix A^{-1} is the inverse element of the matrix $A \in \mathrm{GL}(n)$, noting that this inverse matrix exists since A is nonsingular. Lastly, we know that matrix multiplication is associative, which suffices to show that $(\mathrm{GL}(n),\cdot)$ is a group.

To show that it is a topological group takes more work. To begin, we note that of course $\mathbb{R}^{n^2}$ is Hausdorff and so satisfies the T_1 axiom. Thus so does $\mathrm{GL}(n)$ since $\mathbb{R}^{n^2}$ gives it its topology. Next, we denote a vector in $\mathbb{R}^n$ by $\mathbf{x}_n = x_1\times\cdots\times x_n$, using the subscript on the vector itself to indicate its dimension. We show that the function $s_n : \mathbb{R}^n \to \mathbb{R}$ defined by

$$s_n(\mathbf{x}_n) = \sum_{i=1}^{n} x_i$$

is continuous for all $n \in \mathbb{Z}_+$, which we show by induction. First, for $n = 1$, we clearly have that s_n is simply the identity function from $\mathbb{R}$ to $\mathbb{R}$, which is clearly continuous. Now suppose that s_n is continuous. Define $g : \mathbb{R}^{n+1} \to \mathbb{R}^n$ by

$$g(\mathbf{x}_{n+1}) = \pi_1(\mathbf{x}_{n+1})\times\cdots\times\pi_n(\mathbf{x}_{n+1}) = x_1\times\cdots\times x_n = \mathbf{x}_n\,,$$

which is continuous by Theorem 19.6 since we know that each π_i is continuous.

Then also $s_n\circ g$ is continuous by Theorem 18.2 part (c). It then follows that the function $h : \mathbb{R}^{n+1} \to \mathbb{R}^2$ defined by $h(\mathbf{x}_{n+1}) = (s_n\circ g)(\mathbf{x}_{n+1})\times\pi_{n+1}(\mathbf{x}_{n+1})$ is continuous by Theorem 18.4 since both $s_n\circ g$ and π_{n+1} are continuous. Lastly we then have that $k : \mathbb{R}^{n+1} \to \mathbb{R}$ defined by $+\circ h$ is continuous by Theorem 18.2 part (c), where of course $+$ is the usual addition operation from $\mathbb{R}^2$ to $\mathbb{R}$, which we showed is continuous in Exercise 21.12. Now we claim that $k = s_{n+1}$. For any $\mathbf{x}_{n+1} \in \mathbb{R}^{n+1}$ we have

$$\begin{aligned}k(\mathbf{x}_{n+1}) &= (+\circ h)(\mathbf{x}_{n+1}) = +(h(\mathbf{x}_{n+1})) = +((s_n\circ g)(\mathbf{x}_{n+1})\times\pi_{n+1}(\mathbf{x}_{n+1})) \\ &= +(s_n(g(\mathbf{x}_{n+1})), x_{n+1}) = +(s_n(\mathbf{x}_n), x_{n+1}) = s_n(\mathbf{x}_n) + x_{n+1} \\ &= \sum_{i=1}^{n} x_i + x_{n+1} = \sum_{i=1}^{n+1} x_i \\ &= s_{n+1}(\mathbf{x}_{n+1})\,.\end{aligned}$$

This completes the induction since we have shown that $k = s_{n+1}$ is continuous.

Next we show that the function $p_{ij} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by $p_{ij}(\mathbf{x}_n \times \mathbf{y}_n) = x_i y_j$ is continuous, where of course $ij \in \{1, \ldots, n\}$. Define the function $g_{ij} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^2$ by $g_{ij} = \pi_i \times \pi_j$ as in Exercise 18.10, which we know is continuous by that exercise since the coordinate functions are continuous. Then the function $\cdot \circ g_{ij}$ from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}$ is also continuous by Theorem 18.2 part (c), where of course $\cdot$ is the normal multiplication operation from $\mathbb{R}^2$ to $\mathbb{R}$, which we know is continuous from Exercise 12.12. However, for any $\mathbf{x}_n, \mathbf{y}_n \in \mathbb{R}^n$, we have

$$(\cdot \circ g_{ij})(\mathbf{x}_n \times \mathbf{y}_n) = \cdot(g_{ij}(\mathbf{x}_n \times \mathbf{y}_n)) = \cdot(\pi_i(\mathbf{x}_n) \times \pi_j(\mathbf{y}_n)) = \cdot(x_i \times y_j) = x_i \cdot y_j = p_{ij}(\mathbf{x}_n \times \mathbf{y}_n)$$

so that $\cdot \circ g_{ij} = p_{ij}$ is continuous, which shows the desired result.

Now, by definition, each matrix component of the resultant matrix in matrix multiplication on $\mathrm{GL}(n)$ is a sum of products, where each product involves a term from each of the matrices, and the sum has n terms going across a row of the first matrix and a column of the second. Thus each component is a composition of the sum function $s_n : \mathbb{R}^n \to \mathbb{R}$ with a mapping f from $\mathbb{R}^{n^2} \times \mathbb{R}^{n^2} \to \mathbb{R}^n$, where each element in $\mathbb{R}^n$ of this mapping is a product function p_{ij}. Since have shown above that each p_{ij} is continuous, it follows from Theorem 19.6 that the mapping f is also continuous. Hence the composition $s_n \circ f$, i.e. the matrix component function, is also continuous by Theorem 18.2 part (c) since we have also shown above that s_n is continuous. Since each component function is continuous, it again follows from Theorem 19.6 that the overall matrix multiplication mapping from $\mathbb{R}^{n^2} \times \mathbb{R}^{n^2} \to \mathbb{R}^{n^2}$ is continuous.

Regarding the inverse element function, we recall from linear algebra that in the inverse of a matrix $A \in \mathrm{GL}(n)$ is

$$A^{-1} = \frac{1}{|A|} \mathrm{adj}(A) \,,$$

where $\mathrm{adj}(A)$ is the adjugate matrix of A and $|A|$ is the determinant of A, noting that this is nonzero since A is nonsingular. Now, the determinant is a sum of products so that the function $g : \mathrm{GL}(N) \to \mathbb{R}$ defined by $g(A) = |A|$ is continuous by the same arguments as above for matrix multiplication. Likewise each element of the adjugate matrix is a sum of products as well so that the function $f_{ij} : \mathrm{GL}(A) \to \mathbb{R}$ defined by $f_{ij}(A) = \mathrm{adj}(A)_{ij}$, i.e. the ith row and jth column component of the adjugate matrix, is also continuous.

Then clearly the corresponding component of the inverse matrix is the function $h_{ij} : \mathrm{GL}(A) \to \mathbb{R}$ defined by $h_{ij}(A) = f_{ij}(A)/g(A)$. Since both f_{ij} and g are continuous (and again noting that g is always nonzero), then their quotient h_{ij} is also continuous by Exercise 21.12. Hence, since each component h_{ij} of the inverse matrix is continuous, it follows that the inversion operation as a whole is continuous by Theorem 19.6 as above, considering the matrices as elements of $\mathbb{R}^{n^2}$. Since both multiplication and inversion are continuous, this shows that $\mathrm{GL}(n)$ is a topological group by definition. $\square$

Exercise TG.3

Let H be a subspace of G. Show that if H is also a subgroup of G, then both H and $\overline{H}$ are topological groups.

Solution:

Lemma TG.3.1. *Any subspace of a space satisfying the T_1 axiom also satisfies the T_1 axiom.*

Proof. Suppose that X is a subspace of Y and consider two distinct points $x, y \in X$. Then there is a neighborhood U of x in Y that does not contain y by Exercise 17.15. It then follows that $U \cap X$

is a neighborhood of x in X that does not contain y since $y \notin U$. A similar argument shows that there is a neighborhood of y in X that does not contain x. Hence X also satisfies the T_1 axiom, again by Exercise 17.15. □

Main Problem.

Proof. Presumably here G is a topological group. Hence G satisfies the T_1 axiom so that H and $\overline{H}$ do as well by Lemma TG.3.1 since they are subspaces of G.

We first show that $\overline{H}$ is a subgroup of G, noting that of course $\overline{H}$ is nonempty since H must be (as it is a subgroup) and $H \subset \overline{H}$. Let $f : G \times G \to G$ be the operation defined by $f(x \times y) = x \cdot y^{-1}$ for $x, y \in G$. It is a well known theorem of group theory that $\overline{H}$ is a subgroup of G if and only if $x \cdot y^{-1} \in \overline{H}$ for any $x, y \in \overline{H}$, which is to say that $f(\overline{H} \times \overline{H}) \subset \overline{H}$. This is exactly what we intend to show. We have the following deductions:

- As subsets of $G \times G$, $\overline{H} \times \overline{H} = \overline{H \times H}$ by Exercise 17.9 since $H \subset G$.
- Since f is continuous on $G \times G$, it follows that $f(\overline{H \times H}) \subset \overline{f(H \times H)}$ by Theorem 18.1.
- Since H is a subgroup we have that $x \cdot y^{-1} \in H$ for every $x, y \in H$, which is to say that $f(H \times H) \subset H$. It then follows from Exercise 17.6 part (a) that $\overline{f(H \times H)} \subset \overline{H}$.

Putting these all together, we can conclude that

$$f(\overline{H} \times \overline{H}) = f(\overline{H \times H}) \subset \overline{f(H \times H)} \subset \overline{H}\,,$$

which shows the desired result that $\overline{H}$ is a subgroup of G.

Next, obviously H and $\overline{H}$ are both groups by the definition of a subgroup. The operation of H (or $\overline{H}$) is of course the operation of G with its domain restricted to $H \times H$ (or $\overline{H} \times \overline{H}$). The continuity of this operation on H (or $\overline{H}$) follows from Theorem 18.2 part (d) since it is continuous on G since G is a topological group. Likewise the inversion function on H (or $\overline{H}$) is a restriction of the inversion function on G, and so is also continuous for the same reason. Hence H and $\overline{H}$ are topological groups by definition. □

Exercise TG.4

Let α be an element of G. Show that the maps $f_\alpha, g_\alpha : G \to G$ defined by

$$f_\alpha(x) = \alpha \cdot x \qquad \text{and} \qquad g_\alpha(x) = x \cdot \alpha$$

are homeomorphisms of G. Conclude that G is a *homogeneous space*. (This means that for every pair x, y of points of G, there exists a homeomorphism of G onto itself that carries x to y.)

Solution:

Proof. Clearly G is meant to be a topological group. Let e be the identity element of G. Define the function $f'_\alpha : G \to G$ by $f'_\alpha(x) = \alpha^{-1} \cdot x$. For any any $x \in G$ we then have

$$(f'_\alpha \circ f_\alpha)(x) = f'_\alpha(f_\alpha(x)) = f'_\alpha(\alpha \cdot x) = \alpha^{-1} \cdot (\alpha \cdot x) = (\alpha^{-1} \cdot \alpha) \cdot x = e \cdot x = x$$

and

$$(f_\alpha \circ f'_\alpha)(x) = f_\alpha(f'_\alpha(x)) = f_\alpha(\alpha^{-1} \cdot x) = \alpha \cdot (\alpha^{-1} \cdot x) = (\alpha \cdot \alpha^{-1}) \cdot x = e \cdot x = x\,.$$

This shows that both $f'_\alpha \circ f_\alpha = i_G$ and $f_\alpha \circ f'_\alpha = i_G$ so that f'_α is both a left inverse and a right inverse for f_α (see Exercise 2.5). Hence f_α is bijective and $f_\alpha^{-1} = f'_\alpha$ by Exercise 2.5 part (e). An analogous argument shows g_α is bijective and that $g'_\alpha(x) = x \cdot \alpha^{-1}$ is its inverse function.

To show that they are homeomorphisms, we note that the group operation is a continuous function since G is a topological group. We then have that the operation is continuous in each variable separately by Exercise 18.11. From this it clearly follows that both f_α and g_α are continuous since $f_\alpha(x) = \cdot(\alpha \times x)$ and $g_\alpha(x) = \cdot(x \times \alpha)$. Similarly f_α^{-1} and g_α^{-1} are continuous since we have $f_\alpha^{-1}(x) = f'_\alpha(x) = \cdot(\alpha^{-1} \times x)$ and $g_\alpha^{-1}(x) = g'_\alpha(x) = \cdot(x \times \alpha^{-1})$. This shows that both f_α and g_α are homeomorphisms by definition.

Now, to show that G is a homogeneous space, consider any points $x_0, y_0 \in G$. Set $\alpha = y_0 \cdot x_0^{-1}$, noting that of course $\alpha \in G$. We then claim that f_α as defined above is a homeomorphism that carries x_0 to y_0. Of course we already showed above that f_α is a homeomorphism, so all that remains is to show that $f_\alpha(x_0) = y_0$. To this end we have

$$f_\alpha(x_0) = \alpha \cdot x_0 = (y_0 \cdot x_0^{-1}) \cdot x_0 = y_0 \cdot (x_0^{-1} \cdot x_0) = y_0 \cdot e = y_0 \,,$$

which shows the desired result. □

Exercise TG.5

Let H be a subgroup of G. If $x \in G$, define $xH = \{x \cdot h \mid h \in H\}$; this set is called a ***left coset*** of H in G. Let G/H denote the collection of left cosets of H in G; it is a partition of G. Give G/H the quotient topology.

(a) Show that if $\alpha \in G$, the map f_α of the preceding exercise induces a homeomorphism of G/H carrying xH to $(\alpha \cdot x)H$. Conclude that G/H is a homogeneous space.

(b) Show that if H is a closed set in the topology of G then one-point sets are closed in G/H.

(c) Show that the quotient map $p : G \to G/H$ is open.

(d) Show that if H is closed in the topology of G and is a normal subgroup of G, then G/H is a topological group.

Solution:

First we note that it is a well-known theorem of group theory that any subgroup of a group contains the identity element of the group, which is also the identity element of the subgroup. So, for what follows, let e be the identity element of G and H above, from which it follows that $x \in xH$ for any $x \in G$ since we have $x = x \cdot e$ and $e \in H$. We also have that $H \in G/H$ since clearly $H = eH$. Also let $p : G \to G/H$ denote the quotient map corresponding to the quotient space.

(a)

Proof. Now, for $\alpha \in G$, define the function $h_\alpha : G/H \to G/H$ by mapping the left coset $xH \in G/H$ to $f_\alpha(x)H = (\alpha \cdot x)H$. We note that if $xH = yH$ for $x, y \in G$ then of course $y \in yH = xH$ so that $y = x \cdot h$ for some $h \in H$. Then

$$f_\alpha(y) = \alpha \cdot y = \alpha \cdot (x \cdot h) = (\alpha \cdot x) \cdot h = f_\alpha(x) \cdot h$$

so that $f_\alpha(y) \in f_\alpha(x)H$, which suffices to show that $f_\alpha(x)H = f_\alpha(y)H$ since G/H is a partition. Hence $h_\alpha(yH) = f_\alpha(y)H = f_\alpha(x)H = h_\alpha(xH)$ so that the mapping h_α is a well-defined function.

To show that h_α is a homeomorphism, we first show that it is a bijection. Suppose that xH and yH are left cosets where $h_\alpha(xH) = h_\alpha(yH)$. Then we have $h_\alpha(xH) = f_\alpha(x)H = f_\alpha(y)H = h_\alpha(yH)$, and hence $f_\alpha(x) \in f_\alpha(y)H$. From this it follows that

$$f_\alpha(x) = f_\alpha(y) \cdot h = (\alpha \cdot y) \cdot h = \alpha \cdot (y \cdot h) = f_\alpha(y \cdot h)$$

for some $h \in H$. Since f_α is injective (since it was shown in Exercise TG.4 to be bijective), it has to be that $x = y \cdot h$ so that $x \in yH$. Since also $x \in xH$ and G/H is a partition, it must be that $xH = yH$, which shows that h_α is injective. Now consider any coset $yH \in G/H$. Since f_α is a surjection we have that there is an $x \in G$ where $y = f_\alpha(x)$. Thus it immediately follows that $h_\alpha(xH) = f_\alpha(x)H = yH$, which shows that h_α is surjective since yH was arbitrary. This completes the proof that h_α is a bijection.

Next we digress a bit and show that $\bigcup h_\alpha(\mathcal{H}) = f_\alpha\left(\bigcup \mathcal{H}\right)$ for any subset $\mathcal{H} \subset G/H$, where we use the notation $\bigcup \mathcal{A} = \bigcup_{A \in \mathcal{A}} A$ for a collection of sets $\mathcal{A}$. So first consider any $x_0 \in \bigcup h_\alpha(\mathcal{H})$ so that there is a coset $yH \in h_\alpha(\mathcal{H})$ where $x_0 \in yH$. Then, since $yH \in h_\alpha(\mathcal{H})$, there is another coset $xH \in \mathcal{H}$ where $x_0 \in yH = h_\alpha(xH) = f_\alpha(x)H$. We then have that

$$x_0 = f_\alpha(x) \cdot h = (\alpha \cdot x) \cdot h = \alpha \cdot (x \cdot h) = f_\alpha(x \cdot h)$$

for some $h \in H$. Since clearly $x \cdot h \in xH$ and $xH \in \mathcal{H}$, we have that $x \cdot h \in \bigcup \mathcal{H}$. Then of course $x_0 \in f_\alpha(\bigcup \mathcal{H})$ since $x_0 = f_\alpha(x \cdot h)$, which shows that $\bigcup h_\alpha(\mathcal{H}) \subset f_\alpha\left(\bigcup \mathcal{H}\right)$ since x_0 was arbitrary. Now consider $x_0 \in f_\alpha(\bigcup \mathcal{H})$ so that there is a $y_0 \in \bigcup \mathcal{H}$ where $x_0 = f_\alpha(y_0)$. Then also there is a coset $xH \in \mathcal{H}$ where $y_0 \in xH$ since $y_0 \in \bigcup \mathcal{H}$. Hence $y_0 = x \cdot h$ for some $h \in H$. Define the coset $yH = h_\alpha(xH) = f_\alpha(x)H$ and we have

$$x_0 = f_\alpha(y_0) = \alpha \cdot y_0 = \alpha \cdot (x \cdot h) = (\alpha \cdot x) \cdot h = f_\alpha(x) \cdot h$$

so that $x_0 \in f_\alpha(x)H = yH$. Then, since $yH = h_\alpha(xH)$ and $xH \in \mathcal{H}$, we have that $yH \in h_\alpha(\mathcal{H})$. As we also have $x_0 \in yH$, it follows that $x_0 \in \bigcup h_\alpha(\mathcal{H})$. This shows that $\bigcup h_\alpha(\mathcal{H}) \supset f_\alpha\left(\bigcup \mathcal{H}\right)$ since x_0 was arbitrary, which completes the proof that $\bigcup h_\alpha(\mathcal{H}) = f_\alpha\left(\bigcup \mathcal{H}\right)$.

To return to the main goal, we therefore have that

$$\begin{aligned} \mathcal{U} \text{ is open in } G/H &\Leftrightarrow \bigcup \mathcal{U} \text{ is open in } G && \text{(by the definition of the quotient space)} \\ &\Leftrightarrow f_\alpha\left(\bigcup \mathcal{U}\right) = \bigcup h_\alpha(\mathcal{U}) \text{ is open in } G && \text{(since } f_\alpha \text{ is a homeomorphism)} \\ &\Leftrightarrow h_\alpha(\mathcal{U}) \text{ is open in } G/H\,, && \text{(by the definition of the quotient space)} \end{aligned}$$

noting that f_α was shown to be a homeomorphism in Exercise TG.4. This shows that h_α is a homeomorphism as desired.

Lastly, to show that G/H is homogeneous, consider two cosets $xH, yH \in G/H$. We know from what was shown in Exercise TG.4 that there is an f_α such that $y = f_\alpha(x)$. If h_α is the homeomorphism on G/H induced by f_α as defined above then we have $h_\alpha(xH) = f_\alpha(x)H = yH$. This suffices to show that G/H is homogeneous since xH and yH were arbitrary. □

(b)

Proof. Define the single-point subset $\mathcal{H}_0 = \{H\}$ of G/H, noting that we showed above why $H \in G/H$. We have that $\mathcal{H}_0$ is closed in G/H since we know that $p^{-1}(\mathcal{H}_0) = \bigcup \mathcal{H}_0 = H$ is closed in G, which follows from the alternative definition of a quotient map. Now consider any arbitrary one-point subset $\mathcal{H} = \{xH\} \subset G/H$. Since it was shown in part (a) that G/H is a homogeneous space, there is a homeomorphism $h_\alpha : G/H \to G/H$ that maps H to xH. Then we clearly have that $h_\alpha(\mathcal{H}_0) = h_\alpha(\{H\}) = \{xH\} = \mathcal{H}$. Since $\mathcal{H}_0$ is closed in G/H and h_α is a homeomorphism, it follows that $h_\alpha(\mathcal{H}_0) = \mathcal{H}$ is also closed in G/H. This shows the desired result since $\mathcal{H}$ was an arbitrary single-point subset. □

(c)

Proof. Let $g_\alpha : G \to G$ be the function defined by $g_\alpha(x) = x \cdot \alpha$ for $\alpha, x \in G$, which we know is a homeomorphism for any $\alpha \in G$ by Exercise TG.4. Consider any open set U of G.

We first show that $p^{-1}(p(U)) = \bigcup_{h \in H} g_h(U)$.

($\subset$) Consider arbitrary $x \in p^{-1}(p(U))$ so that $p(x) \in p(U)$. Of course $p(x) = xH$ so that $xH \in p(U)$ so that there is a $y \in U$ where $xH = p(y) = yH$. From this it follows that $x \in yH$ so that $x = y \cdot h_0$ for some $h_0 \in H$, and so $x = g_{h_0}(y)$. Since $y \in U$ we have that $x \in g_{h_0}(U)$, and thus of course $x \in \bigcup_{h \in H} g_h(U)$ since $h_0 \in H$. This shows that $p^{-1}(p(U)) \subset \bigcup_{h \in H} g_h(U)$ since x was arbitrary.

($\supset$) Now consider $x \in \bigcup_{h \in H} g_h(U)$ so that there is an $h_0 \in H$ where $x \in g_{h_0}(U)$. Hence $x = g_{h_0}(y)$ for some $y \in U$, and so $x = y \cdot h_0$. This shows that $x \in yH$ since $h_0 \in H$, and thus it must be that $xH = yH$. However, we have that $xH = yH = p(y)$ and $y \in U$ so that $xH \in p(U)$. Moreover $xH = p(x)$ so that $p(x) \in p(U)$ and hence $x \in p^{-1}(p(U))$. This shows that $p^{-1}(p(U)) \supset \bigcup_{h \in H} g_h(U)$, which shows the desired result.

Now, since each g_h is a homeomorphism for $h \in H$ and U is open in G, it follows that each $g_h(U)$ also open in G. Then of course their union $\bigcup_{h \in H} g_h(U) = p^{-1}(p(U))$ is open in G by the definition of a topology. Since $p^{-1}(p(U))$ is open in G, it follows that $p(U)$ is open in G/H since p is a quotient map. Then, since U was an arbitrary open set of G, this proves that p is an open map. □

(d) Recall from algebra that H being a normal subgroup of G means that $ghg^{-1} \in H$ for any $h \in H$ and $g \in G$. It is also an equivalent definition of that $xy \in H$ if and only if $yx \in H$ for $x, y \in G$.

Proof. First we need to show that we can define an operation on G/H that makes it into a group. This is done in the expected way: for $xH, yH \in G/H$ define $xH \cdot yH = (x \cdot y)H$. To show that this operation is well-defined, suppose that $x_0H = x_1H$ and $y_0H = y_1H$ are elements of G/H. Then of course $x_1 \in x_0H$ so that $x_1 = x_0 \cdot h_x$ for some $h_x \in H$. Similarly $y_1 \in y_0H$ so that $y_1 = y_0 \cdot h_y$ for some $h_y \in H$. Freely utilizing the associativity of the operation of G and suppressing the $\cdot$ by using multiplication notation, we then have that

$$\begin{aligned}
x_1y_1 &= (x_0h_x)(y_0h_y) \\
(x_1y_1)y_0^{-1} &= (x_0h_x)(y_0h_y)y_0^{-1} \\
x_1y_1y_0^{-1} &= x_0h_x(y_0h_yy_0^{-1}) \\
x_1y_1y_0^{-1} &= x_0h_xh_1 && \text{(where } h_1 = y_0h_yy_0^{-1} \in H \text{ since } H \text{ is normal)} \\
x_1y_1y_0^{-1} &= x_0h_2 && \text{(where } h_2 = h_xh_1 \in H \text{ since } H \text{ is a group)} \\
x_1y_1y_0^{-1}x_0^{-1} &= x_0h_2x_0^{-1} \\
x_1y_1y_0^{-1}x_0^{-1} &= h_3 && \text{(where } h_3 = x_0h_2x_0^{-1} \in H \text{ since } H \text{ is normal)}
\end{aligned}$$

Hence $(x_1y_1)(y_0^{-1}x_0^{-1}) = x_1y_1y_0^{-1}x_0^{-1} \in H$ so that also $(y_0^{-1}x_0^{-1})(x_1y_1) \in H$ by the equivalent definition of a normal subgroup. Therefore, for some $h \in H$, we have

$$\begin{aligned}
y_0^{-1}x_0^{-1}x_1y_1 = (x_0^{-1}y_0^{-1})(x_1y_1) &= h \\
x_0^{-1}x_1y_1 &= y_0h \\
x_1y_1 &= x_0y_0h \\
x_1y_1 &= (x_0y_0)h
\end{aligned}$$

so that $x_1 \cdot y_1 \in (x_0 \cdot y_0)H$, which of course shows that $(x_1 \cdot y_1)H = (x_0 \cdot y_0)H$, and hence the operation on G/H is well-defined.

Now we show that G/H with this operation satisfies the group axioms. We first note that, for $x, y \in G$, we have $xH, yH \in G/H$ and $x \cdot y \in G$ since G is a group so that $xH \cdot yH = (x \cdot y)H \in G/H$. Hence the operation is closed in G/H. Next, clearly $eH = H$ itself is the identity element for G/H since we have $eH \cdot xH = (e \cdot x)H = xH$ and $xH \cdot eH = (x \cdot e)H = xH$ for any $xH \in G/H$. We also have that the inverse element of xH is $x^{-1}H$ since $xH \cdot x^{-1}H = (x \cdot x^{-1})H = eH$ and $x^{-1}H \cdot xH = (x^{-1} \cdot x)H = eH$. Lastly, we have that

$$\begin{aligned}(xH \cdot yH) \cdot zH &= (x \cdot y)H \cdot zH = ((x \cdot y) \cdot z)H = (x \cdot (y \cdot z))H \\ &= xH \cdot (y \cdot z)H = xH \cdot (yH \cdot zH)\end{aligned}$$

since of course the operation on G is associative. This shows that the operation on G/H is associative as well, which completes the proof that G/H is a group.

To show that G/H is in fact a topological group, first it was shown in part (b) that one-point subsets of G/H are closed in G/H since H is closed in G. From this is follows that G/H satisfies the T_1 axiom since any finite subset of G/H is a finite union of one-point sets and so is also closed in G/H.

At this point we take a short digression and show that if $f : G \times G \to G$ is continuous, then the function $h : G/H \times G/H \to G/H$ defined by $h(xH, yH) = f(x, y)H$ is also continuous. To see this, we first claim that $h \circ (p \times p) = p \circ f$, where the function $p \times p : G \times G \to G/H \times G/H$ is defined as $(p \times p)(x, y) = (p(x), p(y))$ as in Exercise 18.10. This is easy to show as we have

$$\begin{aligned}(h \circ (p \times p))(x, y) &= h((p \times p)(x, y)) = h(p(x), p(y)) = h(xH, yH) \\ &= f(x, y)H = p(f(x, y)) = (p \circ f)(x, y)\end{aligned}$$

for any $x, y \in G$. Since both p and f are continuous, it follows that $p \circ f = h \circ (p \times p)$ is also continuous by Theorem 18.2 part (c). We also have that $p \times p$ is an open quotient map by the remarks in the text since p is an open quotient map by part (c).

At this point, we use Theorem 22.2 to show that h is continuous. As this can be confusing, we include the following table, which shows how the sets and functions in the statement of Theorem 22.2 map to the sets and functions we are working with:

Type	Theorem 22.2	Ours
Set	X	$G \times G$
Set	Y	$G/H \times G/H$
Set	Z	G/H
Function	$p : X \to Y$	$p \times p$
Function	$g : X \to Z$	$p \circ f = h \circ (p \times p)$
Function	$f : Y \to Z$	h

Now we show that the conditions of the theorem are met. We have already shown that $p \times p$ is a quotient map. Now let $P = \{(xH, yH)\}$ be a one-point subset of $G/H \times G/H$. Since $p \times p$ is a quotient map, it is surjective so that $(p \times p)((p \times p)^{-1}(P)) = P$ by Exercise 2.1. Then clearly

$$\begin{aligned}(p \circ f)((p \times p)^{-1}(P)) &= (h \circ (p \times p))((p \times p)^{-1}(P)) \\ &= h((p \times p)((p \times p)^{-1}(P))) = h(P) \\ &= \{h(xH, yH)\} ,\end{aligned}$$

which shows that $p \circ f$ is constant on the set $(p \times p)^{-1}(P)$. Thus $p \circ f$ induces the function h per Theorem 22.2 since $h \circ (p \times p) = p \circ f$ as shown above. It then follows by the theorem that h is continuous since we have shown above that $p \circ f$ is.

Returning to the main problem, since G is a topological group the function $f : G \times G \to G$ defined by $f(x, y) = x \cdot y^{-1}$ is continuous by Exercise TG.1. It then follows by what was just shown that $h : G/H \times G/H \to G/H$ defined by

$$h(xH, yH) = xH \cdot (yH)^{-1} = xH \cdot y^{-1}H = (x \cdot y^{-1})H = f(x, y)H$$

is also continuous. This suffices to show that G/H is also a topological group as desired, again by Exercise TG.1. □

Exercise TG.6

The integers $\mathbb{Z}$ are a normal subgroup of $(\mathbb{R}, +)$. The quotient $\mathbb{R}/\mathbb{Z}$ is a familiar topological group; what is it?

Solution:

The quotient $\mathbb{R}/\mathbb{Z}$ is the reals with addition modulo one. This is to say that two reals x and y are considered equivalent (i.e. they are in the same equivalence class) if $x - y \in \mathbb{Z}$. This is also homeomorphic and isomorphic (with respect to the group) to the topological group $(S^1, \cdot)$ from Exercise TG.2 part (d) by the map $f : \mathbb{R}/\mathbb{Z} \to S^1$ defined by $f(x\mathbb{Z}) = e^{i2\pi x}$. We shall not show this rigorously as doing so would be tedious, but it is not difficult to see intuitively.

Exercise TG.7

If A and B are subsets of G, let $A \cdot B$ denote the set of all points $a \cdot b$ for $a \in A$ and $b \in B$. Let A^{-1} denote the set of all points a^{-1}, for $a \in A$.

(a) A neighborhood V of the identity element e is said to be ***symmetric*** if $V = V^{-1}$. If U is a neighborhood of e, show that there is a symmetric neighborhood V of e such that $V \cdot V \subset U$. [Hint: If W is a neighborhood of e, then $W \cdot W^{-1}$ is symmetric.]

(b) Show that G is Hausdorff. In fact, show that if $x \neq y$, there is a neighborhood V of e such that $V \cdot x$ and $V \cdot y$ are disjoint.

(c) Show that G satisfies the following separation axiom, which is called the ***regularity axiom***: Given a closed set A and a point x not in A, there exist disjoint open sets containing A and x, respectively. [Hint: There is a neighborhood V of e such that $V \cdot x$ and $V \cdot A$ are disjoint.]

(d) Let H be a subgroup of G that is closed in the topology of G; let $p : G \to G/H$ be the quotient map. Show that G/H satisfies the regularity axiom. [Hint: Examine the proof of (c) when A is saturated.]

Solution:

Lemma TG.7.1. *Suppose that X and Y are topological spaces and $f : X \times X \to Y$ is continuous. Also suppose that $f(x, x) = y$ for some $x \in X$ and $y \in Y$. Then, for any neighborhood V of y, there is a neighborhood U of x such that $f(U \times U) \subset V$.*

Proof. Let V be any neighborhood of $y = f(x, x)$ in Y. Then there is a neighborhood U' of (x, x) in $X \times X$ such that $f(U') \subset V$ by Theorem 18.1 part (4). Now, since U' is an open set of $X \times X$ containing (x, x), there is a basis element $B = U_1 \times U_2$ of $X \times X$ such that $(x, x) \in U_1 \times U_2 = B \subset U'$, where of course U_1 and U_2 are open in X. Then, being a finite intersection of open sets, $U = U_1 \cap U_2$

is also open in X and we have $x \in U$ since $x \in U_1$ and $x \in U_2$. Hence U is a neighborhood of x in X, and of course both $U \subset U_1$ and $U \subset U_2$.

Now consider any $z \in f(U \times U)$ so that there is an $(x_1, x_2) \in U \times U$ where $f(x_1, x_2) = z$. Then $x_1 \in U \subset U_1$ and $x_2 \in U \subset U_2$ so that $(x_1, x_2) \in U_1 \times U_2 = B \subset U'$. Hence $z = f(x_1, x_2) \in f(U')$ so that also $z \in V$ since $f(U') \subset V$. This shows the desired result that $f(U \times U) \subset V$ since z was arbitrary. □

Lemma TG.7.2. $(x \cdot y)^{-1} = y^{-1} \cdot x^{-1}$ *for any x and y in a group.*

Proof. Let e be the identity element of the group. Then we have

$$\begin{aligned}
(x \cdot y)^{-1} \cdot (x \cdot y) &= e && \text{(definition of the inverse)} \\
\big((x \cdot y)^{-1} \cdot x\big) \cdot y &= e && \text{(associativity of the operation)} \\
(x \cdot y)^{-1} \cdot x &= e \cdot y^{-1} && \\
(x \cdot y)^{-1} \cdot x &= y^{-1} && \text{(definition of the identity element)} \\
(x \cdot y)^{-1} &= y^{-1} \cdot x^{-1} \, , &&
\end{aligned}$$

which shows the desired result. □

Lemma TG.7.3. *If G is a topological group and U is an open set of G then the sets $\alpha \cdot U$ and $U \cdot \alpha$ are also open in G for any $\alpha \in G$. Similarly, if C is a closed set of G then $\alpha \cdot C$ and $C \cdot \alpha$ are also closed.*

Proof. For $\alpha \in G$, of course $\alpha \cdot U$ denotes the set $\{\alpha \cdot x \mid x \in U\}$, and analogously $U \cdot \alpha = \{x \cdot \alpha \mid x \in U\}$. The openness of $\alpha \cdot U$ and $U \cdot \alpha$ follow almost immediately from what was shown in Exercise TG.4. It is trivial to show that $\alpha \cdot U = f_\alpha(U)$, where $f_\alpha : G \to G$ is defined by $f_\alpha(x) = \alpha \cdot x$ as in Exercise TG.4. We know from that exercise that f_α is a homeomorphism so that $\alpha \cdot U = f_\alpha(U)$ is open since U is. Analogously $U \cdot \alpha = g_\alpha(U)$ is also open for the same reason, where $g_\alpha(x) = x \cdot \alpha$ as in Exercise TG.4. If C is a closed set of G then $\alpha \cdot C = f_\alpha(C)$ and $C \cdot \alpha = g_\alpha(C)$ are also closed since the homeomorphisms f_α and g_α also of course preserve closed sets. □

Lemma TG.7.4. *If G is a topological group with identity element e and U is a neighborhood of e, then $U \cdot U^{-1}$ is a symmetric neighborhood of e.*

Proof. First we show that $U \cdot U^{-1}$ is indeed a neighborhood of e. We claim that

$$U \cdot U^{-1} = \bigcup_{\alpha \in U^{-1}} (U \cdot \alpha) \, ,$$

We have

$$\begin{aligned}
x \in U \cdot U^{-1} &\Leftrightarrow \exists y \in U \exists \alpha \in U^{-1} \, (x = y \cdot \alpha) \\
&\Leftrightarrow \exists \alpha \in U^{-1} \exists y \in U \, (x = y \cdot \alpha) \\
&\Leftrightarrow \exists \alpha \in U^{-1} \, (x \in U \cdot \alpha) \\
&\Leftrightarrow x \in \bigcup_{\alpha \in U^{-1}} (U \cdot \alpha)
\end{aligned}$$

which of course shows the desired result. Now, we know from Lemma TG.7.3 that each $U \cdot \alpha$ is open since U is open (being a neighborhood of e). Hence their union is open by the definition of a topology, which shows that $U \cdot U^{-1}$ is in fact open. Also, since $e \in U$ and $e^{-1} = e$ (a well-known property of the identity element in any group), we clearly have that $e = e \cdot e = e \cdot e^{-1} \in U \cdot U^{-1}$ and therefore $U \cdot U^{-1}$ is a neighborhood of e.

To show that $U \cdot U^{-1}$ is symmetric, we have

$$\begin{aligned} z \in (U \cdot U^{-1})^{-1} &\Leftrightarrow \exists x \in U \exists y \in U \left(z = (x \cdot y^{-1})^{-1}\right) \\ &\Leftrightarrow \exists x \in U \exists y \in U \left(z = (y^{-1})^{-1} \cdot x^{-1}\right) && \text{(by Lemma TG.7.2)} \\ &\Leftrightarrow \exists x \in U \exists y \in U \left(z = y \cdot x^{-1}\right) \\ &\Leftrightarrow \exists y \in U \exists x \in U \left(z = y \cdot x^{-1}\right) \\ &\Leftrightarrow z \in U \cdot U^{-1}, \end{aligned}$$

which shows that $(U \cdot U^{-1})^{-1} = U \cdot U^{-1}$ so that $U \cdot U^{-1}$ is symmetric by definition. □

Main Problem.

(a)

Proof. Suppose that U is any neighborhood of e. Since G is a topological group, we know that the function $f : G \times G \to G$ defined by $f(x, y) = x \cdot y$ is continuous. Then, since $f(e, e) = e \cdot e = e$, it follows from Lemma TG.7.1 that there is a neighborhood V' of e such that $f(V' \times V') \subset U$. Now we claim that $f(V' \times V') = V' \cdot V'$. For any $z \in f(V' \times V')$ we have that there is an $(x, y) \in V' \times V$ where $f(x, y) = x \cdot y = z$. Since $x, y \in V'$ and $z = x \cdot y$, this shows that $z \in V' \cdot V'$ so that $f(V' \times V') \subset V' \cdot V'$. To show the other direction, for any $z \in V' \cdot V'$ we have that $z = x \cdot y$ for some $x, y \in V'$. Then $(x, y) \in V' \times V'$ and $z = x \cdot y = f(x, y)$ so that $z \in f(V' \times V')$, which shows that $f(V' \times V') \supset V' \cdot V'$. This shows the desired result that $V' \cdot V' = f(V' \times V') \subset U$.

Similarly the function $g : G \times G \to G$ defined by $g(x, y) = x \cdot y^{-1}$ is also continuous by Exercise TG.1 since G is a topological group. Then, since V' is a neighborhood of e and $g(e, e) = e \cdot e^{-1} = e \cdot e = e$, it follows again from Lemma TG.7.1 that there is a neighborhood W of e such that $g(W \times W) \subset V'$. We have that $W \cdot W^{-1} = g(W \times W)$ by an argument analogous to that for f above so that $W \cdot W^{-1} = g(W \times W) \subset V'$. Let $V = W \cdot W^{-1} \subset V'$, which we know is a symmetric neighborhood of e by Lemma TG.7.4 and is the neighborhood we seek.

So consider any $z \in V \cdot V$ so that $z = x \cdot y$ for some $x, y \in V$. Then also $x, y \in V'$ since $V \subset V'$. From this it follows that $z = x \cdot y \in V' \cdot V'$ so that also $z \in U$ since $V' \cdot V' \subset U$. This shows the desired result that $V \cdot V \subset U$ since z was arbitrary, which completes the overall proof. □

(b)

Proof. Suppose that $x, y \in G$ and that $x \neq y$. Then there are neighborhoods U'_x of x and U'_y of y such that $y \notin U'_x$ and $x \notin U'_y$. This follows from Exercise 17.15 since G satisfies the T_1 axiom on account of it being a topological group. Then $U_x = U'_x \cdot x^{-1}$ and $U_y = U'_y \cdot y^{-1}$ are both neighborhoods of e since they are open by Lemma TG.7.3 and we have that $e = x \cdot x^{-1} \in U'_x \cdot x^{-1} = U_x$ since $x \in U'_x$, and analogously $e = y \cdot y^{-1} \in U'_y \cdot y^{-1} = U_y$ since $y \in U'_y$. Note that we also have that

$$\begin{aligned} z \in U_y \cdot y &\Leftrightarrow z \in (U'_y \cdot y^{-1}) \cdot y \\ &\Leftrightarrow \exists y' \in U'_y (z = (y' \cdot y^{-1}) \cdot y) \\ &\Leftrightarrow \exists y' \in U'_y (z = y' \cdot (y^{-1} \cdot y)) \\ &\Leftrightarrow \exists y' \in U'_y (z = y' \cdot e) \\ &\Leftrightarrow \exists y' \in U'_y (z = y') \\ &\Leftrightarrow z \in U'_y \end{aligned}$$

so that clearly $U_y \cdot y = U'_y$.

Now, let $U = U_x \cap U_y$, which is also obviously a neighborhood of e. It then follows from part (a) that there is a symmetric neighborhood V of e such that $V \cdot V \subset U$. Suppose that $V \cdot x$ and $V \cdot y$ are not disjoint so that there is a $z \in V \cdot x$ where also $z \in V \cdot y$. Then we have that $z = v_x \cdot x = v_y \cdot y$ for some $v_x, v_y \in V$. It then follows that $x = (v_x^{-1} \cdot v_y) \cdot y$ and, since $v_x^{-1} \in V^{-1} = V$ as V is symmetric, we have $v_x^{-1} \cdot v_y \in V \cdot V \subset U \subset U_y$. Thus $x = (v_x^{-1} \cdot v_y) \cdot y \in U_y \cdot y = U'_y$, but we know that x cannot be in U'_y by its definition per the T_1 axiom! This contradiction means that it must be that $V \cdot x$ and $V \cdot y$ are disjoint, which shows the desired result.

From this the fact that G is Hausdorff readily follows. Clearly $V \cdot x$ is a neighborhood of x since it is open by Lemma TG.7.3 and we have $x = e \cdot x \in V \cdot x$. Similarly $V \cdot y$ is a neighborhood of y. As we have shown that these are disjoint, this suffices to show that G is a Hausdorff space. □

(c)

Proof. A proof of this is similar to the proof of part (b). Since A is closed, we know that $A \cdot x^{-1}$ is also closed by Lemma TG.7.3. Then $G - A \cdot x^{-1}$ is of course open. Moreover if e were in $A \cdot x^{-1}$ then we would have $e = a \cdot x^{-1}$ for some $a \in A$ so that $a = e \cdot x = x$, which is not possible since we know that $x \notin A$. So it must be that $e \notin A \cdot x^{-1}$ so that $e \in G - A \cdot x^{-1}$ since of course $e \in G$. Hence $G - A \cdot x^{-1}$ is a neighborhood of e, and thus there is a symmetric neighborhood V of e such that $V \cdot V \subset G - A \cdot x^{-1}$ by what was shown in part (a).

We claim that $V \cdot A$ and $V \cdot x$ are disjoint. To see this, suppose to the contrary that there is a $y \in V \cdot A$ where also $y \in V \cdot x$. Then $y = v_a \cdot a = v_x \cdot x$ for some $v_a, v_x \in V$ and $a \in A$. Then we have

$$\begin{aligned} v_x \cdot x &= v_a \cdot a \\ v_a^{-1} \cdot v_x \cdot x &= a \\ v_a^{-1} \cdot v_x &= a \cdot x^{-1} \,. \end{aligned}$$

Since V is symmetric we have that $v_a^{-1} \in V^{-1} = V$ so that $v_a^{-1} \cdot v_x \in V \cdot V \subset G - A \cdot x^{-1}$ and hence $v_a^{-1} \cdot v_x \notin A \cdot x^{-1}$. However, clearly $v_a^{-1} \cdot v_x = a \cdot x^{-1}$ so that $v_a^{-1} \cdot v_x \in A \cdot x^{-1}$ since $a \in A$. This contradiction can only mean that in fact $V \cdot A$ and $V \cdot x$ are disjoint.

Now, for any $a \in A$, we have that $a = e \cdot a \in V \cdot A$ since $e \in V$, which shows that A is contained in $V \cdot A$. Similar to what was done in the beginning of the proof of Lemma TG.7.4, it is easy to show that

$$V \cdot A = \bigcup_{a \in A} V \cdot a \,,$$

which is open since each $V \cdot a$ is open by Lemma TG.7.3. Thus $V \cdot A$ is an open set containing A. We also have that $V \cdot x$ is a neighborhood of x since it is open by Lemma TG.7.3 and $x = e \cdot x \in V \cdot x$ since $e \in V$. Since we have already shown that $V \cdot A$ and $V \cdot x$ are disjoint, this shows the desired result that G satisfies the regularity axiom. □

(d)

Proof. Suppose that A is a closed subset of G/H and xH is an element of G/H not contained in A. Then $p^{-1}(A)$ is a closed subset of G since p is a quotient map. Moreover $p^{-1}(A)$ cannot contain x for, if it did, then we would have $xH = p(x) \in A$. So let V be a symmetric neighborhood of e such that $V \cdot p^{-1}(A)$ and $V \cdot x$ are disjoint, as shown to exist in part (c).

It was also shown in part (c) that $V \cdot p^{-1}(A)$ is an open set in G containing $p^{-1}(A)$ and $V \cdot x$ is a neighborhood of x in G. So, for any $yH \in A$ we have that $p(y) = yH \in A$ so that $y \in p^{-1}(A)$ and hence also $y \in V \cdot p^{-1}(A)$. Then of course $yH = p(y) \in p(V \cdot p^{-1}(A))$, which shows that

$A \subset p(V \cdot p^{-1}(A))$ since yH was arbitrary. We also have that $p(V \cdot p^{-1}(A))$ is open in G/H since $V \cdot p^{-1}(A)$ is open in G and p is an open map by Exercise TG.5 part (c). Similarly, since $x \in V \cdot x$, we have that $xH = p(x) \in p(V \cdot x)$ and $p(V \cdot x)$ is open in G/H since p is an open map. Thus $p(V \cdot x)$ is a neighborhood of xH in G/H.

Now we show that $p(V \cdot p^{-1}(A))$ and $p(V \cdot x)$ are disjoint subsets of G/H. Suppose to the contrary that they are not so that there is a $zH \in p(V \cdot p^{-1}(A))$ where also $zH \in p(V \cdot x)$. Then we have that there is a $y_a \in V \cdot p^{-1}(A)$ where $z = p(y_a)$ and likewise a $y_x \in V \cdot x$ where $z = p(y_x)$. Thus $y_a H = p(y_a) = z = p(y_x) = y_x H$ so that $y_x \in y_a H$, and hence there is an $h \in H$ where $y_x = y_a \cdot h$. Also since $y_a \in V \cdot p^{-1}(A)$ we have $y_a = v_a \cdot a'$ for some $v_a \in V$ and $a' \in p^{-1}(A)$. Putting this together we have

$$y_x = y_a \cdot h = v_a \cdot a' \cdot h\,.$$

Since $a' \in p^{-1}(A)$ we have that $a'H = p(a') \in A$. Also clearly $a' \cdot h \in a'H$ since $h \in H$. Therefore $a' \cdot h \in \bigcup A = p^{-1}(A)$. Hence we have that $y_x = v_a \cdot (a' \cdot h) \in V \cdot p^{-1}(A)$. Since also $y_x \in V \cdot x$, this violates the fact that $V \cdot p^{-1}(A)$ and $V \cdot x$ are disjoint. This contradiction means that it must be that in fact $p(V \cdot p^{-1}(A))$ and $p(V \cdot x)$ are disjoint, which completes the proof that G/H satisfies the regularity axiom. $\square$